第１８回
発展方程式若手セミナー
報告集

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Bilinear Formalism in Soliton Theory

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Abstract

a brief survey on the bilinear formalism originated by Hirota is given. First, the procedure to get soliton solutions of nonlinear evolution equations is discussed. Then algebraic structure of the equations in bilinear form is explained in a simple way. A few extensions of the formalism are also presented.

§1 Introduction

The bilinear formalism, which is originated by Hirota almost a quarter century ago, has played a crucial role in the study of integrable nonlinear systems. The formalism is perfectly suitable for obtaining not only multi-soliton solutions but also several types of special solutions of many nonlinear evolution equations. Moreover, it has been used for investigation of the algebraic structure of evolution equations and extension of the integrable systems.

In this paper, we attempt to present a brief survey on the bilinear formalism and discuss about several recent developments. Main interest is on the solutions of various classes of nonlinear evolution equations. Section 2 is devoted to the explanation of the procedure of obtaining soliton solutions. A few examples , which include the Korteweg-deVries (KdV) equation, the nonlinear Schrödinger (NLS) equation and the Toda equation, are given to show how we get the solutions. In this method, the variable transformation is crucial and the transformed variable becomes a key function. We shall call it the τ function. For multi-soliton solutions, it is written in the form of a polynomial in exponential functions.

The τ function can also be expressed in terms of Wronskian, Paffian or Casorati determinant. In §3, by using this fact we show that the τ functions of soliton equations satisfy algebraic identities in the bilinear form. This result is a reflection of the richness of algebraic structure which the soliton equations possess in common. Some of the indications of the richness will also be briefly mentioned in this section.

In §4, we discuss about a few extensions of the bilinear formalism. The first one is q-discrete soliton equations. It is shown that the Toda equation is naturally q-discretized in its bilinear form keeping the structure of solutions. The second is the trilinear formalism which gives a multi-dimensional extension of the soliton equations. The last is an extension to the ultra-discrete systems. We show that the idea of bilinear formalism is also applied to cellular automata which are the time evolution systems with all the variables discrete.

Finally in §5, we give concluding remarks.

§2 Hirota's Method

The first paper on the bilinear formalism by Hirota [1] considers the KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0. \tag{2.1}$$

Following his idea, let us construct soliton solutions of eq.(2.1). First we introduce dependent variable transformation,

$$u = 2(\log f)_{xx}. \tag{2.2}$$

Then, assuming suitable boundary condition, we obtain the bilinear form,

$$f_{xt}f - f_x f_t + f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2 = 0. \tag{2.3}$$

In order to write this equation in a compact form, we define an operater,

$$D_x^n D_t^m a \cdot b = (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^n (\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})^m a(x,t)b(x',t')\Big|_{x=x',t=t'}, \tag{2.4}$$

which is now called Hirota's operater. The followings are a few simple cases:

$$D_x a \cdot b = a_x b - a b_x,$$
$$D_x^2 a \cdot b = a_{xx} b - 2a_x b_x + a b_{xx},$$
$$D_x^3 a \cdot b = a_{xxx} b - 3a_{xx} b_x + 3a_x b_{xx} - a b_{xxx}.$$

By means of this operater, eq.(2.3) is rewritten by

$$(D_x D_t + D_x^4) f \cdot f = 0. \tag{2.5}$$

In order to obtain soliton solutions, we employ a perturbational technique. Let us expand the variable f as

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \cdots, \tag{2.6}$$

where ϵ is a formal parameter (we take $\epsilon = 1$ later on). Substituting eq.(2.6) into eq.(2.5) and equating terms with the same powers in ϵ, we have

$$\mathcal{O}(\epsilon) \qquad 2(\partial_x \partial_t + \partial_x^4) f_1 = \mathcal{L} f_1 = 0, \tag{2.7}$$

$$\mathcal{O}(\epsilon^2) \qquad \mathcal{L} f_2 = -(D_x D_t + D_x^4) f_1 \cdot f_1, \tag{2.8}$$

$$\mathcal{O}(\epsilon^3) \qquad \mathcal{L} f_3 = -2(D_x D_t + D_x^4) f_1 \cdot f_2, \tag{2.9}$$

$$\vdots$$

If we start with $f_1 = e^{\eta_1}$ in eq.(2.7), then we find that η_1 should be given by $\eta_1 = p_1(x - p_1^2 t) + \eta_1^{(0)}$, where p_1 and $\eta_1^{(0)}$ are arbitrary parameters. Furthermore, by noticing the formula,

$$D_x^n e^{\alpha x} \cdot e^{\beta x} = (\alpha - \beta)^n e^{(\alpha+\beta)x}, \tag{2.10}$$

we see that all the higher order terms of eq.(2.6) can be taken zero in eqs.(2.8, 2.9, $\cdots$). Hence $f = 1 + e^{\eta_1}$ is an exact solution of eq.(2.5), which gives the one soliton solution of the KdV eq.(2.1),

$$u = 2(\log f)_{xx} = \frac{p_1^2}{2} \mathrm{sech}^2 \frac{1}{2} \{p_1(x - p_1^2 t) + \eta_1^{(0)}\}. \tag{2.11}$$

Since eq.(2.7) is linear in f_1, we may take linear sum of exponential functions as a starting fuction. Let us start with $f_1 = e^{\eta_1} + e^{\eta_2}$, where $\eta_j = p_j(x - p_j^2 t) + \eta_j^{(0)}, p_j, \eta_j^{(0)} \in R$. For this function, eq.(2.8) is satisfied by

$$f_2 = e^{\eta_1 + \eta_2 + A_{12}}, \qquad e^{A_{12}} = (\frac{p_1 - p_2}{p_1 + p_2})^2,$$

and again $f_3, f_4, \cdots$ can be taken zero. Thus we have an exact solution,

$$f = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+A_{12}}. \tag{2.12}$$

In the physical variable u, this corresponds to the two soliton solution which describes a collision of two solitons. The parameter A_{12} relates to the phase shift after the collision.

We can obtain a solution describing collision of any number of solitons in principal, if we procede the perturbational calculation to higher orders. It is called the N-soliton solution and will be given in §3 in an elegant form.

The bilinear formalism has been successfully applied to various classes of nonlinear evolution equations. One of the important examples in one spatial dimension is the NLS equation [2],

$$i\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} + 2|\psi|^2\psi = 0. \tag{2.13}$$

For the complex variable ψ, we introduce the variable transformation, $\psi = g/f$ with real f. Then we obtain

$$(iD_t + D_x^2)g \cdot f - \frac{g}{f}(D_x^2 f \cdot f - 2gg^*) = 0,$$

where asterisk denotes complex conjugate. Since we introduced two variables f and g for one variable ψ, we may decouple this equation to yield

$$\begin{cases} (iD_t + D_x^2)g \cdot f = 0, \\ D_x^2 f \cdot f = 2gg^*. \end{cases} \tag{2.14}$$

Again by applying a perturbational technique,

$$f = 1 + \epsilon^2 f_2 + \epsilon^4 f_4 + \cdots, \quad g = \epsilon g_1 + \epsilon^3 g_3 + \cdots,$$

we get soliton solutions.

In particular, the one soliton solution is given by

$$g = e^{\eta}, \quad f = 1 + \frac{1}{(P+P^*)^2} e^{\eta+\eta^*},$$

where $\eta = Px + iP^2 t + \eta^{(0)}, P, \eta^{(0)} \in C$. Rewriting $P = p + ik$ for $p, k \in R$ and using $\psi = g/f$, we have

$$\psi = p \,\mathrm{sech}\, p(x - 2kt - x_0) e^{i\{kx-(k^2-p^2)t+\mathrm{Im}\eta^{(0)}\}}, \tag{2.15}$$

where x_0 is an appropriate phase constant.

Another example is the Toda lattice equation [3],

$$\frac{d^2}{dt^2}\log(1+V_n) = V_{n-1} - 2V_n + V_{n+1}. \tag{2.16}$$

According to Hirota, this is the first equation which he applied the bilinear formalism to obtain soliton solution, although the paper was published two years later than the one for the KdV equation.

Let us substitute

$$V_n = \frac{d^2}{dt^2}\log\tau_n, \tag{2.17}$$

into eq.(2.16). Then assuming a suitable boundary condition, we have

$$\frac{d^2\tau_n}{dt^2}\tau_n - (\frac{d\tau_n}{dt})^2 = \tau_{n+1}\tau_{n-1} - \tau_n^2. \tag{2.18}$$

It is noted that eq.(2.18) may be rewritten by

$$(D_t^2 - 4\sinh^2\frac{D_n}{2})\tau_n\cdot\tau_n = 0, \tag{2.19}$$

where we have introduced the difference operaters,

$$e^{D_n}f_n\cdot f_n = e^{\partial_n - \partial_{n'}}f_nf_{n'}\Big|_{n=n'} = f_{n+1}f_{n-1}, \tag{2.20}$$

with

$$e^{\epsilon\frac{\partial}{\partial x}}f(x) = f(x+\epsilon) \quad \text{or} \quad e^{\partial n}f_n = f_{n+1}. \tag{2.21}$$

The one lattice soliton solution is given by

$$\tau_n = 1 + e^{2\eta}, \quad \eta = Pn - \Omega t + \eta^{(0)}, \quad \Omega^2 = \sinh^2 P, \tag{2.22}$$

or

$$V_n = \Omega^2\mathrm{sech}^2\eta. \tag{2.23}$$

The above three examples are all in two dimensions. The equations extended to 3 dimensions, the Kadomtsev-Petviashvili (KP), the Davey-Stewartson (DS) and the 2-dimensional Toda (2D Toda) equations, have also been successfully treated in the bilinear formalism [4-6]. The algebraic structure of soliton solutions becomes very clear in this formalism, which we shall see in the following section.

§3 Algebraic Identities

The 2D Toda equation,

$$\frac{\partial^2}{\partial x \partial y} \log(1+V_n) = V_{n-1} - 2V_n + V_{n+1}, \tag{3.1}$$

has been first presented by Darboux in 19th century. This is now well known as a generic semi-discrete soliton equation. Equation (3.1) is reduced to

$$D_x D_y \tau_n \cdot \tau_n = 2(\tau_{n+1}\tau_{n-1} - \tau_n^2), \tag{3.2}$$

by substituting

$$V_n = \frac{\partial^2}{\partial x \partial y} \log \tau_n, \tag{3.3}$$

and assuming an appropriate boundary condition.

We now show that eq.(3.2) is nothing but an algebraic identity for determinants [7].

PROPOSITION 3.1

Equation (3.2) is satisfied by the following Casorati determinant:

$$\tau_n(x,y) = \begin{vmatrix} f_n^{(1)} & f_{n+1}^{(1)} & \cdots & f_{n+N-1}^{(1)} \\ f_n^{(2)} & f_{n+1}^{(2)} & \cdots & f_{n+N-1}^{(2)} \\ \vdots & \vdots & & \vdots \\ f_n^{(N)} & f_{n+1}^{(N)} & \cdots & f_{n+N-1}^{(N)} \end{vmatrix}, \tag{3.4}$$

where

$$\frac{\partial}{\partial x} f_m^{(j)} = f_{m+1}^{(j)}, \quad \frac{\partial}{\partial y} f_m^{(j)} = -f_{m-1}^{(j)}, \qquad j = 1, 2, 3, \cdots, N. \tag{3.5}$$

Let us give a rough proof. For $N = 1$, substituting $\tau_n = f_n^{(1)}$ into eq.(3.2), we obtain

$$\begin{aligned} D_x D_y \tau_n \cdot \tau_n &= 2(\tau_{n,xy}\tau_n - \tau_{n,x}\tau_{n,y}) \\ &= 2\left(\frac{\partial^2 f_n^{(1)}}{\partial x \partial y} f_n^{(1)} - \frac{\partial f_n^{(1)}}{\partial x} \cdot \frac{\partial f_n^{(1)}}{\partial y}\right) \\ &= -2(f_n^{(1)} f_n^{(1)} - f_{n+1}^{(1)} f_{n-1}^{(1)}) \\ &= \text{RHS}. \end{aligned}$$

For $N = 2$, we first notice the identity,

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ 0 & a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 & b_3 \end{vmatrix} = 0,$$

for any entries a_j, b_j. Applying a Laplace expansion in 2×2 minors to the left-hand side, we get

$$\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \begin{vmatrix} a_0 & a_2 \\ b_0 & b_2 \end{vmatrix} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \begin{vmatrix} a_0 & a_3 \\ b_0 & b_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0, \qquad (3.6)$$

which is called the Plücker relation. If we simply write eq.(3.6) by

$$(0,1)(2,3) - (0,2)(1,3) + (0,3)(1,2) = 0, \qquad (3.7)$$

and have a correspondence,

$$\tau_n = \begin{vmatrix} f_n^{(1)} & f_{n+1}^{(1)} \\ f_n^{(2)} & f_{n+1}^{(2)} \end{vmatrix} \iff (0,1),$$

then, by noticing the correspondences, $\tau_{n,x} \iff (0,2), \tau_{n,y} \iff -(-1,1), \tau_{n,xy} \iff -(0,1) - (-1,2), \tau_{n+1} \iff (1,2), \tau_{n-1} \iff (-1,0)$, we easily find that eq.(3.2) is equivalent to the identity (3.7). For $N \geq 3$, we may employ the same idea to show that eq.(3.4) satisfies eq.(3.2) or the equivalent indentity (3.7). For example, in the case of $N = 3$, we can start with the identity,

$$\begin{vmatrix} f & a_0 & a_1 & 0 & a_2 & a_3 \\ g & b_0 & b_1 & 0 & b_2 & b_3 \\ h & c_0 & c_1 & 0 & c_2 & c_3 \\ 0 & 0 & a_1 & f & a_2 & a_3 \\ 0 & 0 & b_1 & g & b_2 & b_3 \\ 0 & 0 & c_1 & h & c_2 & c_3 \end{vmatrix} = 0. \qquad (3.8)$$

It is noted however that small modification is necessary to reduce eq.(3.2) to the identity (3.7).

The soliton solutions of the 2D Toda equation are obtained from eq.(3.4) by making a particular choice on the functions $f_n^{(j)}$. The size N of the determinant in eq.(3.4) corresponds to the number of solitons. The one soliton solution is, for example, given by

$$\tau_n = f_n^{(1)} = p^n e^{px - \frac{1}{p}y} + q^n e^{qx - \frac{1}{q}y}, \qquad (3.9)$$

where p and q are arbitrary parameters.

The Plücker relation (3.6) is a key identity for soliton equations. Actually Sato [8, 9] noticed that the bilinear form of the KP equation,

$$(4u_t - 12uu_x - u_{xxx})_x - 3u_{yy} = 0 \tag{3.10}$$

is nothing but the Plücker relation and discovered that the totality of solutions of the KP equation as well as of its generalization constitutes an infinite dimensional Grassmann manifold. The class of equations is now called the KP hierarchy.

Let us briefly sketch a part of his result. Through the variable transformation, $u = (\log \tau)_{xx}$, we have the bilinear form of the KP equation,

$$(4D_xD_t - D_x^4 - 3D_y^2)\tau \cdot \tau = 0. \tag{3.11}$$

By applying the same tecknique as for the 2D Toda equation, it is shown that eq.(3.11) is satisfied by the Wronski determinant,

$$\tau(x,y,t) = \begin{vmatrix} f^{(1)} & \partial_x f^{(1)} & \cdots & \partial_x^{N-1} f^{(1)} \\ f^{(2)} & \partial_x f^{(2)} & \cdots & \partial_x^{N-1} f^{(2)} \\ \vdots & \vdots & & \vdots \\ f^{(N)} & \partial_x f^{(N)} & \cdots & \partial_x^{N-1} f^{(N)} \end{vmatrix}, \tag{3.12}$$

where

$$\frac{\partial}{\partial y} f^{(j)} = \frac{\partial^2}{\partial x^2} f^{(j)}, \quad \frac{\partial}{\partial t} f^{(j)} = \frac{\partial^3}{\partial x^3} f^{(j)}. \tag{3.13}$$

For the purpose, it is convenient to introduce the notation [10], $\tau = (0,1,2,\cdots,N-1)$. Then noticing $\tau_x = (0,1,2,\cdots,N-2,N), \tau_y = -(0,1,2,\cdots,N-3,N-1,N) + (0,1,2,\cdots,N-3,N-2,N+1)$ and so on, we find that eq.(3.11) is essentially the same as eq.(3.7), which means eq.(3.12) automatically satisfies the KP equation.

Shortly after Sato's discovery, Date, Jimbo, Kashiwara and Miwa [11] extended his idea and developed the theory of transformation groups for soliton equations. Moreover, the 2D Toda equation has been shown to belong to an extension of the KP hierarchy [12, 13]. All these results make it possible to understand the soliton theory from a unified point of view. For example, the relationship among the inverse scattering transform, Hirota's method and the Bäcklund transformation is clearly explained by the infinite dimensional Lie algebra and its representation on a function space.

As we see from eqs.(3.4) and (3.12), the semi-discrete 2D Toda and the continuous KP equations possess solutions with common structure. The Casorati determinant is the discrete version of Wronski determinant. Moreover, the Casorati determinant (3.4) itself is considered to be the Wronski determinant if we employ the linear relation (3.5) for the entries. Actually both equations are the relatives because the KP is obtained by taking a proper continuous limit of the 2D Toda equaton.

Then a natural question is whether there exists a full-discrete equation which has the same type of solutions. One answer was given by Hirota[14]. The equation, which Hirota called the discrete analogue of generalized Toda equation, is written in bilinear form by

$$\tau_n(l+1,m+1)\tau_n(l,m) - \tau_n(l+1,m)\tau_n(l,m+1)$$

$$= ab\{\tau_{n+1}(l,m+1)\tau_{n-1}(l+1,m) - \tau_n(l+1,m+1)\tau_n(l,m)\}, \tag{3.14}$$

where a and b are parameters related to the difference interval (see below). Since the algebraic structure of this equation was studied by Miwa [15] shortly after Hirota's finding, we call eq.(3.14) the Hirota-Miwa equation.

As expected, the solution of eq.(3.14) is again given by the Casorati determinant. Its explicit form is exactly the same as eq.(3.4). Only difference is the linear equations which should be satisfied by the entries. In this case they are given by

$$\Delta_l f_n^{(j)}(l,m) \equiv \frac{1}{a}\{f_n^{(j)}(l+1,m) - f_n^{(j)}(l,m)\} = f_{n+1}^{(j)}(l,m), \tag{3.15}$$

$$\Delta_m f_n^{(j)}(l,m) \equiv \frac{1}{b}\{f_n^{(j)}(l,m+1) - f_n^{(j)}(l,m)\} = -f_{n-1}^{(j)}(l,m). \tag{3.16}$$

If we read l, m as x, y, respectively, and take a continuous limit, then we obtain the 2D Toda eq.(3.2).

The Hirota-Miwa equation may be considered as one of master equations in soliton theory, since we recover many of soliton equations by taking proper continuous limits [14].

Finally in this section, we comment on another class of solutions of the 2D Toda equation. The Casorati determinant solution (3.4) is obtained by assuming the suitable boundary condition in an infinite lattice. We may instead consider a finite lattice. If we impose the boundary condition $V_0 = V_M = 0$ for some positive integer M, the system is called the 2D Toda molecule equation. In this context, we call the infinite lattice system the 2D Toda lattice equation.

The Toda molecule equation is reduced to its bilinear form,

$$D_x D_y \tau_n \cdot \tau_n = 2\tau_{n+1}\tau_{n-1}, \tag{3.17}$$

with $\tau_{-1} = \tau_{M+1} = 0$, by introducing the variable transformation (3.3). It is known [7] that eq.(3.17) admits the solution,

$$\tau_n(x,y) = \begin{vmatrix} f(x,y) & \partial_x f & \cdots & \partial_x^{n-1} f \\ \partial_y f & \partial_x \partial_y f & \cdots & \partial_x^{n-1}\partial_y f \\ \vdots & \vdots & & \vdots \\ \partial_y^{n-1} f & \partial_x \partial_y^{n-1} f & \cdots & \partial_x^{n-1}\partial_y^{n-1} f \end{vmatrix}, \tag{3.18}$$

for $n \geq 1$ and $\tau_0 = 1$, where the function $f(x,y)$ is given by

$$f(x,y) = \sum_{k=1}^{M} f_k(x) g_k(y), \tag{3.19}$$

for arbitrary f_k and g_k. Since this solution is a Wronskian with respect to x in the horizontal direction and with respect to y in the vertical direction, we call the determinant a two-directional Wronskian. The proof is given by using the Laplace expansion or the Jacobi identity for determinants. It should be remarked that the solution (3.18) is meaningful only for discrete system since the discrete variable n determines the size of determinant.

§4 Extensions

The algebraic structure of determinant solutions discussed in the preceding section is crucial to consider extensions of nonlinear integrable systems. In this section we present a few examples which have been obtained based on the structure.

4-1 q-discrete Toda equation

As we have seen in §3, the solutions of the (continuous) KP, the (semi-discrete) 2D Toda and the (full-discrete) Hirota-Miwa have the same structure. Only the difference is the linear equations which the entries of determinant should satisfy. This fact suggests that if we can generalize the linear equations we may have another integrable system. In this case integrable means that the equation admits the similar type of determinant solutions. The q-difference version of 2D Toda equation is just such a case [16].

Let us introduce an operator,

$$\delta_{q^\alpha,x} f(x,y) = \frac{f(x,y) - f(q^\alpha x, y)}{(1-q)x}, \tag{4.1}$$

which reduces to $\alpha\partial/\partial x$ in the limit of $q \to 1$. Note that this operator reduces to the original q-difference operator if α is taken to be 1.

The q-difference version of 2D Toda equation is given by

$$\{\delta_{q^2,x}\delta_{q^2,y}\tau_n(x,y)\}\tau_n(x,y) - \{\delta_{q^2,x}\tau_n(x,y)\}\{\delta_{q^2,y}\tau_n(x,y)\}$$
$$= \tau_{n+1}(x,q^2y)\tau_{n-1}(q^2x,y) - \tau_n(q^2x,q^2y)\tau_n(x,y). \qquad (4.2).$$

Again by using a Laplace expansion, we can show that eq.(4.2) admits the solution of Casorati determinant type, eq.(3.4). The linear eqs.(3.5) now become

$$\delta_{q^2,x}f_n^{(j)}(x,y) = f_{n+1}^{(j)}(x,y) \qquad (4.3)$$

and

$$\delta_{q^2,y}f_n^{(j)}(x,y) = -f_{n-1}^{(j)}(x,y). \qquad (4.4)$$

It is noted that the q-discrete version of the 2D Toda equation is considered to be an extension of Hirota-Miwa equation. The former equation is obtained by reading $l+1 \to q^2x,\quad m+1 \to q^2y,\quad a \to (q-1)x,\quad b \to (q-1)y$ in the latter.

If we impose a restriction on variables in eq.(4.2), we are able to obtain a reduced system. Let us introduce a variable r by $xy = r^2$. Then, for example, we have

$$\{\delta_{q^2,x}\tau_n(x,y)\}\{\delta_{q^2,y}\tau_n(x,y)\} = \{\delta_{q,r}\tau_n(r)\}^2.$$

By using this kind of reduction, we obtain from eq.(4.2),

$$\begin{aligned}(\frac{1}{r}\delta_{q,r} + q^2\delta_{q,r}^2)\tau_n(r)\cdot\tau_n(r) - \{\delta_{q,r}\tau_n(r)\}^2 \\ = \tau_{n+1}(qr)\tau_{n-1}(qr) - \tau_n(q^2r)\tau_n(r),\end{aligned} \qquad (4.5)$$

which is considered to be the q-difference version of the cylindrical Toda equation. We find that the solution for eq.(4.5) is given by the Casorati determinant whose entries are expressed by the q-Bessel function.

Finally in this subsection, we remark that a q-discrete version of the Toda molecule equation and its solution can also be constructed by extending eqs.(3.17) and (3.18) [17].

4-2 Trilinear Formalism

In order to prove Proposition 3.1, we have used identities for determinants. We have seen that the Plücker relation (3.6) is obtained by applying a Laplace expansion to the determinants and that the 2D Toda equation is equivalent to the relation. One extension of soliton equations is possible by following this simple idea. It is the trilinear formalism [18-20]. By this formalism, we can costruct four dimensional nonlinear equations which admit solutions expressed by Wronski or Casorati determinants. We here show the procedure for the semi-discrete case [19].

First we consider the following identities for $(3N+3)\times(3N+3)$ determinant:

$$\left|\begin{array}{cccc}
\begin{matrix} A & \begin{matrix} f_{m,n-1} \\ f_{m+1,n-1} \\ \vdots \\ f_{m+N-2,n-1} \\ f_{m+N-1,n-1} \\ f_{m+N,n-1} \end{matrix} \end{matrix} & 0 & 0 & \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \\
\hline
0 & \begin{matrix} A & \begin{matrix} f_{m,n+N-1} \\ f_{m+1,n+N-1} \\ \vdots \\ f_{m+N-2,n+N-1} \\ f_{m+N-1,n+N-1} \\ f_{m+N,n+N-1} \end{matrix} \end{matrix} & 0 & \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \\
\hline
0 & 0 & \begin{matrix} A & \begin{matrix} f_{m,n+N} \\ f_{m+1,n+N} \\ \vdots \\ f_{m+N-2,n+N} \\ f_{m+N-1,n+N} \\ f_{m+N,n+N} \end{matrix} \end{matrix} & \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}
\end{array}\right| = 0, \tag{4.6}$$

where A is the matrix given by

$$A = \begin{pmatrix} f_{m,n} & f_{m,n+1} & \cdots & f_{m,n+N-2} \\ f_{m+1,n} & f_{m+1,n+1} & \cdots & f_{m+1,n+N-2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m+N-2,n} & f_{m+N-2,n+1} & \cdots & f_{m+N-2,n+N-2} \\ f_{m+N-1,n} & f_{m+N-1,n+1} & \cdots & f_{m+N-1,n+N-2} \\ f_{m+N,n} & f_{m+N,n+1} & \cdots & f_{m+N,n+N-2} \end{pmatrix}. \tag{4.7}$$

Applying a Laplace expansion in $(N+1)\times(N+1)$ minors to the left-hand side of eq.(4.6), we have a trilinear form,

$$\begin{vmatrix} \partial_y \tau_{m,n-1} & \tau_{m,n-1} & \tau_{m+1,n-1} \\ \partial_y \tau_{m,n} & \tau_{m,n} & \tau_{m+1,n} \\ \partial_y \partial_x \tau_{m,n} & \partial_x \tau_{m,n} & \partial_x \tau_{m+1,n} \end{vmatrix} = 0, \tag{4.8}$$

where

$$\tau_{m,n} = \begin{vmatrix} f_{m,n} & f_{m,n+1} & \cdots & f_{m,n+N-1} \\ f_{m+1,n} & f_{m+1,n+1} & \cdots & f_{m+1,n+N-1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m+N-1,n} & f_{m+N-1,n+1} & \cdots & f_{m+N-1,n+N-1} \end{vmatrix}, \tag{4.9}$$

and f satisfies

$$\frac{\partial}{\partial x} f_{m,n} = f_{m,n+1}, \quad \frac{\partial}{\partial y} f_{m,n} = f_{m+1,n}. \tag{4.10}$$

This result shows that the τ function (4.9) in the form of two-directional Casorati determinant is a solution of the four (two discrete + two continuous) dimensional eq.(4.8).

If we introduce the dependent variables ψ, ϕ by

$$\psi_{m,n} = \log \frac{\tau_{m,n}}{\tau_{m,n-1}}, \quad \phi_{m,n} = \log \frac{\tau_{m,n}}{\tau_{m+1,n}}, \tag{4.11}$$

then eq.(4.8) is reduced to a coupled system,

$$\partial_x \partial_y \phi_{m,n} = \frac{\partial_x \phi_{m,n} \partial_y \psi_{m,n}}{e^{\phi_{m,n} - \phi_{m,n-1}} - 1} - \frac{\partial_x \phi_{m+1,n} \partial_y \psi_{m+1,n}}{e^{\phi_{m+1,n} - \phi_{m+1,n-1}} - 1}, \tag{4.12a}$$

$$\partial_x \partial_y \psi_{m,n} = \frac{\partial_x \phi_{m,n} \partial_y \psi_{m,n}}{e^{\phi_{m,n} - \phi_{m+1,n}} - 1} - \frac{\partial_x \phi_{m,n-1} \partial_y \psi_{m,n-1}}{e^{\phi_{m,n-1} - \phi_{m+1,n-1}} - 1}, \tag{4.12b}$$

with a constraint

$$\psi_{m+1,n} - \psi_{m,n} = \phi_{m,n-1} - \phi_{m,n}. \tag{4.13}$$

Furthermore, if the reduction, $\phi(x,y)_{m,n} = q(x+y)_{m+n}$, $\psi(x,y)_{m,n} = q(x+y)_{m+n-1}$, is imposed, then eqs.(4.12) reduce to

$$\partial_x^2 q_n = -\partial_x q_n \left(\frac{\partial_x q_{n-1}}{e^{q_n - q_{n-1}} - 1} - \frac{\partial_x q_{n+1}}{e^{q_{n+1} - q_n} - 1} \right), \tag{4.14}$$

which is nothing but the relativistic Toda equation proposed by Ruijsenaars. Therefore, eqs.(4.12) is considered to be a 2+2 dimensional extension of the relativistic Toda equation [21, 22].

In the continuous case, we have a hierarchy of trilinear equations [18],

$$\begin{vmatrix} p_i(\tilde{\partial}) p_l(-\tilde{\partial}')\tau & p_i(\tilde{\partial}) p_m(-\tilde{\partial}')\tau & p_i(\tilde{\partial}) p_n(-\tilde{\partial}')\tau \\ p_j(\tilde{\partial}) p_l(-\tilde{\partial}')\tau & p_j(\tilde{\partial}) p_m(-\tilde{\partial}')\tau & p_j(\tilde{\partial}) p_n(-\tilde{\partial}')\tau \\ p_k(\tilde{\partial}) p_l(-\tilde{\partial}')\tau & p_k(\tilde{\partial}) p_m(-\tilde{\partial}')\tau & p_k(\tilde{\partial}) p_n(-\tilde{\partial}')\tau \end{vmatrix} = 0, \tag{4.15}$$

for arbitrary nonnegative integers i, j, k, l, m, n, where τ is a function of $x_1, x_2, x_3, \cdots$, $y_1, y_2, y_3, \cdots$, $\tilde{\partial}, \tilde{\partial}'$ are defined by

$$\tilde{\partial} = (\frac{\partial}{\partial x_1}, \frac{1}{2}\frac{\partial}{\partial x_2}, \frac{1}{3}\frac{\partial}{\partial x_3}, \cdots), \tag{4.16a}$$

$$\tilde{\partial}' = (\frac{\partial}{\partial y_1}, \frac{1}{2}\frac{\partial}{\partial y_2}, \frac{1}{3}\frac{\partial}{\partial y_3}, \cdots), \tag{4.16b}$$

respectively and $p_j, j = 1, 2, \cdots$, are polynomials defined by

$$\exp\left(\sum_{n=1}^{\infty} x_n \lambda^n\right) = \sum_{j=0}^{\infty} p_j(x)\lambda^j. \tag{4.17}$$

The simplest case of eq.(4.15) ($i = l = 0, j = m = 1, k = n = 2$) gives a 2+2 dimensional extension of the Brouer-Kaup system,

$$h_t = (h_x + 2hu)_x, \tag{4.18a}$$

$$u_t = (u^2 + 2h - u_x)_x, \tag{4.18b}$$

and the solution is again given by a two-directional Wronskian [18, 23].

4-3 Ultra-discrete Systems

As was mentioned in §3, the Hirota-Miwa equation is one of the master equations, in the sense that it reduces to the KP equation via the 2D Toda equation in the continuous limit. Very recently we found a very interesting fact that there exists another limit, from which we obtain cellular automata systems [24-26]. Since we obtain discrete systems in which all the variables including dependent ones are discrete, we call it an ultra-discrete limit (the name is due to B. Grammaticos). In this subsection, we explain how to get a cellular automaton and its solutions starting from eq.(3.14) [26].

The Hirota-Miwa equation may be written in a symmetric form [14],

$$\{Z_1 \exp(D_1) + Z_2 \exp(D_2) + Z_3 \exp(D_3)\} f \cdot f = 0, \tag{4.19}$$

where $Z_i (i = 1, 2, 3)$ are arbitrary parameters and $D_i (i = 1, 2, 3)$ stand for Hirota's operators with respect to variables of the unknown function f. We here consider a particular case of eq.(4.19),

$$\left\{\exp(D_t) - \delta^2 \exp(D_x) - (1 - \delta^2)\exp(D_y)\right\} f \cdot f = 0, \tag{4.20}$$

or equivalently,

$$f(t-1,x,y)f(t+1,x,y)-\delta^2 f(t,x-1,y)f(t,x+1,y)-(1-\delta^2)f(t,x,y+1)f(t,x,y-1)=0. \tag{4.21}$$

If we introduce a variable S by

$$f(t,x,y)=\exp[S(t,x,y)], \tag{4.22}$$

then eq.(4.21) is reduced to

$$\exp[(\Delta_t^2-\Delta_y^2)S(t,x,y)]=(1-\delta^2)\left(1+\frac{\delta^2}{1-\delta^2}\exp[(\Delta_x^2-\Delta_y^2)S(t,x,y)]\right), \tag{4.23}$$

where Δ_t^2, Δ_x^2 and Δ_y^2 represent central difference operators defined, for example, by

$$\Delta_t^2 S(t,x,y)=S(t+1,x,y)-2S(t,x,y)+S(t-1,x,y). \tag{4.24}$$

Taking a logarithm of eq.(4.23) and operating $(\Delta_x^2-\Delta_y^2)$, we have

$$(\Delta_t^2-\Delta_y^2)u(t,x,y)=(\Delta_x^2-\Delta_y^2)\log\left(1+\frac{\delta^2}{1-\delta^2}\exp[u(t,x,y)]\right), \tag{4.25}$$

where

$$u(t,x,y)=(\Delta_x^2-\Delta_y^2)S(t,x,y). \tag{4.26}$$

Ultra-discretization is defined by the following formula:

$$\lim_{\varepsilon\to+0}\varepsilon\log(1+\mathrm{e}^{X/\varepsilon})=F(X)=\max[0,X]. \tag{4.27}$$

It is noted that the function $F(x)$ maps positive integers to themselves. Let us take an ultra-discrete limit of eq.(4.25). Putting

$$u(t,x,y)=\frac{v_\varepsilon(t,x,y)}{\varepsilon},\quad \frac{\delta^2}{1-\delta^2}=\mathrm{e}^{-\theta_0/\varepsilon}, \tag{4.28}$$

and taking the small limit of ε, we obtain the following equation:

$$(\Delta_t^2-\Delta_y^2)v(t,x,y)=(\Delta_x^2-\Delta_y^2)F(v(t,x,y)-\theta_0), \tag{4.29}$$

where we have rewritten $\lim_{\varepsilon\to+0}v_\varepsilon(t,x,y)$ as $v(t,x,y)$.

Equation (4.29) is considered to be an (extended) filter cellular automaton. This system is in 2 (spatial) and 1 (time) dimension and may take only integer values. Since eq.(4.29) is an ultra-discrete limit of the Hirota-Miwa equation, we expect that it admits soliton solutions. We here show that they are obtained also by taking an ultra-discrete limit of those for eq.(4.21).

The one soliton solution of eq.(4.21) is given by

$$f(t,x,y) = 1 + \mathrm{e}^{\eta}, \;\; \eta = px + qy + \omega t, \tag{4.30}$$

where the set of parameters (p, q, ω) satisfies

$$(\mathrm{e}^{-\omega} + \mathrm{e}^{\omega}) - \delta^2(\mathrm{e}^{-p} + \mathrm{e}^{p}) - (1-\delta^2)(\mathrm{e}^{-q} + \mathrm{e}^{q}) = 0. \tag{4.31}$$

Then by means of eqs.(4.22) and (4.26), we have

$$u(t,x,y) = \log(1 + \mathrm{e}^{\eta+p}) + \log(1 + \mathrm{e}^{\eta-p}) - \log(1 + \mathrm{e}^{\eta+q}) - \log(1 + \mathrm{e}^{\eta-q}). \tag{4.32}$$

Introducing new parameters and variables by

$$P = \varepsilon p, \;\; Q = \varepsilon q, \;\; \Omega = \varepsilon\omega,$$

$$K = Px + Qy + \Omega t, \;\; v_\varepsilon(t,x,y) = \varepsilon u(t,x,y),$$

and taking the limit $\varepsilon \to +0$, we obtain

$$v(t,x,y) = F(K+P) + F(K-P) - F(K+Q) - F(K-Q). \tag{4.33}$$

The dispersion relation (4.31) reduces, through the same limiting procedure, to

$$|\Omega| = \max[|P|, |Q| + \theta_0] - \max[0, \theta_0]. \tag{4.34}$$

This solution describes a solitary wave propagating in xy plane at a speed without changing its shape.

The two-soliton solution describing a nonlinear interaction of two solitary wave is obtained starting from that of eq.(4.21), which is expressed by

$$f(t,x,y) = 1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_1+\eta_2+\theta_{12}}, \;\; \eta_i = p_i x + q_i y + \omega_i t, \tag{4.35}$$

$$(\mathrm{e}^{-\omega_i} + \mathrm{e}^{\omega_i}) - \delta^2(\mathrm{e}^{-p_i} + \mathrm{e}^{p_i}) - (1-\delta^2)(\mathrm{e}^{-q_i} + \mathrm{e}^{-q_i}) = 0, \quad (i = 1, 2), \tag{4.36}$$

$$e^{\theta_{12}} = -\frac{(e^{-\omega_1+\omega_2} + e^{\omega_1-\omega_2}) - \delta^2(e^{-p_1+p_2} + e^{p_1-p_2}) - (1-\delta^2)(e^{-q_1+q_2} + e^{q_1-q_2})}{(e^{\omega_1+\omega_2} + e^{-\omega_1-\omega_2}) - \delta^2(e^{p_1+p_2} + e^{-p_1-p_2}) - (1-\delta^2)(e^{q_1+q_2} + e^{-q_1-q_2})}. \quad (4.37)$$

The variable θ_{12} stands for a phase shift. Again introducing new parameters and variables by

$$P_i = \varepsilon p_i,\ Q_i = \varepsilon q_i,\ \Omega_i = \varepsilon \omega_i,$$

$$K_i = P_i x + Q_i y + \Omega_i t,\ (i = 1,2),\ v_\varepsilon(t,x,y) = \varepsilon u(t,x,y),\ \Theta_{12} = \varepsilon\theta_{12},$$

and taking the same limit of $\varepsilon \to +0$, we have

$$\begin{aligned} v(t,x,y) = \max[0, K_1 + P_1, K_2 + P_2, K_1 + K_2 + P_1 + P_2 + \Theta_{12}] \\ + \max[0, K_1 - P_1, K_2 - P_2, K_1 + K_2 - P_1 - P_2 + \Theta_{12}] \\ - \max[0, K_1 + Q_1, K_2 + Q_2, K_1 + K_2 + Q_1 + Q_2 + \Theta_{12}] \\ - \max[0, K_1 - Q_1, K_2 - Q_2, K_1 + K_2 - Q_1 - Q_2 + \Theta_{12}], \end{aligned} \quad (4.38)$$

where

$$|\Omega_i| = \max[|P_i|, |Q_i| + \theta_0] - \max[0, \theta_0] \quad (i = 1,2), \quad (4.39)$$

and

$$\begin{aligned} \max\left[\Theta_{12} + \max[0,\theta_0] + |\Omega_1 + \Omega_2|, \max[0,\theta_0] + |\Omega_1 - \Omega_2|\right] \\ = \max\left[\Theta_{12} + |P_1 + P_2|, \Theta_{12} + \theta_0 + |Q_1 + Q_2|, |P_1 - P_2|, \theta_0 + |Q_1 - Q_2|\right]. \end{aligned} \quad (4.40)$$

The following figure demonstrates a snapshot of the two-soliton solution (4.38) at $t = -4$ for $P_1 = 6, Q_1 = 1, P_2 = 6, Q_2 = 5$.

```
15 0000000000000000000004200000000
14 0000000000000000000003300000000
13 1000000000000000000002400000000
12 1100000000000000000001500000000
11 0110000000000000000000500000000
10 0011000000000000000005100000000
 9 0001100000000000000004200000000
 8 0000100000000000000003300000000
 7 0000110000000000000002400000000
 6 0000011000000000000001500000000
 5 0000001100000000000000500000000
```

```
  4 000000011000000005100000000000
  3 000000001100000004200000000000
  2 000000000100000003300000000000
  1 000000000110000002400000000000
  0 000000000011000015000000000000
 -1 000000000001100005000000000000
 -2 000000000000110005100000000000
 -3 000000000000011004200000000000
 -4 000000000000001003300000000000
 -5 000000000000001102400000000000
 -6 000000000000000111500000000000
 -7 000000000000000011500000000000
 -8 000000000000000001610000000000
 -9 000000000000000000520000000000
-10 000000000000000000431000000000
-11 000000000000000000331000000000
-12 000000000000000000241100000000
-13 000000000000000000150110000000
-14 000000000000000000050011000000
y/x //////////////0123456789******
```

At the bottom of this figure, negative values of x coordinate are expressed as "/" and values greater than 10 are expressed as "*" for convenience sake.

It is noted that N-soliton solution is obtained through the same limiting procedure. It is also remarked that we can costruct other types of not only integrable but also nonintegrable cellular automata by using the ultra-discrete limit on several full-discrete systems.

§5 Concluding Remarks

In this paper, we have given a brief survey of Hirota's bilinear formalism and presented a few extensions. By virtue of the advantage of obtaining explicit solutions and of making the algebraic structure of equations clear, there are many other applications of this formalism. Here we just mention only one example, the Painlevé equations.

It has been shown by Okamoto [27] that the explicit solutions of Painlevé equations are expressed in terms of the τ functions. For example, the Painlevé II equation,

$$w_x x - 2w^3 + 2xw + \alpha = 0, \tag{5.1}$$

admits a solution for $\alpha = -(2N+1)$,

$$w = \frac{d}{dx}\left(\log \frac{\tau_{N+1}}{\tau_N}\right), \tag{5.2}$$

where τ_N is given by an $N \times N$ two-directional Wronski determinant of the Airy function. Recent finding of discrete analogue of the Painlevé equations [28] gave rise to a problem whether there exist corresponding solutions for the discrete case. An answer was given by the bilinear formalism. For example, it has been shown through the formalism [29] that the discrete Painlevé II equation,

$$w_{n+1} + w_{n-1} = \frac{(\alpha n + \beta) w_n + \gamma}{1 - w_n^2}, \tag{5.3}$$

admits particular solutions written by Casorati determinants whose entries are the discrete analogue of the Airy functions.

This example as well as the results in the preceding sections indicates that the bilinear formalism would be one of the most powerful tools to treat discrete problems, which the author believes to be an important subject in 21st century.

References

[1] R. Hirota, Phys. Rev. Lett. **27**(1971), 1192.

[2] R. Hirota, J. Math. Phys. **14**(1973), 805.

[3] R. Hirota, J. Phys. Soc. Jpn. **35**(1973), 289.

[4] J. Satsuma, J. Phys. Soc. Jpn. **40**(1976), 286.

[5] J. Hietarinta and R. Hirota, Phys. Lett. A **145**(1990), 237.

[6] R. Hirota, M. Ito and F. Kako, Prog. Theor. Phys. Suppl. **94** (1988), 42.

[7] R. Hirota, Y. Ohta and J. Satsuma, J. Phys. Soc. Jpn. **57**(1988), 1901; Prog. Theor. Phys. Suppl. **94**(1988), 59.

[8] M. Sato, RIMS Kokyuroku **439**(1981), 30.

[9] Y. Ohta, J. Satsuma, D. Takahashi and T. Tokihiro, Prog. Theor. Phys. Suppl. **94**(1988), 210.

[10] N. C. Freeman, IMA J. Appl. Math. **32**(1984), 125.

[11] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, in *Non-linear Integrable Systems—Classical Theory and Quantum Theory*, ed. by M. Jimbo and T. Miwa (World Scientific, Singapore, 1983), 39.

[12] K. Ueno and K. Takasaki, RIMS Kokyuroku **472**(1982), 62.

[13] M. Jimbo and T. Miwa, Publ. RIMS, Kyoto Univ. **19**(1983), 943.

[14] R. Hirota, J. Phys. Soc. Jpn. **50**(1981), 3785.

[15] T. Miwa, Proc. Jpn. Acad. **58A**(1982), 9.

[16] K. Kajiwara and J. Satsuma, J. Phys. Soc. Jpn. **60**(1991), 3986.

[17] K. Kajiwara, Y. Ohta and J. Satsuma, Phys. Lett. A **180**(1993), 249.

[18] J. Matsukidaira, J. Satsuma and W. Strampp, Phys. Lett. A. **147**(1990) 467.

[19] J. Matsukidaira and J.Satsuma, J. Phys. Soc. Jpn. **59**(1990), 3413.

[20] J. Matsukidaira and J. Satsuma, Phys. Lett. A. **158**(1991) 366.

[21] J. Hietarinta and J. Satsuma, Phys. Lett. A. **161**(1991) 267.

[22] Y. Ohta, K. Kajiwara, J. Matsukidaira and J. Satsuma, J. Math. Phys. **34**(1993) 5190.

[23] J. Satsuma, K. Kajiwara, J. Matsukidaira and J. Hietarinta, J. Phys. Soc. Jpn. **61**(1992) 3096.

[24] T. Tokihiro, D. Takahashi, J. Matsukidaira and J. Satsuma, Phys. Rev. Lett. **76**(1996) 3247.

[25] J. Matsukidaira, J. Satsuma, D. Takahashi, T. Tokihiro and J. Satsuma, Phys. Lett. A. **225**(1997) 287.

[26] S. Moriwaki, A. Nagai, J. Satsuma, T. Tokihiro, M. Torii, D. Takahashi and J. Matsukidaira, to appear in London Math. Soc. Lecture Notes Series, Cambridge Univ. Press.

[27] K. Okamoto, Math. Ann. **275**(1986) 221; Japan J. Math. **13**(1987) 47; Ann. Mat. Pura. Appl. **146**(1987) 337; Funkcial. Ekvac. **30**(1987) 305.

[28] B. Grammaticos, A. Ramani and J. Hietarinta, Phys. Rev. Lett. **67**(1991) 1825:

[29] K. Kajiwara, Y. Ohta, J. Satsuma, B. Grammaticos and A. Ramani, J. Phys. A: Math. Gen. **27**(1994) 915: See also articles by B. Grammaticos and A. Ramani in this proceeding.

非線形力学的境界条件をもつ相転移問題の解の存在とその一意性について

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1.Introduction

本稿においては、非線形力学的境界条件をもつ制限されたPhase Field方程式の解の存在と一意性について考える:次のシステムをみたす3つの関数$u:[0,T]\to L^2(\Omega), w:[0,T]\to L^2(\Omega), v:[0,T]\to L^2(\Gamma)$を求めよ;

$$u_t + w_t - \Delta u = f(t,x) \quad \text{in } Q := (0,T)\times\Omega, \tag{1.1}$$

$$\nu w_t - \kappa \Delta w + \beta(w) + g(w) \ni u \quad \text{in } Q, \tag{1.2}$$

$$u = v \quad \text{a.e. on } \Sigma := (0,T)\times\Gamma, \tag{1.3}$$

$$\frac{\partial u}{\partial n} + c\frac{\partial v}{\partial t} + h(v) = 0 \quad \text{on } \Sigma, \tag{1.4}$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{on } \Sigma, \tag{1.5}$$

$$u(0,\cdot) = u_0,\ w(0,\cdot) = w_0 \quad \text{in } \Omega, \tag{1.6}$$

$$v(0,\cdot) = v_0 \quad \text{on } \Gamma, \tag{1.7}$$

ここで、$0 < T < +\infty$, Ωはなめらかな境界ΓをもつR^Nの有界領域; ν,κ,cは正定数; βは$R\times R$上の極大単調作要素; g,hはR上のLipschitz連続関数; fはQ上の与えられた関数; u_0, w_0, v_0は初期関数である。$\frac{\partial}{\partial n}$は$\Gamma$上における外向き法線方向の微分を表す。上のシステム(1.1)~(1.7)をCP=CP(u_0,w_0,v_0)で表すことにする。

相転移現象を記述するモデルの1つであるphase field方程式はCaginalp [4], Fix[5], Visintin[11], Kenmochi[9]などによって既に研究されている。

境界条件(1.3),(1.4)を、未知関数uに関する第3種境界条件にかえた初期値境界値問題についてはその解が一意に存在することが示されている。(Damlamian-Kenmochi-Sato[7])

境界条件(1.4)のように、固定境界において未知関数の時間tに関する微分を含む境界条件を力学的境界条件と呼ぶ。この境界条件をもつ2相ステファン問題に対してはAiki[1],[2]で解の存在、その一意性、解の周期的安定性、周期解の集合の構造について研究されている。

本稿では、システムCP(u_0,w_0,v_0)の解の存在と一意性に関する結果について述べる。

本稿で用いる凸関数などに関する性質はBrézis[3]等によるものである。

2.Main result

本稿を通じて次の条件(β), (f), (I)を仮定する。

(β) R上の適正下半連続凸関数$\hat{\beta}$と正定数C_1が存在して、

$$\beta = \partial\hat{\beta} \quad \text{かつ} \quad \hat{\beta}(r) \geq C_1|r|^2 \quad \text{for any } r\in R,$$

ここで、$\partial\hat{\beta}$は$\hat{\beta}$の劣微分を表す。

(*f*) $f \in L^2(0,T;L^2(\Omega))$,

(I) $u_0 \in L^2(\Omega), w_0 \in L^2(\Omega), v_0 \in L^2(\Gamma)$.

次の記号を用いる;

$$V = H^1(\Omega), \quad W = L^2(\Omega) \times L^2(\Gamma),$$

$$A(y,z) = \int_\Omega \nabla y \cdot \nabla z dx \quad y,z \in V,$$

$$(y,z)_{L^2(\Omega)} = \int_\Omega yzdx \quad y,z \in L^2(\Omega),$$

$$(y,z)_{L^2(\Gamma)} = \int_\Gamma yzd\Gamma \quad y,z \in L^2(\Gamma),$$

$$Zz = \int_\Omega zdx + c\int_\Gamma zd\Gamma \quad z \in V,$$

ここで、正定数cは(1.3)のものと同じである。

$$(y,z)_V = A(y,z) + ZyZz \quad y,z \in V,$$

$$([y,y_\Gamma],[z,z_\Gamma])_W = (y,z)_{L^2(\Omega)} + (y_\Gamma,z_\Gamma)_{L^2(\Gamma)} \qquad [y,y_\Gamma],[z,z_\Gamma] \in W.$$

V,Wはそれぞれ内積$(\cdot,\cdot)_V,(\cdot,\cdot)_W$によってHilbert空間になる。$W$と$W$の双対空間を同一視し、$V$の双対空間を$V^*$で表すことにする。$F_V : V \to V^*$を双対性作用素とする。i.e.

$$\langle F_V y, z\rangle_V = A(y,z) + ZyZz \quad y,z \in V,$$

ここで、$\langle\cdot,\cdot\rangle_V$は$V^*$と$V$のduality pairingを表す。$V^*$に次のような内積を定義するとHilbert空間になる;

$$(y^*,z^*)_{V^*} = \langle y^*, F_V^{-1}z^*\rangle_V \ (= \langle z^*, F_V^{-1}y^*\rangle_V) \quad y^*,z^* \in V^*.$$

$|\cdot|_V, |\cdot|_W, |\cdot|_{V^*}$でそれぞれの空間のノルムを表すことにする。

作用素$E : V \to W$を次のように定める:

$$Ez := [z, cz|_\Gamma] \quad z \in V,$$

ここで、正定数cは(1.3)のものと同じである。Eの値域$R(E)$はWの稠密な部分空間であり、Eは線形完全連続作用素である。このとき、Eの双対作用素E^*について次の等式が成立する。

$$\langle E^*[z,z|_\Gamma],\eta\rangle_V = (z,\eta)_{L^2(\Omega)} + c(z_\Gamma,\eta)_{L^2(\Gamma)} \quad [z,z|_\Gamma] \in W, \eta \in V,$$

ここで、正定数cは(1.3)のものと同じである。

次に$\mathrm{CP}(u_0,w_0,v_0)$の解の定義を述べる;

Definition 2.1. 3つの関数$u : [0,T] \to L^2(\Omega), w : [0,T] \to L^2(\Omega), v : [0,T] \to L^2(\Gamma)$の組$\{u,w,v\}$が次の条件(1)~(4)をみたすとき、$\{u,w,v\}$を$[0,T]$上の$\mathrm{CP}(u_0,w_0,v_0)$の解であるという;

(1) $u \in C_w([0,T];L^2(\Omega)) \cap L^2(0,T;V) \cap L^2_{loc}((0,T];V)$,
$w \in C([0,T];L^2(\Omega)) \cap W^{1,2}_{loc}((0,T];L^2(\Omega)) \cap L^2(0,T;V) \cap L^\infty_{loc}((0,T];V)$,
$v \in C_w([0,T];L^2(\Gamma)) \cap W^{1,2}_{loc}((0,T];L^2(\Gamma))$,
$E^*[u+w,v] \in W^{1,2}_{loc}((0,T];V^*), \hat{\beta}(w) \in L^1(Q), u(t,x) = v(t,x)$ a.e. on Σ,

(2)
$$\langle \frac{d}{dt}E^*[u(t)+w(t),v(t)],z\rangle_V + A(u(t),z) + (h(v(t)),z)_{L^2(\Gamma)} = (f(t),z)_{L^2(\Omega)} \qquad (2.1)$$
for all $z \in V$ and a.e. $t \in [0,T]$,

(3) Q 上いたるところ$\xi \in \beta(w)$ となる$\xi \in L^2_{loc}((0,T];L^2(\Omega))$が存在して、

$$\nu(\frac{d}{dt}w(t),\eta)_{L^2(\Omega)} + \kappa A(w(t),\eta) + (\xi(t),\eta)_{L^2(\Omega)} + (g(w(t)),\eta)_{L^2(\Omega)} = (f(t),\eta)_{L^2(\Omega)} \quad (2.2)$$

for all $\eta \in V$ and a.e. $t \in [0,T]$,

(4) $u(0,x) = u_0(x), w(0,x) = w_0(x)$ for a.e. $x \in \Omega$ and $v(0,x) = v_0(x)$ for a.e. $x \in \Gamma$.

以下のような解の存在と一意性に関する結果を得た。

Theorem 2.1. $0 < T < +\infty$. 条件 *(β),(f),(I)*が成り立つとき、*CP*(u_0,w_0,v_0)の $[0,T]$ 上の解 $\{u,w,v\}$ は一意に存在する。

以下の節でこの定理の証明を簡単に述べる。

3. 一意性の証明

$\{u_1,w_1,v_1\},\{u_2,w_2,v_2\}$ をそれぞれ CP$(u_{1,0},w_{1,0},v_{1,0})$, CP$(u_{2,0},w_{2,0},v_{2,0})$ の解とすると、(2.1), (2.2) 式から次の 2 式を得る;

$$\langle \frac{d}{dt}(E_1^*(t)-E_2^*(t)),z\rangle_V + A(u_1(t)-u_2(t),z) + (h(v_1(t))-h(v_2(t)),z)_{L^2(\Gamma)} = 0 \qquad (3.1)$$

for all $z \in V$ and a.e. $t \in [0,T]$,

$$\begin{aligned}\nu(\frac{d}{dt}(w_1(t)-w_2(t)),\eta)_{L^2(\Omega)} + \kappa A(w_1(t)-w_2(t),\eta) + (\xi_1(t)-\xi_2(t),\eta)_{L^2(\Omega)} + \\ +(g(w_1(t))-g(w_2(t)),\eta)_{L^2(\Omega)} = (u_1(t)-u_2(t),\eta)_{L^2(\Omega)}\end{aligned} \qquad (3.2)$$

for all $\eta \in V$ and a.e. $t \in [0,T]$,
ただし、$E_i^*(t) := E^*[u_i(t)+w_i(t),v_i(t)], \xi_i(t) \in \beta(w_i(t)) (i=1,2)$ とする。ここで、$z = F_V^{-1}(E_1^*(t)-E_2^*(t)), \eta = w_1(t)-w_2(t)$ をそれぞれ (3.1),(3.2) 式に代入しいくつかの変形を行うと次の不等式を得る;

Lemma 3.1. $0 < T < +\infty$. 条件 *(β),(f),(I)*が成り立つとき、$\{u_i,w_i,v_i\}(i=1,2)$ をそれ

ぞれ $CP(u_{i0}, w_{i0}, v_{i0})$ $(i=1,2)$ の解とするとき、t に依存しない正定数 C_2 が存在して次の不等式を満たす;

$$|E_1^*(t)-E_2^*(t)|_{V^*}^2+|w_1(t)-w_2(t)|_{L^2(\Omega)}^2+\int_0^t|v_1(\tau)-v_2(\tau)|_{L^2(\Gamma)}^2d\tau+\int_0^t|u_1(\tau)-u_2(\tau)|_{L^2(\Omega)}^2d\tau$$
$$\le C_2\{|E^*[u_{1,0}+w_{1,0},v_{1,0}]-E^*[u_{2,0}+w_{2,0},v_{2,0}]|_{V^*}^2+|w_{1,0}-w_{2,0}|_{L^2(\Omega)}^2\}$$
$$\text{for all } t\in[0,T].$$

Proof of Theorem 2.1.(uniqueness) Lemma 3.1 より明らか。 □

このように、双対性作用素 F_V を利用して解の一意性を導く方法は Damlamian[6] による。Damlamian-Kenmochi-Sato[7], Aiki[1],[2] も同様な手法で解の一意性を示している。

4. 存在性の証明

CP(u_0, w_0, v_0) の解の存在を示す前に、次の問題を考える:次のシステムをみたす3つの関数 $u:[0,T]\to L^2(\Omega), w:[0,T]\to L^2(\Omega), v:[0,T]\to L^2(\Gamma)$ を求めよ;

$$u_t+w_t-\Delta u=f(t,x) \quad \text{in } Q, \tag{4.1}$$

$$\nu w_t-\kappa\,\Delta\, w+\beta(w)+\tilde{g}(t,x)\ni u \quad \text{in } Q, \tag{4.2}$$

$$u=v \quad \text{a.e. on } \Sigma, \tag{4.3}$$

$$\frac{\partial u}{\partial n}+c\frac{\partial v}{\partial t}+\tilde{h}(t,x)=0 \quad \text{on } \Sigma, \tag{4.4}$$

$$\frac{\partial w}{\partial n}=0 \quad \text{on } \Sigma, \tag{4.5}$$

$$u(0,\cdot)=u_0,\ w(0,\cdot)=w_0 \qquad \text{in } \Omega, \tag{4.6}$$

$$v(0,\cdot)=v_0 \qquad \text{on} \quad \Gamma, \tag{4.7}$$

ここで、$\tilde{g}$ は Q 上の与えられた関数;$\tilde{h}$ は Σ 上の与えられた関数である。上のシステム (4.1)~(4.7) を CP1=CP1$(u_0, w_0, v_0; \tilde{g}, \tilde{h})$ で表すことにする。この CP1 の解の存在性と Schauder の不動点定理を用いて CP の解の存在を示す。まず、CP1 の解の存在と一意性について述べる。

初期関数 u_0, w_0, v_0 は次の条件 (I)' をみたすものとする;

(I)' $u_0\in V, u_0|_\Gamma=v_0, w_0\in V$ with $\hat{\beta}(w_0)\in L^1(\Omega)$.

さらに、$\tilde{g},\tilde{h}$ は次の条件をみたすものとする;

$(\tilde{g})$ $\tilde{g}\in L^2(0,T;L^2(\Omega))$,

$(\tilde{h})$ $\tilde{h}\in L^2(0,T;L^2(\Gamma))$.

次に CP1$(u_0, w_0, v_0; \tilde{g}, \tilde{h})$ の解の定義を述べる;

Definition 4.1. 3つの関数 $u:[0,T]\to L^2(\Omega), w:[0,T]\to L^2(\Omega), v:[0,T]\to L^2(\Gamma)$ の組 $\{u,w,v\}$ が次の条件 (1)~(4) をみたすとき、$\{u,w,v\}$ を $[0,T]$ 上の CP1$(u_0, w_0, v_0; \tilde{g}, \tilde{h})$ の解であるという;

(1) $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V), w \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V), v \in W^{1,2}(0,T;L^2(\Gamma))$, $E^*[u+w,v] \in W^{1,2}(0,T;V^*), \hat{\beta}(w) \in L^1(Q), u(t,x) = v(t,x)$ a.e. on Σ.

(2)

$$\langle \frac{d}{dt} E^*[u(t)+w(t),v(t)], z \rangle_V + A(u(t),z) + (\tilde{h}(t),z)_{L^2(\Gamma)} = (f(t),z)_{L^2(\Omega)} \tag{4.8}$$

for all $z \in V$ and a.e. $t \in [0,T]$,

(3) Q 上いたるところ$\xi \in \beta(w)$ となる$\xi \in L^2(0,T;L^2(\Omega))$ が存在して、

$$\nu(\frac{d}{dt}w(t),\eta)_{L^2(\Omega)} + \kappa A(w(t),\eta) + (\xi(t),\eta)_{L^2(\Omega)} + (\tilde{g}(t),\eta)_{L^2(\Omega)} = (f(t),\eta)_{L^2(\Omega)}$$

for all $\eta \in V$ and a.e. $t \in [0,T]$,

(4) $u(0,x) = u_0(x), w(0,x) = w_0(x)$ for a.e. $x \in \Omega$ and $v(0,x) = v_0(x)$ for a.e. $x \in \Gamma$.

次に、CP1$(u_0,w_0,v_0;\tilde{g},\tilde{h})$ の解の存在を示すために新たな関数空間を導入する。そのための準備として、

Lemma 4.1. $0 < T < +\infty$. 条件 *(β), (f), (I)'*が成り立つとする。$\{u,w,v\}$ を *CP1*$(u_0,w_0,v_0;\tilde{g},\tilde{h})$ の解とすると、任意の $t \in [0,T]$ に対して次の等式が成り立つ;

$$\int_\Omega u(t,x)dx + \int_\Omega w(t,x)dx + c\int_\Gamma v(t,x)d\Gamma$$

$$= \int_\Omega u_0(x)dx + \int_\Omega w_0(x)dx + c\int_\Gamma v_0(x)d\Gamma + \int_0^t\int_\Omega f(\tau,x)dxd\tau - \int_0^t\int_\Gamma \tilde{h}(\tau,x)d\Gamma d\tau.$$

Proof. 等式 (4.8) において $z \equiv 1$ として、区間 $[0,t]$ で積分すればよい。 □

区間 $[0,T]$ 上の関数 a を次のように定義する;

$$a(t) = \frac{1}{|\Omega| + c|\Gamma|}\{\int_\Omega u_0(x)dx + \int_\Omega w_0(x)dx + c\int_\Gamma v_0(x)d\Gamma +$$

$$+ \int_0^t\int_\Omega f(\tau,x)dxd\tau - \int_0^t\int_\Gamma \tilde{h}(\tau,x)d\Gamma d\tau\} \quad t \in [0,T],$$

ただし、$|\Omega|$ はΩの体積、$|\Gamma|$ はΓの面積を表す。この a を用いると Lemma 4.1 より CP1$(u_0,w_0,v_0;\tilde{g},\tilde{h})$ の解 $\{u,w,v\}$ について次の等式が成り立つ;

$$\int_\Omega (u(t,x)+w(t,x)-a(t))dx + c\int_\Gamma (v(t,x)-a(t))d\Gamma = 0 \quad t \in [0,T].$$

ここで次のような関数空間 Y_1, Y_2を考える;

$$Y_1 := \{z \in V; Zz = 0\}, \quad Y_2 := \{[z,z_\Gamma] \in W; \int_\Omega zdx + c\int_\Gamma z_\Gamma d\Gamma = 0\}.$$

Y_1はノルム $|z|_{Y_1} := |\nabla z|_{L^2(\Omega)}$により Banach 空間になる。$Y_1^*$で Y_1の双対空間を表す。
$F_{Y_1} : Y_1 \to Y_1^*$を双対性作用素とする。i.e.

$$\langle F_{Y_1} y, z\rangle_{Y_1} = A(y,z) \quad y,z \in Y_1,$$

ここで、$\langle\cdot,\cdot\rangle_{Y_1}$ はY_1^*と Y_1の duality pairing を表す。Y_1^*に次のような内積を定義すると Hilbert 空間になる;

$$(y^*,z^*)_{Y_1^*} = \langle y^*, F_{Y_1}^{-1} z^*\rangle_{Y_1} \ (= \langle z^*, F_{Y_1}^{-1} y^*\rangle_{Y_1}) \quad y^*,z^* \in Y_1^*.$$

Y_2は W と同じ内積により Hilbert 空間になる。
$P_{Y_1} : V \to Y_1, \hat{E} : Y_1 \to Y_2$をそれぞれ次のように定義する。

$$P_{Y_1} z := z - \frac{Zz}{|\Omega| + c|\Gamma|} \quad z \in V, \quad \hat{E}z := [z, cz|_\Gamma] \quad z \in Y_1.$$

このとき、$\hat{E}$の双対作用素$\hat{E}^*$について次の等式が成立する;

$$\langle \hat{E}^*[z,z_\Gamma], \eta\rangle_{Y_1} = (z,\eta)_{L^2(\Omega)} + c(z_\Gamma,\eta)_{L^2(\Gamma)} \quad [z,z_\Gamma] \in Y_2, \eta \in Y_1,$$

ここで、正定数cは (1.3) のものと同じである。$R(\hat{E})$ で$\hat{E}$の値域を表す。
次のことが示される;

Proposition 4.1. $0 < T < +\infty$. 条件 *(β), (f), (I)'* が成り立つとする。$\{u,w,v\}$ を *CP1*$(u_0, w_0, v_0; \tilde{g}, \tilde{h})$ の解とする。このとき、$q(t) := u(t) + w(t) - a(t)$, $q_\Gamma(t) := v(t) - a(t)$, $q_0 := u_0 + w_0 - a(0)$, $q_{\Gamma 0} := v_0 - a(0)$ と置き換えると *CP1*$(u_0,w_0,v_0; \tilde{g}, \tilde{h})$ の解 $\{u,w,v\}$ は次のシステム *(4.9)〜(4.15)* の解 $\{q,w,q_\Gamma\}$ と同値である;

$$\hat{E}^*[q,q_\Gamma] \in W^{1,2}(0,T;Y_1^*), \tag{4.9}$$

$$[q,q_\Gamma] \in L^\infty(0,T;Y_2), \quad q \in L^2(0,T;Y_1), \tag{4.10}$$

$$w \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V), \quad \hat{\beta}(w) \in L^1(Q), \tag{4.11}$$

$$\frac{d}{dt}\hat{E}^*[q(t),q_\Gamma(t)] = -F_{Y_1} P_{Y_1}(q(t) + a(t) - w(t)) + f^* \text{ in } Y_1^* \text{ a.e. } t \in (0,T), \tag{4.12}$$

$$\nu \frac{d}{dt} w(t) - \kappa \triangle w(t) + \beta(w(t)) + \tilde{g}(t) \ni q(t) + a(t) - w(t) \text{ in } L^2(\Omega) \text{ a.e. } t \in (0,T), \tag{4.13}$$

$$q(t,x) - w(t,x) = q_\Gamma(t,x) \text{ a.e. on } \Sigma, \tag{4.14}$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{on } \Sigma, \tag{4.15}$$

$$q(0) = q_0, w(0) = w_0 \text{ in } L^2(\Omega), q_\Gamma(0) = q_{\Gamma 0} \text{ in } L^2(\Gamma), \tag{4.16}$$

ここで、$\langle f^*(t),\eta\rangle_{Y_1} := (f(t),\eta)_{L^2(\Omega)} - (\tilde{h}(t),\eta)_{L^2(\Gamma)} \quad \eta \in Y_1$.

上のシステム (4.9)〜(4.16) を CP2=CP2$(q_0, w_0, q_{\Gamma 0}; \tilde{g}, \tilde{h})$ で表すことにする。
直積空間 $X := Y_1^* \times L^2(\Omega)$ は次のような内積 $(\cdot,\cdot)_X$によって Hilbert 空間になる;

$$([y_1,y_2],[z_1,z_2])_X = (y_1,z_1)_{Y_1^*} + \nu(y_2,z_2)_{L^2(\Omega)}.$$

この直積空間 X 上の適正下半連続凸関数φ^tを各 $t \in [0,T]$ に対して次のように定義する;

$$\varphi^t(U) := \begin{cases} \frac{1}{2}|q+a(t)-w|^2_{L^2(\Omega)} + \frac{c}{2}|q_\Gamma + a(t)|^2_{L^2(\Gamma)} + \frac{\kappa}{2}|\nabla w|^2_{L^2(\Omega)} + \int_\Omega \hat{\beta}(w)dx \\ \quad \text{if } U = [q^*, w] \in R(\hat{E}^*) \times H^1(\Omega) \text{ with } \hat{\beta}(w) \in L^1(\Omega) \text{ and } [q, q_\Gamma] = \hat{E}^{*-1}q^*, \\ +\infty \qquad \text{otherwise.} \end{cases}$$

すると、

Proposition 4.2. $\{q, w, q_\Gamma\}$ を *CP2*$(q_0, w_0, q_{\Gamma 0}; \tilde{g}, \tilde{h})$ の解とする。$U(t) = [\hat{E}^*[q(t), q_\Gamma(t)], w(t)]$, $U(0) = [\hat{E}^*[q_0, q_{\Gamma 0}], w_0]$ と置き換えると、*CP2*$(q_0, w_0, q_{\Gamma 0};\tilde{g}, \tilde{h})$ の解 $\{q, w, q_\Gamma\}$ は次の発展方程式 *(4.16)* $\sim$ *(4.17)* の解 U と同値である;

$$\frac{d}{dt}U(t) + \partial\varphi^t(U(t)) = F^*(t) \text{ in } X \text{ a.e. } t \in (0,T), \tag{4.16}$$

$$U(0) = [\hat{E}^*[q_0, q_{\Gamma 0}], w_0], \tag{4.17}$$

ただし、$F^*(t) = [f^*(t), -\frac{1}{\nu}\tilde{g}(t)] \in L^2(0,T;X)$.

このタイプの発展方程式は剣持先生の一般論[8] より解の存在と一意性が示される。従って、

Proposition 4.3. $0 < T < +\infty$. 条件 *(β), (f), (I)'* が成り立つとする。システム *CP1*$(u_0, w_0, v_0;\tilde{g},\tilde{h})$ の解 $\{u, w, v\}$ は一意に存在する。

Proof of Theorem 2.1.(existence) 初期関数が条件 (I)' をみたすときは、$L^2(0,T;L^2(\Omega)) \times L^2(0,T;L^2(\Gamma))$ の適当な部分コンパクト集合 B を選び、各 $[\overline{w}, \overline{v}] \in B$ に対し CP1$(u_0, w_0, v_0; g(\overline{w}), h(\overline{v}))$ を解く。このとき、$[\overline{w}, \overline{v}] \in B$ を CP1$(u_0, w_0, v_0; g(\overline{w}), h(\overline{v}))$ の解の $[w, v]$ に対応させる写像 S は B から B への連続写像であることがわかる。ここで、Schauder の不動点定理から S は不動点を許す。これが CP の解である。

初期関数が条件 (I) をみたすときは、初期関数が条件 (I)' をみたす近似解を構成するとその極限関数が CP の解になっていることが示される。 □

参考文献

[1] T. Aiki, Periodic stability of solutions to some degenerate parabolic equations with dynamic boundary conditions, J. Math. Soc. Japan, **48**(1996), 37-59.

[2] T. Aiki, Multi-dimentional two-phase Stefan problems with nonlinear dynamic boundary conditions, to appear in Gakuto Inter. Ser. Math. Sci. Appl. Vol.7.

[3] H. Brézis, Opérateurs maximaux monototones et semi-groupes de contractions dans les espases de Hilbert, North-Holland, Amsterdam, 1973.

[4] G. Caginalp, An analysis of a phase field model of a free boundary, Arch. Rat. Mech. Anal., **92**(1986), 205-245.

[5] G. J. Fix, Phse field models for free boundary problems, *Free Boundary Problems: Theory and Applocation*, pp.580-589, Pitman Reserch Notes in Math. Ser. Vol. 79, 1983.

[6] A. Damlamian, Some results on the multi-phase Stefan problem, Comm. P.D.E., **2**(1977), 1017-1044.

[7] A. Damlamian, N. Kenmochi and N. Sato, Phase field equation with constraints, *'Nonlinear Mathematical Problems in Industry"*, pp. 391-404, Gakuto Inter. Ser. Math. Sci. Appl. Vol. **2**, Gakkōtosho, Tokyo, 1993.

[8] N. Kenmochi, Solvability of nonlinear evolution equations with time-dependent constraints and applications, Bull. Fac. Education, Chiba Univ., **30**(1981), 1-87.

[9] N. Kenmochi, Systems of nonlinear PDEs arising from dynamical phase transitions, *Phase Transitions and Hysteresis*, pp.39-86, Lecture Notes in Math. 1584, Springer-Verlag, Berlin, 1993.

[10] R. E. Langer, A problem in diffusion or in the flow of heat for a solid in contact with fluid, Tôhoku Math. J., Ser. 1,**35**(1932), 260-275.

[11] A. Visintin, Stefan problems with phse relaxation, IMA J. Appl. Math., **34**(1985), 225-245.

非自励系に対する大域的アトラクターの存在について

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1. 序

実ヒルベルト空間 H 上で定義された時間依存適正凸下半連続関数 φ^t の劣微分 $\partial\varphi^t$ に支配された非線形発展方程式

$$u'(t) + \partial\varphi^t(u(t)) + g(u(t)) \ni f(t) \quad \text{in } H, \quad t > 0, \tag{1.1}$$

を考える。ここで、$u' = \frac{du}{dt}$、 $g(u)$ はリップシッツ摂動とし、f を forcing term とする。この方程式は、非自励系と呼ばれる。

時間 t が $+\infty$ としたとき、φ^t が、ある適正凸下半連続関数 φ^∞ にある適当な意味で収束し、$f(t)$ がヒルベルト空間 H のある 1 点 f^∞ にある適当な意味で収束すると仮定したとき、(1.1) の解の漸近挙動について考察する。(1.1) の極限方程式は、自励系

$$u'(t) + \partial\varphi^\infty(u(t)) + g(u(t)) \ni f^\infty, \quad t > 0. \tag{1.2}$$

である。

ここでは「非自励系 (1.1) の解の漸近挙動は、自励系 (1.2) により特徴づけることができるか？」つまり、「非自励系 (1.1) に対する大域的アトラクターは存在するのか、存在するならば、自励系 (1.2) の大域的アトラクターとの関係はどのようになっているか？」ということについて論ずる。

第 5 章までは、定義域 $\overline{D(\varphi^t)}$ が時間依存する非自励系 (1.1) を扱う。しかし特に第 6 節では、定義域 $\overline{D(\varphi^t)}$ が時間依存しない場合を考える。なぜならばこの場合、skew-product flows を利用して大域的アトラクターを構成することができるからである。(cf. [1]). そこで第 6 節では、実際に skew-product flows を構成し非自励系 (1.1) に対する大域的アトラクターの存在を示す。

2. 仮定

以下の条件を仮定する。

(A1) $\{\varphi^t; 0 \le t \le +\infty\}$ を H 上で定義された適正凸下半連続の族とし、 $\partial\varphi^t$ で φ^t の劣微分を表す;

(A2) パラメータ $r \in R_+ := [0, +\infty)$ をもつ R_+ 上で定義された絶対連続関数の族 $\{a_r;\ r \ge 0\}$ と $\{b_r;\ r \ge 0\}$ が存在し次の条件 $(a1)$ と $(a2)$ を満たす。

(a1) $a_r' \in L^1(R_+) \cap L^2(R_+)$, $b_r' \in L^1(R_+)$;

(a2) 任意の定数 $r \ge 0$、任意の時間 $s, t \in [0, +\infty]$ かつ任意の $z \in D(\varphi^s)$; $|z|_H \le r$ に対し、次を満たす $\tilde{z} \in D(\varphi^t)$ が存在する；

$$|\tilde{z} - z|_H \le |a_r(t) - a_r(s)|(1 + |\varphi^s(z)|^{\frac{1}{2}}),$$

かつ、

$$\varphi^t(\tilde{z}) - \varphi^s(z) \le |b_r(t) - b_r(s)|(1 + |\varphi^s(z)|);$$

更に、上の条件を満たす適正凸下半連続関数の族 $\{\varphi^t;\ 0 \le t \le \infty\}$ 全体を $\Phi(\{a_r\},\{b_r\})$ で表す；

(A3) g は $D(g) = H$ から H への作用素であり、次を満たす；

(g1) g はリップシッツ定数 $L(g)$ をもつリップシッツ連続である。つまり、

$$|g(u) - g(v)|_H \le L(g)|u - v|_H, \qquad \text{for } {}^\forall u,\ v \in H;$$

(g2) H の任意の有界部分集合 B に対し、ある2つの定数 $C_0(B) > 0$ と $C_1(B) > 0$ が存在して次の不等式が成立する；

$$\varphi^t(z) + (g(z), z - v) \ge C_0(B)|z|_H^2 - C_1(B),$$

$$\text{for all } t \in R_+,\ z \in D(\varphi^t) \text{ and } v \in B;$$

(A4) 任意の時間 $t; 0 \le t \le \infty$ と任意の定数 $r; 0 < r < +\infty$ に対し、

レベル集合 $\{z \in H;\ |z|_H \le r,\ \varphi^t(z) \le r\}$ が H でコンパクトである；

(A5) $f \in L^2_{loc}(R_+; H)$、 $f^\infty \in H$ 、$\sup_{t\ge0} |f|_{L^2(t,t+1;H)} < +\infty$ かつ、

$$|f(t + \ \cdot\ ) - f^\infty|_{L^2(0,1;H)} \longrightarrow 0, \qquad \text{as } t \to +\infty;$$

(A6) $u_0 \in \overline{D(\varphi^s)}$、$v_0 \in \overline{D(\varphi^\infty)}$

3. 解の大域的存在性と有界性

この節では、それぞれの初期時間 $s \ge 0$ に対する次の発展方程式の解の存在性、一意性そして有界性を考察する。

$$(\mathrm{E}_s) \qquad u'(t) + \partial\varphi^t(u(t)) + g(u(t)) \ni f(t), \qquad t \ge s.$$

定義 3.1. $[s,T]$ 上の関数 u が (E_s) の解であるとは、つぎを満たすときをいう。

$$u \in C([s,T]; H) \cap W^{1,2}_{loc}((s,T]; H),\ \varphi^{(\cdot)}(u(\cdot)) \in L^1(s,T),$$

$$f(t) - u'(t) - g(u(t)) \in \partial\varphi^t(u(t)) \text{ for a.e. } t \in [s,T].$$

また、u が (E_s) の大域的解であるとは、$u : [s, +\infty) \to H$ が任意の $T(> s)$ に対し $[s,T]$ 上で (E_s) の解であるときをいう。

定理 3.1. $\{\varphi^t\} \in \Phi(\{a_r\},\{b_r\})$ とし、 g は仮定 *(g1)* と *(g2)* を満たすとする。また、$f \in L^2_{loc}(R_+; H)$ とする。このとき、任意の $0 \le s < +\infty$ と $u_0 \in \overline{D(\varphi^s)}$ に対し、初期値 $u(s) = u_0$ をもつ *(E_s)* の大域的解 u が一意に存在する。

定理 3.1 より、我々は (E_s) に対する 解（発展）作用素を定義することができる。

定義 3.2. 任意の $0 \le s \le t < +\infty$ に対し、$E(t,s)$ を次のように定義する；
$E(t,s)$ は、$\overline{D(\varphi^s)}$ から $\overline{D(\varphi^t)}$ への作用素で、初期値 $u_0 \in \overline{D(\varphi^s)}$ に対し $u(t) \in \overline{D(\varphi^t)}$ を対応させる。こ

こで、u は初期値 $u(s)=u_0$ をもつ (E_s) の大域的解である。

このとき、解作用素 $E(t,s)$ は次の性質をもつ。

(E1) $E(s,s)=I$ on $\overline{D(\varphi^s)}$ for any $s \geq 0$.

(E2) $E(t_2,s)z=E(t_2,t_1)E(t_1,s)z$ for any $0 \leq s \leq t_1 \leq t_2 < +\infty$ and $z \in \overline{D(\varphi^s)}$.

この解作用素 $E(t,s)$ を用いて、解の大域的有界性定理を述べる。

定理 3.2. $\{\varphi^t\} \in \Phi(\{a_r\},\{b_r\})$ とする。このとき、次を満たすある定数 $N_0>0$ が存在する；

$$|E(t,s)u_0|_H \leq N_0(|u_0|_H+1), \qquad \textit{for all } 0 \leq s \leq t < +\infty \textit{ and } u_0 \in \overline{D(\varphi^s)}.$$

また、定理 3.2. の系として次の 2 つが成り立つ。

系 1. 定理 *3.2* と同じ仮定のもとで、ある定数 $N_1>0$ が存在して次の不等式が成り立つ。

$$\int_t^{t+1} |\varphi^\tau(E(\tau,s)u_0)|d\tau \leq N_1(|u_0|_H^2+1),$$

$$\textit{for all } 0 \leq s \leq t < +\infty \textit{ and } u_0 \in \overline{D(\varphi^s)}.$$

系 2. 定理 *3.2* と同様な仮定のもとで、任意の H の有界部分集合 B とそれぞれの定数 $\delta>0$ に対して、次を満たすある定数 $M_{B,\delta}>0$ が存在する；

$$\sup_{t \geq \delta+s} |\varphi^t(E(t,s)u_0)| + \sup_{t \geq \delta+s} \left|\frac{d}{d\tau}E(\cdot,s)u_0\right|^2_{L^2(t,t+1;H)} \leq M_{B,\delta},$$

$$\textit{for all } s \geq 0 \textit{ and } u_0 \in \overline{D(\varphi^s)} \cap B.$$

定理 3.1 及び定理 3.2 とその系の詳細な証明は、[2] を参照する。

4. 自律系に対する大域的アトラクター

この節で、(E_s) の極限方程式 (E_∞)

$$(\mathrm{E}_\infty) \qquad u'(t)+\partial\varphi^\infty(u(t))+g(u(t)) \ni f^\infty, \qquad t \geq 0$$

を考える。このとき、$\{\varphi^t=\varphi^\infty;\ 0 \leq t \leq \infty\} \in \Phi(\{a_r\},\{b_r\})$ なので、3 節の議論はすべて成り立つ。更に、我々は、(E_∞) に対する 解作用素として $\overline{D(\varphi^\infty)}$ 上で定義された半群 $\{S(t)\} := \{S(t);\ 0 \leq t\}$ を考えることができる。つまり、それぞれの $t \in R_+$ に対し、$S(t)$ は $\overline{D(\varphi^\infty)}$ から $\overline{D(\varphi^\infty)}$ への写像で、それぞれの初期値 $z \in \overline{D(\varphi^\infty)}$ に対し $u(t) \in \overline{D(\varphi^\infty)}$ を対応させる。ここで、$u(t)$ は初期値 $u(0)=z$ をもつ (E_∞) の解である。

このとき、適正凸下半連続関数 φ^∞ に対するコンパクト性を仮定すれば、次の半群 $\{S(t)\}$ に対する大域的アトラクターの存在定理を得ることができる。

定理 4.1. φ^∞ が次を満たすと仮定する；

$$\left\{\begin{array}{l} \text{それぞれの } 0 < r < +\infty \text{ に対し、集合 } \{z \in H;\ |z|_H \le r,\ \varphi^\infty(z) \le r\} \\ \text{が } H \text{ のコンパクト部分集合である。} \end{array}\right.$$

そして、g は *(g1)* と次の *(4.1)* を満たすとする；

$$\left\{\begin{array}{l} \text{それぞれの } H \text{ の有界部分集合 } B \text{ に対し、ある 2 つの定数} \\ C_0(B) > 0 \text{ と } C_1(B) > 0 \text{ が存在し次を満たす；} \\ \qquad \varphi^\infty(z) + (g(z), z - v) \ge C_0(B)|z|_H^2 - C_1(B), \\ \textit{for all } z \in D(\varphi^\infty) \textit{ and } v \in B. \end{array}\right. \tag{4.1}$$

そのとき、以下の 3 つの条件を満たす $\overline{D(\varphi^\infty)}$ の部分集合 $\mathcal{A}_\infty$ が存在する；

(i) $\mathcal{A}_\infty$ は空でない H のコンパクトで連結な部分集合である；

(ii) 任意の $t \in R_+$ に対し、$S(t)\mathcal{A}_\infty = \mathcal{A}_\infty$；

(iii) 任意の H の有界部分集合 B と定数 $\epsilon > 0$ に対し、有限時間 $T_{B,\epsilon} > 0$ が存在して次が成立する；

$$dist_H(S(t)z, \mathcal{A}_\infty) < \epsilon,$$

$$\textit{for all } z \in \overline{D(\varphi^\infty)} \cap B \textit{ and all } t \ge T_{B,\epsilon}.$$

定理 4.1. の (i)-(iii) の性質をもつとき、$\mathcal{A}_\infty$ を半群 $\{S(t)\}$ に対する大域的アトラクターとよぶ。もし大域的アトラクター が存在するなら、明らかに大域的アトラクター は一意である。

大域的アトラクターを構成するために補題を用意する。

補題 4.1. 定理 *4.1* の仮定の下で次を得る；

(1) $S(\cdot)(\cdot)$ は、$R_+ \times \overline{D(\varphi^\infty)}$ から $\overline{D(\varphi^\infty)}$ への連続写像である。ここで、$R_+ \times \overline{D(\varphi^\infty)}$ と $\overline{D(\varphi^\infty)}$ の位相は、それぞれ $R_+ \times H$ と H の部分位相である。

(2) 任意の H の有界部分集合 B に対し、$\bigcup_{t \in R_+} S(t)(\overline{D(\varphi^\infty)} \cap B)$ は H の有界部分集合である。

(3) それぞれの H の有界部分集合 B と任意の定数 $\delta > 0$ に対し、

$$C_\delta := \bigcup_{t \ge \delta} S(t)(\overline{D(\varphi^\infty)} \cap B),$$

は H の相対コンパクト集合で、φ^∞ は C_δ 上で有界である。

(4) ある $\overline{D(\varphi^\infty)}$ のコンパクト凸部分集合 B_0 が存在し、次が成り立つ；

$$\sup_{z \in B_0} \varphi^\infty(z) < +\infty. \tag{4.2}$$

そして、それぞれの H の有界部分集合 B に対し、次を満たす有限時間 $T_B > 0$ が存在する；

$$S(t)(\overline{D(\varphi^\infty)} \cap B) \subset B_0, \qquad \textit{for all } t \ge T_B. \tag{4.3}$$

証明. (1)、(2) そして (3) は、それぞれ定理 3.2 とその系 2 より成り立つ。実際、族 $\{\varphi^\infty\}; \varphi^t \equiv \varphi^\infty$ $(t \in \overline{R_+})$ に対して、定理 3.2 とその系 2 を適用すればよい。

そこで、(4) のみ示す。

$z \in \overline{D(\varphi^\infty)}$ とし、それぞれの時間 $t \geq 0$ に対し $u(t) := S(t)z$ と定める。このとき、$z_0 \in D(\varphi^\infty)$ を固定し (E_∞) に $u(t) - z_0$ をかけると、次を得る。

$$\frac{1}{2}\frac{d}{dt}|u(t) - z_0|_H^2 + \varphi^\infty(u(t)) + (g(u(t)), u(t) - z_0)$$
$$\leq \varphi^\infty(z_0) + (f^\infty, u(t) - z_0).$$

ここで条件 (4.1) に注意すると、次の不等式が導かれる。

$$\frac{d}{dt}|u(t) - z_0|_H^2 + \delta_1|u(t) - z_0|_H^2 \leq R_1 \quad \text{for a.e. } t \geq 0, \tag{4.4}$$

ここで δ_1 と R_1 は、$z \in \overline{D(\varphi^\infty)}$ に依存しない正の定数である。(4.4) から、

$$|u(t) - z_0|_H^2 \leq \mathrm{e}^{-\delta_1 t}|z - z_0|_H^2 + \frac{R_1}{\delta_1} \quad \text{for all } t \geq 0. \tag{4.5}$$

を得る。ここで、

$$B_1 := \overline{D(\varphi^\infty)} \cap \{v \in H;\ |v - z_0|_H^2 \leq 1 + \frac{R_1}{\delta_1}\}.$$

とおく。(4.5) から、B_1 が次の性質をもつことがわかる；

それぞれの H の有界部分集合 B に対し、次の条件を満たす有限時間 $t_B > 0$ が存在する；

$$S(t)(\overline{D(\varphi^\infty)} \cap B) \subset B_1, \qquad \text{for all } t \geq t_B, \tag{4.6}$$

そこで、$B_0 := \overline{\mathrm{conv}}(S(1)B_1)$ と定める。ここで $\overline{\mathrm{conv}}(\ \cdot\ )$ は、$(\ \cdot\ )$ に対する convex hull である。実際、補題の (3) により、$S(1)B_1$ は H で相対コンパクトで φ^∞ は、$S(1)B_1$ 上で有界である。それゆえ B_0 は H でコンパクト凸集合で、(4.2) が成立し、(4.3) は (4.6) を考慮すると $T_B = t_B + 1$ に対し成立する。◇

定理 4.1 の証明: 補題 4.1 を考慮すると、半群 $\{S(t)\}$ に対する大域的アトラクター $\mathcal{A}_\infty$ は、アトラクターの一般論を用いることにより得られる。実際、補題 4.1 の (4) で得られた吸収集合 B_0 に対し、大域的アトラクター $\mathcal{A}_\infty$ は、

$$\mathcal{A}_\infty = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)(B_0)},$$

で与えられる。

◇

5. 非自励系に対する大域的アトラクター

この節で、我々が得た定理を述べる。

定理 5.1. *(A1)-(A6)* を仮定する。また、$\mathcal{A}_\infty$ を定理 *4.1.* で得られた半群 $\{S(t)\}$ に対する大域的アトラクターとする。このとき、次が成り立つ。

(i) H の任意の有界部分集合 B に対して、

$$\bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau,s\geq 0} E(t+s,s)(\overline{D(\varphi^s)}\cap B)}\subset\mathcal{A}_\infty.$$

または同値的に、任意の $\epsilon>0$ に対し、$T_{B,\epsilon}>0$ が存在して次を満たす；

$$dist_H(E(t+s,s)u_0,\mathcal{A}_\infty)\leq\epsilon,\qquad \textit{for all } s\geq 0,\ u_0\in\overline{D(\varphi^s)}\cap B \textit{ and } t\geq T_{B,\epsilon}. \tag{5.1}$$

(ii) 次をみたす H の有界部分集合 B^* が存在する；

$$\bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau,s\geq 0} E(t+s,s)(\overline{D(\varphi^s)}\cap B^*)}=\mathcal{A}_\infty.$$

これと同値的に、任意の $v_\infty\in\mathcal{A}_\infty$ に対し、2つの列 $\{t_n\}$；$t_n\uparrow+\infty$ と $\{s_n\}\subset R_+$ そして点列 $\{z_n\}\subset\overline{D(\varphi^{s_n})}\cap B^*$ が存在して次が成り立つ；

$$E(t_n+s_n,s_n)z_n\longrightarrow v_\infty\qquad \textit{in } H.$$

しばらくの間、H の 有界部分集合 B を固定する。そこで、定理 3.2 と系 2 により、

$$|E(t+s,s)z|_H\leq r_B,\qquad \text{for any } s,\ t\geq 0 \text{ and } z\in\overline{D(\varphi^s)}\cap B,$$

and

$$|\varphi^{t+s}(E(t+s,s)z)|\leq M_B,\qquad \text{for any } s\geq 0,\ t\geq 1 \text{ and } z\in\overline{D(\varphi^s)}\cap B,$$

を満たす定数 $r_B>0$ と $M_B>0$ が存在する。

次に時間依存の仮定 (A2) から、それぞれの初期時間 $s\geq 0$ と初期値 $z\in\overline{D(\varphi^s)}\cap B$ そして $t\geq 0$ に対し、次をみたすある点 $\tilde{z}:=\tilde{z}_{s,z,t}\in D(\varphi^\infty)$ が存在する；

$$\begin{aligned}&|\tilde{z}-E(t+s,s)z|_H\leq\left(\int_{t+s}^{\infty}|a'_{r_B}(\sigma)|d\sigma\right)(1+M_B^{\frac{1}{2}}),\\ &(\text{従って、}|\tilde{z}|_H\leq r_B+\left(\int_0^{\infty}|a'_{r_B}(\sigma)|d\sigma\right)(1+M_B^{\frac{1}{2}})=r'_B.)\end{aligned} \tag{5.2}$$

そして、

$$\begin{aligned}\varphi^\infty(\tilde{z})&\leq M_B+\left(\int_{t+s}^{\infty}|b'_{r_B}(\sigma)|d\sigma\right)(1+M_B)\\ &\leq M_B+\left(\int_0^{\infty}|b'_{r_B}(\sigma)|d\sigma\right)(1+M_B)=M'_B.\end{aligned}$$

ここで、$\tilde{z}_{s,z,t}$ の集合を $\tilde{B}$ で表すことにする。つまり、

$$\tilde{B}:=\{\tilde{z}_{s,z,t};\ s\geq 0,\ z\in\overline{D(\varphi^s)}\cap B,\ t\geq 1\}, \tag{5.3}$$

と定義する。このとき $\tilde{B}$ は、$D(\varphi^\infty)$ の部分集合で H で相対コンパクトであることに注意する。

定理 5.1 を証明するために次の補題を用意する。

補題 5.1. B を H の有界集合とし、$\tilde{B}$ は上で定義したものとする。また T は任意の正定数とする。このとき、それぞれの $\epsilon > 0$ に対し、

$$\sup_{\tau\in[0,T]} |E(\tau+t+s,s)z - S(\tau)\tilde{z}_{s,z,t}|_H \le \epsilon,$$

$$\textit{for all } s \ge 0,\ z \in \overline{D(\varphi^s)} \cap B \textit{ and } t \ge t^*,$$

を満たす $t^* = t^*(B,T,\epsilon) \ge 1$ が存在する。

証明 背理法によりこの補題を証明する。結論が成り立たないと仮定する。つまり、ある定数 $\epsilon_0 > 0$ に対し、

$$\sup_{\tau\in[0,T]} |E(\tau+t_n+s_n,s_n)z_n - S(\tau)\tilde{z}_n|_H \ge \epsilon_0, \tag{5.4}$$

となる列 $\{s_n\} \subset R_+$ と $\{z_n\} \subset B$; $z_n \in \overline{D(\varphi^{s_n})}$ そして $\{t_n\}$; $t_n \ge n$ $(n = 1,2,\ldots)$ が存在する。
ここで、$\tilde{z}_n := \tilde{z}_{s_n,z_n,t_n} \in \tilde{B}$ である。$\tilde{B}$ は H で相対コンパクトなので、

$$\tilde{z}_n \longrightarrow \tilde{z}_\infty \quad \text{in } H,$$

となる $\tilde{z}_\infty$ が存在すると仮定してもよい。すると、(5.2) により明らかに、

$$E(t_n+s_n,s_n)z_n \longrightarrow \tilde{z}_\infty \quad \text{in } H,$$

となる。

さて、次の 2 つの初期値問題を考える。

$$\begin{cases} v_n'(\tau) + \partial\varphi^\infty(v_n(\tau)) + g(v_n(\tau)) \ni f^\infty, & 0<\tau<T, \\ v_n(0) = \tilde{z}_n, \end{cases}$$

そして、

$$\begin{cases} u_n'(\tau) + \partial\varphi^{\tau+t_n+s_n}(u_n(\tau)) + g(u_n(\tau)) \ni f(\tau+t_n+s_n), & 0<\tau<T, \\ u_n(0) = E(t_n+s_n,s_n)z_n. \end{cases}$$

すると v_n と u_n はともに、初期値 $\tilde{z}_\infty$ の $[0,T]$ 上の (E_∞) の解に収束する。(cf. [2]). それゆえ、

$$\sup_{\tau\in[0,T]} |u_n(\tau) - v_n(\tau)|_H \longrightarrow 0.$$

$u_n(\tau) = E(\tau+t_n+s_n,s_n)z_n$ で $v_n(\tau) = S(\tau)\tilde{z}_n$ なので、これは (5.4) に矛盾する。♢.

定理 5.1 の (i) の証明： B を H の任意の有界集合とし、$\tilde{B}$ は (5.3) で与えられた H の相対コンパクト部分集合とする。$\mathcal{A}_\infty$ は半郡 $\{S(t)\}$ に対する大域的アトラクターなので、その定義からそれぞれの $\epsilon > 0$ に対し、

$$\mathrm{dist}_H(S(\tau)\tilde{z}, \mathcal{A}_\infty) \le \frac{\epsilon}{2}, \quad \text{for all } \tilde{z} \in \tilde{B} \text{ and } \tau \ge \tau_0,$$

となる有限時間 $\tau_0 = \tau_0(B,\epsilon) > 0$ が存在する。すなわち、

$$\mathrm{dist}_H(S(\tau)\tilde{z}_{s,z,t}, \mathcal{A}_\infty) \le \frac{\epsilon}{2}, \tag{5.5}$$

$$\text{for all } s \ge 0,\ z \in \overline{D(\varphi^s)} \cap B,\ t \ge 1 \text{ and } \tau \ge \tau_0.$$

ここで、$T=\tau_0$ に対し補題 5.1 を適用すると、

$$\sup_{\tau\in[0,\tau_0]} |E(\tau+t+s,s)z - S(\tau)\tilde{z}_{s,z,t}|_H \le \frac{\epsilon}{2}, \tag{5.6}$$

$$\text{for all } s\ge 0,\ z\in \overline{D(\varphi^s)}\cap B \text{ and } t\ge t^*.$$

となる $t^*=t^*(B,\tau_0,\epsilon)\ge 1$ が存在する。それゆえ、$\tau=\tau_0$ ととると、(5.5) と (5.6) から、

$$\begin{aligned}\mathrm{dist}_H(E(\tau_0+t+s,s)z,\mathcal{A}_\infty) &\le |E(\tau_0+t+s,s)z - S(\tau_0)\tilde{z}_{s,z,t}|_H \\ &\quad +\mathrm{dist}_H(S(\tau_0)\tilde{z}_{s,z,t},\mathcal{A}_\infty) \\ &\le \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon,\end{aligned}$$

$$\text{for all } s\ge 0,\ z\in \overline{D(\varphi^s)}\cap B \text{ and } t\ge t^*.$$

となる。従って、$T_{B,\epsilon}:=t^*+\tau_0$ に対し、(5.1) が成り立つ。 ◇

定理 5.1 の (ii) の証明： B_0 を補題 4.1 の (4) で得られた半群 $\{S(t)\}$ に対する吸収集合とする。さて、定理 5.1 の (i) の証明で集合 B と対応して 集合 $\tilde{B}$ を考えたように、それぞれの時間 $s\ge 0$ に対し吸収集合 B_0 と対応して $D(\varphi^s)$ の部分集合 B_s を次のように定める。まず、

$$r_0 := \sup_{z\in B_0}|z|_H < +\infty,\quad M_0 := \sup_{z\in B_0}|\varphi^\infty(z)| < +\infty,$$

とおく。このとき、それぞれの $z\in B_0$ に対し、次を満たす $v=v_{z,s}\in D(\varphi^s)$ を選ぶ；

$$\begin{aligned}&|v-z|_H \le \left(\int_s^\infty |a'_{r_0}(\sigma)|d\sigma\right)(1+M_0^{\frac{1}{2}}),\\ &(\text{hence } |v|_H \le r_0 + \left(\int_0^\infty |a'_{r_0}(\sigma)|d\sigma\right)(1+M_0^{\frac{1}{2}})),\end{aligned} \tag{5.7}$$

そして、

$$\begin{aligned}\varphi^s(v) &\le \varphi^\infty(z) + \left(\int_s^\infty |b'_{r_0}(\sigma)|d\sigma\right)(1+M_0)\\ &\le M_0 + \left(\int_0^\infty |b'_{r_0}(\sigma)|d\sigma\right)(1+M_0) =: M_0'.\end{aligned} \tag{5.8}$$

そこで、B_s を

$$B_s := \{v_{z,s};\ s\ge 0,\ z\in B_0\}\subset D(\varphi^s),$$

と定める。

このとき、B^* を

$$B^* := \bigcup_{0\le s<+\infty} B_s,$$

と定めると、この集合が求めるものであることを以下示す。(5.7) と (5.8) から、明らかに B^* は、H で相対コンパクトである。

定理 4.1 の証明で $\mathcal{A}_\infty = \bigcap_{\tau\ge 0}\overline{\bigcup_{t\ge\tau} S(t)B_0}$ であることがわかった。これは次と同値である；それぞれの $v_\infty\in\mathcal{A}_\infty$ に対し、

$$S(t_n)z_n \longrightarrow v_\infty, \qquad \text{in } H, \tag{5.9}$$

となる2つの列 $\{t_n\}$；$t_n \uparrow +\infty$ (as $n \to +\infty$) と $\{z_n\} \subset B_0$ が存在する。

ここで、それぞれの自然数 n と時間 $s \geq 0$ に対し、上の $z_n \in B_0$ と対応する $v_{z_n,s} \in B_s$ をとる。すると $\varphi^{t_n+s} \longrightarrow \varphi^\infty$ on H as $s \to +\infty$ そして $v_{z_n,s} \longrightarrow z_n$ in H as $s \to +\infty$ なので、

$$\sup_{\tau\in[0,t_n]} |E(\tau+s,s)v_{z_n,s} - S(\tau)z_n|_H \longrightarrow 0 \quad \text{as } s \to +\infty.$$

それゆえ、十分大きい時間 s_n に対し次が成立する；

$$|E(t_n+s_n,s_n)v_{z_n,s_n} - S(t_n)z_n|_H \leq \frac{1}{n}. \tag{5.10}$$

(5.9) と (5.10) により、

$$E(t_n+s_n,s_n)v_{z_n,s_n} \longrightarrow v_\infty, \qquad \text{in } H \text{ as } n \to +\infty,$$

を得る。つまり、

$$v_\infty \in \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau,\ s\geq 0} E(t+s,s)(\overline{D(\varphi^s)} \cap B^*)},$$

である。

◇

6. Skew product flows に対する大域的アトラクター

この節では、skew-product flows を構成し非自励系 (1.1) に対する大域的アトラクターの存在を示す。その為、次のような適正凸下半連続関数の集合を定義する。

十分大きい定数 $R_0 > 0$ に対し、集合 Ψ を次のように定義する。

$$\Psi := \left\{ \psi; \begin{array}{c} \psi\text{は適正凸下半連続 on } H \text{ で、} \\ |z|_H \leq R_o \text{ そして } \psi(z) \leq R_0 \\ \text{となる点 } z \in D(\psi) \text{ が少なくとも1点存在する} \end{array} \right\}.$$

[3] に従って、Ψ 上の距離位相 $d_\Psi(\cdot,\cdot)$ を考える。$d_\Psi(\cdot,\cdot)$ は、$\Psi \times \Psi$ 上の擬距離 $d_r(\cdot,\cdot)$, $R_0 \leq r < +\infty$,

$$d_r(\psi_1,\psi_2) := H(L_r(\psi_1), L_r(\psi_2)), \quad \psi_1,\ \psi_2 \in \Psi,$$

により与えられる。ここで、$H(X_1,X_2)$ は、$H \times R$ の2つの空でない有界閉部分集合 X_1 と X_2 のハウスドルフ距離であり、$L_r(\psi) := \{(z,\sigma \in H \times R;\ |z|_H \leq r,\ \psi(z) \leq \sigma \leq r\}$ である。

この位相をもつ Ψ において、列 $\{\psi_n\}$ が ψ に収束する、つまり $D_\Psi(\psi_n,\psi) \longrightarrow 0$ (as $n \to \infty$) であることは次と同値である；

$$d_r(\psi_n,\psi) \longrightarrow 0, \qquad \text{for every } r \geq R_0.$$

さて、ここで第2節で与えた仮定 (A1)-(A6) に次のような仮定 (A7) を加える。

(A7) $\overline{D(\varphi^t)} =: D_0$ は、時間 $t \in [0,+\infty]$ に依存しない。

定義 6.1. $C(R_+;\Psi) \times L^2_{loc}(R_+;H)$ の部分集合 $\mathcal{K}$ を次のように定義する。

$$\mathcal{K} := \{(\tau_s\varphi^{(\cdot)}, \tau_s f);\ 0 \leq s \leq +\infty\}.$$

ここで、τ_s はそれぞれの時間 s に対しシフト作用素を表すものとする。つまり、τ_s はそれぞれの$\varphi^{(\cdot)} := \{\varphi^t;\ 0 \le t \le +\infty\}$ (resp. f) に対し、族 $\varphi^{(\cdot+s)} := \{\varphi^{t+s};\ 0 \le t + \infty\}$ (resp. $f(\cdot+s)$) を対応させる作用素である。言い換えれば、

$$\tau_s\varphi^{(\cdot)} = \varphi^{(\cdot+s)}, \qquad (\text{resp. } \tau_s f = f(\cdot+s)),$$

となる。特に、もし $s = +\infty$ ならば、$\tau_\infty\varphi^{(\cdot)} \equiv \varphi^\infty$ そして $\tau_\infty f \equiv f^\infty$ と定める。

定義 6.2. 次のように作用素の族 $\{T(t);\ t \ge 0\}$ を定義する；

それぞれの $t \ge 0$ に対し、

$$T(t)(\psi^{(\cdot)}, p(\cdot)) = (\psi^{(\cdot+t)}, p(\cdot+t)), \qquad \text{for any } (\psi^{(\cdot)}, p(\cdot)) \in \mathcal{K}.$$

注意 6.1. $\mathcal{K}$ の定義から、それぞれの $(\psi^{(\cdot)}, p(\cdot)) \in \mathcal{K}$ に対し、$(\psi^{(\cdot)}, p(\cdot)) \in \mathcal{K} = (\tau_\sigma\varphi^{(\cdot)}, \tau_\sigma f)$ を満たすある定数 $\sigma \ge 0$ が存在することに注意する。

$\mathcal{K}$ と $\{T(t);\ t \ge 0\}$ の定義から、次の補題が成立することがわかる。

補題 6.1. 次の性質が成り立つ；

(1) $\mathcal{K}$ は、$C(R_+;\Psi) \times L^2_{loc}(R_+;H)$ のコンパクト部分集合である。

(2) $\{T(t);\ t \ge 0\}$ は、$\mathcal{K}$ 上で定義された半群である。

(3) 任意の時間 $t \ge 0$ に対し、$T(t)\mathcal{K} \subset \mathcal{K}$.

そこで、それぞれの $(u_0, \psi^{(\cdot)}, p(\cdot)) \in D_0 \times \mathcal{K}$ に対し、次の発展方程式を考える。

$$\begin{cases} u'(t) + \partial\psi^t(u(t)) + g(u(t)) \ni p(t), & t > \sigma, \\ u(\sigma) = u_0. \end{cases} \tag{6.1}$$

このとき、定理 3.1 により大域的解が存在するので、そのときの解作用素を $U_{(\psi^{(\cdot)},p(\cdot))}(t,\tau)$ とする。この作用素は、次の性質をもつことがわかる。

Lemma 6.2. 解作用素は、次の性質をもつ；

$$U_{(\psi^{(\cdot)},p(\cdot))}(t+s, \tau+s) = U_{T(s)(\psi^{(\cdot)},p(\cdot))}(t,\tau) \quad \textit{for any } t \ge \tau \ge 0,\ s \ge 0 \textit{ and } (\psi^{(\cdot)}, p(\cdot)) \in \mathcal{K}$$

証明. 注意 6.1 からこの補題は簡単に示せる。 ◊.

さて、 skew product flows を構成するために次のような $D_0 \times \mathcal{K}$ 上の作用素を定義する。

定義 6.3. 作用素の族 $\{F(t);\ t \ge 0\}$ を次のように定義する。
それぞれの時間 $t \ge 0$ に対し、

$$F(t)(u_0, \psi^{(\cdot)}, p(\cdot)) := (U_{(\psi^{(\cdot)},p(\cdot))}(t,0)u_0, T(t)(\psi^{(\cdot)}, p(\cdot))),$$

for any $(u_0, \psi^{(\cdot)}, p(\cdot)) \in D_0 \times \mathcal{K}$.

このとき、次の半群の存在定理を得る。

定理 6.3. $\{F(t);\ t \geq 0\}$ は、半群の性質をもつ。
Proof. 補題 6.1 から、次を得る；

$$\begin{aligned} F(0)(u_0, \psi^{(\cdot)}, p(\cdot)) &= (U_{(\psi^{(\cdot)}, p(\cdot))}(0,0)u_0, T(0)(\psi^{(\cdot)}, p(\cdot))) \\ &= (u_0, \psi^{(\cdot)}, p(\cdot)), \end{aligned}$$

つまり、$F(0)$ は、$D_0 \times \mathcal{K}$ 上の恒等作用素である。
次に、任意の $t, s \geq 0$ に対して、$F(t+s) = F(t) \circ F(s)$ が成立することを示す。補題 6.2 から、次が成立する。

$$\begin{aligned} F(t+s)(u_0, \psi^{(\cdot)}, p(\cdot)) &= (U_{(\psi^{(\cdot)}, p(\cdot))}(t+s, 0)u_0, T(t+s)(\psi^{(\cdot)}, p(\cdot))) \\ &= (U_{(\psi^{(\cdot)}, p(\cdot))}(t+s, s) \circ U_{(\psi^{(\cdot)}, p(\cdot))}(s, 0)u_0, T(t) \circ T(s)(\psi^{(\cdot)}, p(\cdot))) \\ &= (U_{(\psi^{(\cdot)}, p(\cdot))}(t+s, s)(U_{(\psi^{(\cdot)}, p(\cdot))}(s, 0)u_0), T(t)(T(s)(\psi^{(\cdot)}, p(\cdot)))) \\ &= F(t)(U_{(\psi^{(\cdot)}, p(\cdot))}(s, 0)u_0, T(s)(\psi^{(\cdot)}, p(\cdot))) \\ &= F(t)(F(s)(u_0, \psi^{(\cdot)}, p(\cdot))) \\ &= F(t) \circ F(s)(u_0, \psi^{(\cdot)}, p(\cdot)). \end{aligned}$$

従って、この定理が証明された。 ◇

この定理により、アトラクターの一般論が使える。よって、アトラクターの一般論により次の定理を得る。

Theorem 6.4. $\{F(t)\}$ に対する大域的アトラクター $\mathcal{A} \subset D_0 \times \mathcal{K}$ が存在する。その上、$\mathcal{A}$ は、$\mathcal{A}_\infty \times \{(\varphi^\infty, f^\infty)\}$ と等しい。ここで $\mathcal{A}_\infty$ は、定理 *4.1* で得られた自励系に対する大域的アトラクターである。

References

1. V. V. Chepyzhov and M. I. Vishik, Attractors of non-autonomous dynamical systems and their dimension, J. Math. Pures Appl. **73** (1994), 279-333.
2. A. Ito, N. Kenmochi and N. Yamazaki, Attractors of nonlinear evolution systems generated by time-dependent subdifferentials in Hilbert spaces, Technical Reports of Mathematical Sciences, Chiba Univ., Vol. 12 (1996), No. 7.
3. N. Kenmochi and M. Ôtani, Nonlinear evolution equations governed by subdifferential operators with almost periodic time-dependence, Rend, Acc. Naz. Sci. XL, Memorie di Mat. **140** (1986), 65-91.

Degree for Subdifferential Operators with Nonmonotone Perturbations*

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1 Introduction and Browder's Results

Browder [2, 3, 4] は, 回帰的 Banach 空間 X からその双対空間 X^* への, ある種の単調性をもついくつかの写像のクラスに写像度を定義した. その中の 1 つに極大単調作用素に対する写像度がある. 彼は, $A+f$ (A は極大単調作用素, f は有界な 1 価の擬単調作用素) というクラスに写像度を定義したが, それとまったく同様の方法で, $A+k$ (k はコンパクトな写像) というクラスにも写像度が定義できる.

THEOREM 1.1 X を実回帰的 Banach 空間で, X と X^* が locally uniformly convex (一様凸なら十分, Aspland [1] を見よ) となるもの, G を X の有界開集合とせよ. A を X から X^* への極大単調作用素で $0 \in A0$ を満たすもの, $k : \overline{G} \to X^*$ をコンパクトな写像とする. $p^* \in X^* \setminus \overline{(A+k)(\partial G)}$ のとき, 整数 $\deg(A+k, G, p^*)$ が定義され, 次を満たす.

1. (Normalization) $F : X \to X^*$ を duality map とする. $p^* \in F(G)$ ならば, $\deg(F, G, p^*) = 1$.
2. (Existence of solution) $\deg(A+k, G, p^*) \neq 0$ ならば, $p^* \in \overline{(A+k)(G)}$.
3. (Domain decomposition and excision) $G_1,\ G_2 \subset G$ を互いに素な開集合とする. $p^* \notin \overline{(A+k)(\overline{G} \setminus (G_1 \cup G_2))}$ のとき, $\deg(A+k, G, p^*) = \deg(A+k, G_1, p^*) + \deg(A+k, G_2, p^*)$.
4. (Invariance under homotopy) $\{A^t : t \in [0,1]\}$ を pseudo-monotone homotopy of maximal monotone oparators, $\{k_t : t \in [0,1]\}$ を $\overline{G}$ 上のコンパクトホモトピー, $\{p_t^* : t \in [0,1]\}$ を X^* における連続な曲線とせよ. $r > 0$ が存在して, 任意の $t \in [0,1]$ に対し, $B(p_t^*, r) \cap (A^t + k_t)(\partial G) = \emptyset$ が成立すると仮定する. このとき $\deg(A^t + k_t, G, p_t^*)$ は t に依らず一定である.

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pseudo-monotone homotopy の定義は Browder [2, 4] を見よ.

この理論には次のような欠点が見られる.

(a) 写像度を定義するための条件が, $p^* \notin \overline{(A+k)(\partial G)}$ のように閉包を必要とする. 従って, 領域の分解定理やホモトピー不変性の定理も使いにくいものとなっている.

(b) $\deg(A+k,G,p^*) \neq 0$ から, 方程式 $Ax + kx \ni p^*$ $(x \in G)$ の解の存在が直接は得られない.

(c) 写像度を normalize する duality map F と極大単調作用素 A との間の凸結合の族 $\{(1-t)F+tA\}$ が, 写像度を不変とするホモトピーのクラス (pseudo-monotone homotopy) に入るかどうか一般にはわからない.

(Browder の $A+f$ (f は擬単調作用素) というクラスに対する写像度も, ほぼ同様の欠点をもつ)
これらの点を改善するため, A が実 Hilbert 空間における劣微分作用素となっている場合について考察する.

2 Degree for Subdifferential Operators

H を可分な実 Hilbert 空間, G を H の有界な開集合とする. H の内積, ノルムをそれぞれ $(\cdot,\cdot)_H$, $|\cdot|_H$, 又は簡単に $(\cdot,\cdot)$, $|\cdot|$ で表す.

以後 H から H への多価写像を, $H \times H$ の部分集合であるそのグラフと同一視する.

$A \subset H \times H$ を極大単調作用素とせよ. A のレゾルベントと吉田近似をそれぞれ J_λ^A, A_λ で表す.

Definition 2.1 次の条件 (i)-(iii) を満たす H から $[0,+\infty]$ への下半連続凸関数 φ 全体の集合を $\Phi(H)$ で表す.

(i) $\varphi(0) = 0$.

(ii) $D(\varphi) = \{u \in H : \varphi(u) < +\infty\}$ が H で稠密.

(iii) 任意の $L \in]0,+\infty[$ に対して, 集合 $\{u \in H : \varphi(u) + |u|_H^2 \leq L\}$ が H でコンパクト.

(i) より φ の劣微分 $\partial\varphi$ は, $[0,0] \in \partial\varphi$ を満たす極大単調作用素となる. k をコンパクトな写像とすれば, $\partial\varphi + k$ の写像度を考えることができる. (iii) より $\partial\varphi + k$ は, 有界閉集合を閉集合に写す写像となり, 1 章で述べた欠点 (a), (b) が解決する (THEOREM 2.4 を見よ). (c) のホモトピーについては, (i), (ii) より次のような結果を得る.

PROPOSITION **2.2** $\varphi,\ \psi \in \Phi(H)$ とせよ.

1. $\{(1-t)\mathrm{Id} + t\partial\varphi : t \in [0,1]\}$ (Id は H 上の恒等写像) は pseudo-monotone homotopy である.

2. φ と ψ がさらに次のような角度条件

$$(\partial\varphi_\lambda(u), \partial\psi_\mu(u)) \geq 0 \qquad {}^\forall u \in H,\ {}^\forall\lambda > 0,\ {}^\forall\mu > 0 \tag{1}$$

を満たすならば, $\{(1-t)\partial\varphi + t\partial\psi : t \in [0,1]\}$ は pseudo-monotone homotopy である.

REMARK (1) より $\partial\{(1-t)\varphi + t\psi\} = (1-t)\partial\varphi + t\partial\psi$ となる. (1) と同値な条件については Brézis [5, THEOREM4.4] をみよ.

ホモトピーのクラスも定義しておく.

DEFINITION **2.3** 次の条件 (i)-(iv) を満たす H から $[0,+\infty]$ への下半連続凸関数の族 $\{\varphi^t : t \in [0,1]\}$ 全体の集合を $\Phi^t(H)$ で表す.

(i) 任意の $t \in [0,1]$ に対し, $\varphi^t(0) = 0$.

(ii) 任意の $t \in [0,1]$ に対し, $D(\varphi^t)$ が H で稠密.

(iii) 任意の $L \in]0,+\infty[$ に対して,

$$\{[u,t] \in H \times [0,1] : \varphi^t(u) + |u|_H^2 \leq L\}$$

が相対コンパクト.

(iv) $\{\partial\varphi^t : t \in [0,1]\}$ が pseudo-monotone homotopy.

REMARK $\varphi,\ \psi \in \Phi(H)$ が (1) を満たすとせよ. このとき, $\{(1-t)\varphi + t\psi\} \in \Phi^t(H)$ となる.

$\Phi(H)$ の元の劣微分の場合, THEOREM 1.1 は次のようになる.

THEOREM 2.4 $\varphi \in \Phi(H)$ とせよ. k を $\overline{G}$ から H へのコンパクトな写像とする. $p \in H \setminus (\partial\varphi + k)(\partial G)$ のとき, 整数 $\deg(\partial\varphi + k, G, p)$ が定義され, 次を満たす.

1. さらに, $\partial\varphi$ が狭義単調であるとせよ. $p \in \partial\varphi(G)$ ならば, $\deg(\partial\varphi, G, p) = 1$.

2. $\deg(\partial\varphi + k, G, p) \neq 0$ ならば, 方程式 $\partial\varphi(u) + ku \ni p$ は G に解をもつ.

3. G_1, G_2 を互いに素な G に含まれる開集合とする. $u \notin (\partial\varphi + k)(\overline{G} \setminus (G_1 \cup G_2))$ のとき, $\deg(\partial\varphi + k, G, p) = \deg(\partial\varphi + k, G_1, p) + \deg(\partial\varphi + k, G_2, p)$.

4. $\{\varphi^t : t \in [0,1]\} \in \Phi^t(H)$ とせよ. $\{k_t : t \in [0,1]\}$ を $\overline{G}$ 上のコンパクトホモトピー, $\{p_t : t \in [0,1]\}$ を H における連続な曲線とする. 任意の $t \in [0,1]$ に対し, $p_t \notin (\partial\varphi^t + k_t)(\partial G)$ が成立するとき, $\deg(\partial\varphi^t + k_t, G, p_t)$ は t に依らず一定である.

3 Perturbation Problems

2 章において Hilbert 空間上で $\partial\varphi + k$ という形の写像のクラス ($\varphi \in \Phi(H)$, k はコンパクトな写像) に限定して考えることにより, Browder の極大単調作用素に対する写像度の欠点を改善した. しかし φ のレベルセットのコンパクト性をすでに仮定しているため, 実際の偏微分方程式への応用を考えた場合, $\partial\varphi$ の摂動のクラスを, より弱いクラスへ拡張する必要がある. この様な視点から, この章では摂動の一般化を考える.

まず, 摂動のクラスをコンパクトな多価写像に拡張することから始める. 基本的な考え方は, Leray-Schauder の写像度を多価写像に対するものに拡張する方法の一つと同様である. 詳細は Lloyd [8, 7 章] 参照.

THEOREM 3.1 K を $\overline{G}$ から H へのコンパクトな多価写像で, 任意の $u \in \overline{G}$ に対し, Ku は空でない閉凸集合となるものとする. THEOREM 2.4 は, 一価のコンパクトな写像 k を K に, 一価のコンパクトホモトピー $\{k_t\}$ を多価のコンパクトホモトピー $\{K_t\}$ に, それぞれ置き換えても成立する.

次に, 摂動のクラスを, φ に付随したある種の有界性と demiclosed 性をもつ多価写像のクラスにまで拡張する.

$[0,+\infty[$ 上の単調増加関数全体の集合を $\mathcal{M}$ とおく.

DEFINITION **3.2** $\varphi \in \Phi(H)$ に対し, 次の条件 (i)-(iv) を満たす H から H への多価写像 B 全体の集合を $\mathcal{BD}_\varphi(H)$ で表す.

(i) 任意の $u \in D(\partial\varphi)$ に対し, Bu は空でない閉凸集合.

(ii) B は次の意味で demiclosed: $[u_n, v_n] \in \partial\varphi$, $b_n \in B(u_n)$ で, かつ, $u_n \to u$, $v_n \rightharpoonup v$, $b_n \rightharpoonup b$ とすると, $b \in Bu$.

(iii) $k_0 \in \,]0,1[\,$, $\alpha \in \,]0,2[\,$, $\ell_0 \in \mathcal{M}$ が存在し,

$$|b|_H^2 \le k_0|v|_H^2 + \ell_0(|u|_H)(\varphi(u)^\alpha + 1) \quad {}^\forall[u,v] \in \partial\varphi,\ {}^\forall b \in Bu.$$

(iv) $k_1 \in \,]0,1[\,$, $\ell_1 \in \mathcal{M}$ が存在し,

$$-(b,u)_H \le k_1\varphi(u) + \ell_1(|u|_H) \quad {}^\forall u \in D(\partial\varphi),\ {}^\forall b \in Bu.$$

ホモトピーのクラスも同様に定義する.

DEFINITION **3.3** $\{\varphi^t : t \in [0,1]\} \in \Phi^t(H)$ に対し, 次の条件 (i)-(iv) を満たす H から H への多価写像の族 $\{B^t : t \in [0,1]\}$ 全体の集合を $\mathcal{BD}^t_{\varphi^t}(H)$ で表す.

(i) 任意の $t \in [0,1]$, $u \in D(\partial\varphi)$ に対し, $B^t u$ は空でない閉凸集合.

(ii) $t_n \in [0,1]$, $[u_n, v_n] \in \partial\varphi^{t_n}$, $b_n \in B^{t_n}u_n$ で, かつ, $t_n \to t$, $u_n \to u$, $v_n \rightharpoonup v$, $b_n \rightharpoonup b$ ならば $b \in B^t u$.

(iii) $k_0 \in \,]0,1[\,$, $\alpha \in \,]0,2[\,$, $\ell_0 \in \mathcal{M}$ が存在し,

$$|b|_H^2 \le k_0|v|_H^2 + \ell_0(|u|_H)(\varphi^t(u)^\alpha + 1)$$
$${}^\forall t \in [0,1],\ {}^\forall[u,v] \in \partial\varphi^t,\ {}^\forall b \in B^t u.$$

(iv) $k_1 \in \,]0,1[\,$, $\ell_1 \in \mathcal{M}$ が存在し,

$$-(b,u)_H \le k_1\varphi^t(u) + \ell_1(|u|_H)$$
$${}^\forall t \in [0,1],\ {}^\forall u \in D(\partial\varphi^t),\ {}^\forall b \in B^t u.$$

さて, H の可分性より, 次のような有限次元部分空間の列 $\{H_i\}$ が存在する.

$$\begin{cases} H_1 \subset H_2 \subset \cdots \subset H_i \subset \cdots \\ \bigcup_{i \in \mathbf{N}} H_i = H \end{cases}$$

P_i を H から H_i への射影とする.

$\varphi \in \Phi(H)$, $B \in \mathcal{BD}_\varphi(H)$ とせよ. $i \in N$, $\lambda > 0$ に対し, $\partial\varphi^t$ のレゾルベント $J_\lambda^{\partial\varphi}$ と B と P_i の合成

$$B_{i,\lambda} \equiv P_i \circ B \circ J_\lambda^{\partial\varphi}$$

を考える. $B_{i,\lambda}$ はコンパクトな多価写像となる.

$$p \notin (\partial\varphi + B)(\partial G)$$

とせよ. このとき十分小さな $\lambda > 0$ と (λ に依存した) 十分大きな $i \in \mathbb{N}$ に対して $\deg(\partial\varphi + B_{i,\lambda}, G, p)$ が定義され, しかも λ, i に依らないことが示される. そこで

$$\deg(\partial\varphi + B, G, p) \equiv \lim_{\lambda\to 0} \lim_{i\to\infty} \deg(\partial\varphi + B_{i,\lambda}, G, p)$$

と定義する.

THEOREM 3.4 $\varphi \in \Phi(H)$, $B \in \mathcal{BD}_\varphi(H)$ とせよ. $p \in H \setminus (\partial\varphi + B)(\partial G)$ のとき, 整数 $\deg(\partial\varphi + B, G, p)$ が定義され, 次を満たす.

1. (Normalization) さらに, $\partial\varphi$ が狭義単調であるとせよ. $p \in \partial\varphi(G)$ ならば, $\deg(\partial\varphi, G, p) = 1$.

2. (Existece of solution) $\deg(\partial\varphi + B, G, p) \neq 0$ ならば, 方程式 $\partial\varphi(u) + Bu \ni p$ は G に解をもつ.

3. (Domain decomposition and excision) G_1, G_2 を互いに素な G に含まれる開集合とする. $u \notin (\partial\varphi + B)(\overline{G} \setminus (G_1 \cup G_2))$ のとき, $\deg(\partial\varphi + B, G, p) = \deg(\partial\varphi + B, G_1, p) + \deg(\partial\varphi + B, G_2, p)$.

4. (Invariance under homotopy) $\{\varphi^t : t \in [0,1]\} \in \Phi^t(H)$, $\{B^t : t \in [0,1]\} \in \mathcal{BD}^t_{\varphi^t}(H)$ とせよ. $\{p_t : t \in [0,1]\}$ を H における連続な曲線とする. 任意の $t \in [0,1]$ に対し, $p_t \notin (\partial\varphi^t + B^t)(\partial G)$ が成立するとき, $\deg(\partial\varphi^t + B^t, G, p_t)$ は t に依らず一定である.

参考文献

[1] E. ASPLAND, *Averaged norms*, Israel J. Math., **5** (1967), 227-233.

[2] F. E. BROWDER, *Fixed point theory and nonlinear problems*, Bull. Amer. Math. Soc. **9** (1983), 1-39.

[3] —-, *Degree of mappings for nonlinear mappings of monotone type*, Proc. Nat. Acad. Sci. USA, **80** (1983), 1771-1773.

[4] —-, *Degree of mappings for nonlinear mappings of monotone type; Densely defined mappings*, Proc. Nat. Acad. Sci. USA, **80** (1983), 2405-2407.

[5] H. BRÉZIS, *Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert,* Math. Studies, vol. 5, North-Holland, Amstardam/ New York, 1973.

[6] M. ÔTANI, *Nonmonotone perturbations for nonlinear evolution equations associated with subdifferential operators, Cauchy problems,* J. Differential Equations **46** (1982), 268-299.

[7] —-, *Nonmonotone perturbations for nonlinear evolution equations associated with subdifferential operators, periodic problems*, J. Differential Equations **54** (1984), 248-273.

[8] N. G. LLOYD, *Degree theory*, Cambridge University Press, Cambridge, 1978.

[9] V. BARBU, *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff, Leyden, 1976.

LOCAL SOLVABILITY AND SMOOTHING EFFECTS OF NONLINEAR SCHRÖDINGER EQUATIONS WITH MAGNETIC FIELDS

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1. INTRODUCTION

In this paper, we consider the following nonlinear Schrödinger equations with a potential in a magnetic field,

$$i\partial_t u = \frac{1}{2}\sum_{j=1}^{n}(-i\partial_j - A_j(t,x))^2 u + V(t,x)u + F(u)$$

,

$$= H(t)u + F(u), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{1}$$

where $A(t,x) = (A_1(t,x), A_2(t,x), \dots A_n(t,x))$ is a vector potential, $V(t,x)$ is a scalar potential, $F(u)$ is a local nonlinear operator.

We consider local Cauchy ploblem of Eqs.(1), in this paper,we regard $\partial_t u$ as distribution on $\mathbb{R}^n$, we construct weak solutions u that $u(t) \in L^2$ or H^1, and consider local smoothing effects for H^1 solutions. Then, we need to consider linear part of Eqs(1),

$$i\partial_t u = \frac{1}{2}\sum_{j=1}^{n}(-i\partial_j - A_j(t,x))^2 u + V(t,x)u \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{2}$$

Assumption A For $j = 1, ..n$, $A_j(t,x)$ is a real-valued function of $(t,x) \in \mathbb{R}^1 \times \mathbb{R}^n$ such that $\partial_x^\alpha A_j(t,x)$ is C^1 for any multi-index α. For $|\alpha| \geq 1$ we have, with some $\varepsilon > 0$

$$|\partial_x^\alpha B_{jk}(t,x)| \leq C_\alpha (1+|x|)^{-1-\epsilon}, \ j,k = 1,..,n, \tag{3}$$

$$|\partial_x^\alpha A(t,x)| + |\partial_x^\alpha \partial_t A(t,x)| \leq C_\alpha, \ (t,x) \in \mathbb{R}^1 \times \mathbb{R}^n, \tag{4}$$

where $B_{jk}(t,x) = \partial_j A_k(t,x) - \partial_k A_j(t,x)$.

Assumption V $V(t,x)$ is a real-valued function of $(t,x) \in \mathbb{R}^1 \times \mathbb{R}^n$ such that $\partial_x^\alpha V(t,x)$ is continuous eor every α. For $|\alpha| \geq 2$ we have

$$|\partial_x^\alpha V(t,x)| \leq C_\alpha, \ (t,x) \in \mathbb{R}^1 \times \mathbb{R}^n. \tag{5}$$

Assumption F1 $F \in C^1(\mathbb{C};\mathbb{C})$, with $F(0) = 0$.

Assumption F2 $|F'(\zeta)| \le M(1 + |\zeta|^{p-1})$, $\zeta \in \mathbb{C}$, $1 \ge p \le \infty$.

Theorem 1. Asaume (A,V,F1,F2) with $1 < p < 1 + 4/n$. For each $\phi \in L^2$,there is $T > 0$,depending only on $\|\phi\|_2$, and a unique solution $u \in C(I; L^2)$ of the Eq(1) with $u(0) = \phi$.

To construct H^1−solution and consider local smoothing effects, we change assumptions of A and V.

Assumption A' For $j = 1,..n, A_j(x)$, which is independent of t, is a real-valued function of $x \in \mathbb{R}^n$. For multi-index $|\alpha| \ge 1$ we have, with some $\varepsilon > 0$,

(6) $$|\partial^\alpha B_{jk}(x) \le C_\alpha(1 + |x|)^{-1-\varepsilon},\ j,k = 1,..n,$$

(7) $$|\partial^\alpha A(x)| \le C_\alpha,\ x \in \mathbb{R}^n,$$

where $B_{jk}(x) = \partial_j A_k(x) - \partial_k A_j(x)$.

Assumption V' $V(x)$, which is independent of t, is real-valued function and bounded from below. For $|\alpha| = 2$ we have,

(8) $$\partial^\alpha V \in L^\infty$$

Remark Then there is V_0 which is satisfied with (V), we have $V = V_0 + V_1$.

Assumption A" Under (A',V') we have,

(9) $$|A(x)| \le C|V_0(x)^{1/2}|,\ x \in \mathbb{R}^n.$$

Then we write Eqs(1) by

(10) $$i\partial_t u = H_1 u + V_1 u + F(u), t \ge 0, x \in \mathbb{R}^n,$$

(11) $$H_1 = \frac{1}{2}\sum_{j=1}^{n}(-i\partial_j - A_j(x))^2 + V_0(x)$$

Theorem 2. Assume (A',A",V',F1) and, if $n \ge 2$, (F2) with $1 < p < 1+4/(n-2)$. For each $\phi \in D(H_1^{1/2}) = H^1 \cap D(V_0^{1/2})$,there is $T > 0$, depending only on $\|H_1^{1/2}\phi\|_2$, and a unique solution $u \in C(I; D(H_1^{1/2}))$ to the Eq(1) $u(0) = \phi$. Where $H_1 = (1/2)\sum_{j=1}^{n}(-\partial_j - A_j(x))^2 - V_0(x)$.

Theorem 3. Let u denote the solution to the Eq(1) in Th.2. Suppose $\mu > 1/2$.

In the case $1 \leq n \leq 6$, the following holds.

$$\int_I \|\langle x\rangle^{\mu-3/2}\langle D\rangle^{3/2}u\|_2^2 dt < \infty \tag{12}$$

where $\|\cdot\|_2$ is the L^2 -norm.
In the case $n \geq 7$, if $p < 1 + 2/(n-4)$, then the above inequality holds.

The following notations are used in this paper.

$I = [0, T]$
$(\ ,\ ) : L^2-$ inner product.
$\|\ \|_q : L^q-$ norm.
$L^{q,s} = L^s(I; L^q)$
$\|\ \|_{q,s} : L^{q,s}-$norm.
$r \equiv 4(p+1)/n(p-1)$
$r' = r/(r-1)$
$S = S(\mathbb{R}^n)$: the space of rapidly decreasing functions.

2. Preliminaries

Proposition2.1 (Yajima[3]) Let $T > 0$ be sufficient small. The family of oscillatory integral operators $U(t,s)$ defined for $0 < |t-s| < T$ by

$$U(t,s)f(x) = (2\pi i(t-s))^{-n/2} \int e^{iS(t,s,x,y)} e(t,s,x,y) f(y) dy \quad f \in S, \tag{13}$$

where $S(t,s,x,y), e(t,s,x,y)$ are proper functions the below equation,

$$|\partial_x^\alpha \partial_y^\beta e(t,s,x,y)| \leq C_{\alpha\beta}, \quad 0 < |t-s| < T, \quad x, y \in \mathbb{R}^n,$$

is a unique prpagator for Eqs.(2) with the following properties:

(a)For every $t \neq s$,$U(t,s)$ maps $S(\mathbb{R}^n)$ into $S(\mathbb{R}^n)$ continuously and extends to a unitary operator in $L^2(\mathbb{R}^n)$.

(b)If we set $U(t,t) = 1$, the identify operator, then $\left\{U(t,s) \Big| |t-s| < T, \quad t,s \in \mathbb{R}^1\right\}$ is strongly continuous in $L^2(\mathbb{R}^n)$ and satisfies $U(t,r)U(r,s) = U(t,s)$.

(c)For$f \in \Sigma(2) = \left\{f \in L^2 \Big| \|f\|_{\Sigma(2)}^2 = \sum_{|\alpha+\beta|\leq 2} \|x^\alpha \partial_x^\beta f\|_2^2 < \infty\right\}$,$U(t,s)f$ is a $\Sigma(2)-$valued continuous and L^2-valued C^1 function of (t,s). It sutisfies the following equations,$i\partial_t U(t,s)f = H(t)U(t,s)f$,$i\partial_s U(t,s)f = -U(t,s)H(s)f$.

Lemma2.2(Yajima[3])Let $T > 0$ be sufficient small, $0 < |t-s| < T$. Then for $2 \leq q \leq \infty$,

$$\|U(t,s)f\|_q \leq C|t-s|^{-n(1/2-1/n)}\|f\|_{q'} \tag{14}$$

where q' is the index conjugate to q: $1/q + 1/q' = 1$, and the constant C does not depend on t, s, and $f \in S(\mathbb{R}^n)$.

We introduce the following spaces as Kato[1].

$$X = X(I) = L^{2,\infty} \cap L^{p+1,r}$$
$$\bar{X} = \bar{X}(I) = C(I; L^2) \cap L^{p+1,r}$$
$$X' = X'(I) = L^{2,1} + L^{1+1/p,r'}$$
$$\text{norm} \|u\|_X = \|u\|_{2,\infty} \vee \|u\|_{p+1,r}$$
$$\|f\|_{X'} = \inf\Big\{ \|f_1\|_{2,1} + \|f_2\|_{1+1/p,r'} \Big| f = f_1 + f_2 \Big\}$$

Remark2.3(i)$\bar{X}$ is a closed subspace of X.

(ii)X' is the dual of X.

(iii)The above three spaces are defined on $I = [0, T]$, that is, depend on T.Hence for a different T, a different space is defined. But these are no influence of the estimate for $\|\cdot\|_X, \|\cdot\|_{X'}$, we discuss now.

Set $s = 0$, for simplify, we define two linear operators Γ and G by

$$(\Gamma\phi)(t) = U(t,0)\phi, \qquad t \in I, \tag{15}$$

$$(Gf)(t) = \int_0^t U(t,\tau)f(\tau)d\tau, \qquad t \in I, \tag{16}$$

Here $\Sigma(k) = \Big\{ f \in L^2 \Big| \sum_{|\alpha+\beta|\le k} \|x^\alpha \partial_x^\beta f\|_2^2 = \|f\|_{\Sigma(k)^2} < \infty \Big\}, k = 0, 1, , , ,, \Sigma(-k)$ is the dual space of $\Sigma(k)$. AC denotes the class of absolutely continuous functions.

Lemma2.4 Γ is a bounded operator from L^2 into $C(I; L^2) \cap C^1(I; \Sigma(-2))$, with

$$i\partial_t \Gamma\phi = H(t)\Gamma\phi. \tag{17}$$

Lemma2.5 Let $f \in L^1(I; L^2)$. Then $Gf \in C(I; L^2) \cap AC(I; \Sigma(-2))$, with

$$i\partial_t Gf = H(t)Gf + if. \tag{18}$$

Lemma2.6

$$\|\Gamma\phi\|_X \le C\|\phi\|_2, \qquad \phi \in L^2, \tag{19}$$

$$\|Gf\|_X \le C'\|f\|_{X'}, \qquad f \in X', \tag{20}$$

where C, C' are independent of T.

Lemma2.7(Kato[1]) Assume that F satisfies (F1,2). Then $F \in C^1(X; X')$ with

$$\|F(u)\|_{X'} \le M_1 T\|u\|_X = M_2 T^\theta \|u\|_X^p, \quad u \in X, \tag{21}$$

$$\|F'(u)v\|_{X'} \le (M_1 T + M_2 T^\theta \|u\|_X^{p-1})\|v\|_X, \quad u, v \in X, \tag{22}$$

where M_1, M_2 are some constants, $\theta = 1/r - 1/r' = 1 - 2/r > 0$.

For Banach spaces X, Y, we say $F \in C^1(X;Y)$ if $F \in C(X;Y)$ and has a Gâteaux derivative $DF(u) \in L(X;Y)$ that depends on u strongly continuously. $L(X;Y)$ is the space of linear operators from X to Y.

3. The proof of Theorem.1.

Lemma3.1 Let $v, f \in L^1(I;L^2)$. Assume that v satisfies the differential equation;

$$i\partial_t v = H(t)v + f. \tag{23}$$

Then $v \in AC(I;\Sigma(-2))$, so that $v(0) \in \Sigma(-2)$ exists, with $v = \Gamma v(0) + Gf$.

We prove the theorem by solving the integral equation

$$u = \Phi(u) \equiv \Gamma\phi - iGF(u). \tag{24}$$

Let $E[\bar{E}]$ be the closed ball in $X[\bar{X}]$ with radius R and center at the origin.

Lemma3.2(Kato[1]) Φ maps E into $\bar{E}$ if R is sufficiently large and T is sufficiently small, both depending only on $\|\phi\|_2$.

Lemma3.3(Kato[1]) Φ is a contraction map in the $X-$metric.

A solution u of (1) coincides with the unique fixed point of Φ in E.

4. The proof of Theorem.2

Lemma4.1(Kato[1]) Assume (V'). V can be written in the form $V = V_0 + V_1$, where both V_0, V_1 are real-valued, and

$$V_0 \in C^\infty; V_0 \geq 1; \partial^k V_0 \in L^\infty (k \geq 2), \tag{25}$$

$$\partial^k V_1 \in L^\infty (k \leq 2). \tag{26}$$

Lemma4.2 $D(H_1^{1/2}) = H^1 \cap D(V_0^{1/2})$.
Set $\partial_A = \partial - iA, Q(x) = V_0(x)^{1/2}$, we introduce the following function spaces
$Y = \{u \in X | \partial u \in X, Qu \in X\}, \|u\|_Y = \|u\|_X \vee \|\partial u\|_X \vee \|Qu\|_X,$
$Y' = \{f \in X | \partial f \in X', Qu \in X'\}, \|f\|_{Y'} = \|u\|_{X'} \vee \|\partial f\|_{X'} \vee \|Qu\|_{X'},$
and prove that Y coincide with $H^1 \cap D(V_0{}^{1/2})$.

Lemma4.3 Let $T > 0$ be sufficient small, $0 < |t-s| < T$. Then

$$\Gamma_1 : D(H_1^{1/2}) \to Y, \|\Gamma_1\phi\| \leq c\|H_1^{1/2}\phi\|_2, \phi \in D(H_1^{1/2}), \tag{27}$$

$$G_1 : Y' \to Y, \|G_1 f\| \leq c\|f\|_{Y'}, f \in Y. \tag{28}$$

Let Γ_1, G_1 be operators defined by() with $U_1(t,s)$, a propagator for H_1.

Lemma4.4(Kato[1])Let $u \in Y, f = V_1 u + F(u)$. Then $f \in Y'$ with

$$\|f\|_{Y'} \leq c(M_1 + K_1)T\|u\|_Y + M_2 T^{\theta}\|u\|_Y^p \tag{29}$$

where $\theta = 1 - 2/r > 0$, $K_1 < \infty$ is a constant depending on V_1, and M_1, M_2 are as in(21).

Let $E_1[\bar{E}_1]$ be the closed ball in $Y[\bar{Y}]$ with radius R and center at the origin. $(\bar{Y} = \{u \in \bar{X} | \partial u \in \bar{X}, Qu \in \bar{X}\}.)$

Lemma4.5(Kato[1]) E_1 is a complete metric space in the $X-$metric.

$$u = \Phi_1(u) \equiv \Gamma_1 \phi - iG_1 F(u). \tag{30}$$

A solution u of (10) coincides with the unique fixed point of Φ_1 in E_1.

5. The proof of Theorem.3

Proposition5.1(Yajima[3])Suppose that (A,V) be satisfied for Eqs.(). Let $T > 0$ be sufficient small, $\mu > 1/2$ and $\rho > 0$. Then there exists a constant $C_{\rho\mu} > 0$ such that for $s \in \mathbb{R}^1$

$$\int_{s-T}^{s+T} \|\langle x\rangle^{-\rho-\mu}\langle D\rangle^{\rho} U(t,s) f\|_2^2 dt \leq C_{\rho\mu}\|\langle D\rangle^{\rho-1/2} f\|_2^2 \qquad f \in S(\mathbb{R}^n). \tag{31}$$

Lemma5.2(Sjölin[2]) Assume (F1) and, if $n \geq 2$, (F2) with $1 \leq p < 1 + 4/(n-2)$. Then $F(u) \in L^1(I; H^1)$ for $1 \leq n \leq 6$. Under the additional assumption $p < 1 + 2/(n-4)$, $F(u) \in L^1(I; H^1)$ for $n \geq 7$.

References

1. T.Kato, Nonlinear Schrödinger equations,in Schrödinger operators, Lecture Notes in Physics 345, Springer-Verlag, Berlin-New York, 1989
2. P.Sjölin, Regularity of solutions to nonlinear equations of Schrödinger type, Tohoku Math. J. 45 (1993)
3. K.Yajima, Schrödinger evolution equations with magnetic fields,J.d'Analyse Math. 56 (1991)

Exponential Stabilization of Flexible Structures using Static Output Feedback

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Abstract

In this research,we consider the exponential stabilization of flexible structures by using the static output feedback control. Sufficient conditions of structures and observation mechanisms are obtained for the closed loop systems to be exponentially stable. The energy multiplier method is the key idea in the proof of exponential stability.

Keywords : Distributed parameter sysems, Flexible structures, Flexible robot arms, Flexible manipulators, Direct strain feedback

1 Introduction

There are many researches to suppress vibrations of flexible structures such as light weight manipulators and large space structures . In the practical view point, it is convenient to design controller based on the finite dimensional approximation model obtained by experimental modal analysis or finite element method.

Since these flexible structures are continua phisically, the dynamics of flexible structures are described by infinite dimensional model, i.e. evolution equations whose state spaces are infinite dimensional Hilbert spaces or Banach spaces. The finite dimensional model cannot describe the dynamics of the flexible structures sufficiently. The above controller is valid for the finite dimensional model not for the original infinite dimensional system.

It is difficult to show in what mean and how this finite dimensional model approximates the original infinite dimensional system. In order to show the validity of the controller based on the finite dimensional model, it is necessary to choose adequate norm in the infinite dimensional state space and estimate the approximation error with this norm.

Thus, in this paper, we construct a controller based on the original infinite dimensional system. In many control methods proposed in the previous researches, static output feedback control is the most fundamental. In this control method, we design a bounded operator from a measurement space to a state space, and feed the measurement data back to the plant directly. This method is simple, but the asymptotic stability holds under the condition that the parameters of the system is unkown.

Theses static output feedback control laws are introduced by Gressang and Lamont [2] and Sakawa and Matsushita[3], with the application to the infinite dimensional observers. These researches deal with the distributed paramete systems described by diffusion equations as examples. For the system described by wave equations, Sakawa studied the relation between the infinite dimensional obsevavility and the static output feedback control.[4] This control based on boundary outputs (boundary observation) are studied by Nambu[5][6].

As static output feedback control laws for flexible structures, direct velocity feedback [7][8][9][10] and direct strain feedback are studied[11]. Recently, direct shear force feedback control are studied [13].

The theoretical essences of the static output feedback control laws which contain direct velocity feedback control, direct strain feedback control, and other genrral output feedback control was clarified in [1]. In these researches, the static GENERAL output feedback control is introduced and the class of structures and measurements for which the closed loop system become asymptotically stable. In this paper, the exponential stability is proved based on the energy multiplier method.

2 Dynamics and Observation Mechanisms of Flexible Structures

Consider a flexible structure and let $u_1(t,x) \in R^q$ $(q = 1,2,3)$ is an elastic displacement at time t, position $x \in \Omega$ where Ω represents a domain of structures and $\Omega \subset R^p$ $(p = 1,2,3)$. In general, the displacement $u_1(t,x)$ of a flexible structure at time t at position x satisfies a partial differential equation

$$\frac{\partial^2 u_1}{\partial t^2} + D_x \frac{\partial u_1}{\partial t} + A_x u_1(t,x) = f_2(t,x) \qquad (1)$$

for suitable partial differential operators D and A. For fixed time t, the spatial function $u_1(t,\cdot)$ can be thought as an element of a suitable Hilbert space H. The operator A represents stiffness and it is assumed that A is a self-adjoint positive definite operator with the compact resolvent. The operator D represents damping of the flexible structure, and we assume that it is Rayleigh damping, i.e.

$$D = \alpha I + \beta A \qquad (2)$$

follows for some non-negative constants α and β. $f_2(t,x)$ is a control input.

By introducing $u_2(t,x) = \partial u_1/\partial t$, the dynamics Eq.(1) of the flexible structure becomes

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix} = \begin{pmatrix} u_2(t,x) \\ -A_x u_1(t,x) - D_x u_2(t,x) \end{pmatrix} + \begin{pmatrix} 0 \\ f_2(t,x) \end{pmatrix} \qquad (3)$$

Let $u(t,x) = (u_1(t,x),\ u_2(t,x))^T$, $f(t,x) = (0,\ f_2(t,x))^T$, and

$$\mathcal{A}_x u(x) = (u_2(x),\ -A_x u_1(x) - D_x u_2(x))^T \quad \text{for } u(x) = (u_1(x), u_2(x))^T \qquad (4)$$

Then, Eq.(3) can be rewritten as

$$\frac{\partial u}{\partial t} = \mathcal{A}_x u(t,x) + f(t,x) \qquad (5)$$

Consider the following observation mechanism, i.e. suppose we can measure the following data

$$u_{(obs)}(t) = Cu(t,\cdot) \qquad (6)$$

$$= \begin{pmatrix} (c_1(\cdot), Cu_2(t,\cdot))_H \\ \vdots \\ (c_N(\cdot), Cu_2(t,\cdot))_H \end{pmatrix} \qquad (7)$$

where $c_k(\cdot)$ is a spatial weighting function $(k = 1,2,\cdots,N)$, $(\cdot,\cdot)_H$ is an inner product in a Hilbert space H, and the observation operator C is defined as follows.

$$Cu(\cdot) = \begin{pmatrix} (c_1(\cdot), Cu_2(\cdot))_H \\ \vdots \\ (c_N(\cdot), Cu_2(\cdot))_H \end{pmatrix} \qquad (8)$$

CONDITIONS We suppose that the following 4 conditions hold. For a flexible structure which satisfies the following conditions, the static output feedback control is valid, as we show later.

1. C^{-1} is a bounded operator on H.
2. $\tilde{A} = CAC^{-1}$ is a self-adjoint positive definite operator on H.
3. $\tilde{D} = CDC^{-1}$ is a positive definite operator on H.
4. $\tilde{\mathcal{A}}$ generates an analytic semigroup on $\mathcal{D}(\tilde{A}^{1/2}) \times H$.

where the operator $\tilde{\mathcal{A}}$ on $\mathcal{D}(\tilde{A}^{1/2}) \times H$ and $\tilde{\mathcal{C}} : \mathcal{D}(\tilde{A}^{1/2}) \times H \to R^N$

$$\tilde{\mathcal{A}}y = \left(y_2, -\tilde{A}y_1 - \tilde{D}y_2\right)^T \qquad (9)$$

$$\tilde{\mathcal{C}}y = [(c_k, y_2)_H; k \downarrow 1,2,\cdots,N] \qquad (10)$$

for $y = (y_1, y_2)^T \in \mathcal{D}(\tilde{A}^{1/2}) \times H$,

Examles of A and C which satisfies these conditions are as follows.

Cantilevers When the flexible structure is a cantilever, the spatial domain Ω of the structure is one-dimensional and Ω becomes an open interval $(0,L)$ where L is a length of the cantilever. Let $H = L^2((0,L);R)$, $A_x = EId^4/dx^4$, $D = \delta A$, and $C_x = d^2/dx^2$, where δ and EI are positive constants and

$$\mathcal{D}(A) = \{u \in H^4(0,L); 0 = u(0) = u'(0) = u''(L) = u'''(L)\} \qquad (11)$$

$$\mathcal{D}(C) = \left\{u \in H^2(0,L); 0 = u(0) = u'(0)\right\} \qquad (12)$$

which reflect the boundary conditions of the both ends $(x = 0, L)$ of the structure. $H^m(0,L)$ is the Sobolev space of order m on an open interval $(0,L)$ and EI is a positive constant.

Let $c_k(x)$ be a positive continuous function which approximates $\delta(x - a_k)$ $(k = 1,2,\cdots,N)$. The time derivative of the strains (the bending moments) of the structure at $x = a_1, a_2, \cdots, a_N$ are measured. A and CAC^{-1} becomes positive definite self-adjoint operators on H and the above conditions are satisfied.

Free Beams When the flexible structure is a free beam (a beam whose both ends are free), the spatial domain Ω of the structure is one-dimensional and Ω becomes an open interval $(0, L)$ where L is a length of the beam. Let $H = L^2((0,L);R)$, $A_x = EId^4/dx^4$, $D = \delta A$, and $C_x = d^2/dx^2$, where δ and EI are positive constants and

$$\mathcal{D}(A) = \{u \in H^4(0,L); 0 = u''(0) = u'''(0) = u''(L) = u'''(L)\} \quad (13)$$

$$\mathcal{D}(C) = \{u \in H^2(0,L); 0 = u(0) = u'(0)\} \quad (14)$$

which reflect the boundary conditions of the both ends ($x = 0, L$) of the structure. $H^m(0,L)$ is the m-th order Sobolev space on an open interval $(0,L)$ and EI is a positive constant.

Let $c_k(x)$ be a positive continuous function which approximates $\delta(x - a_k)$ ($k = 1, 2, \cdots, N$). The time derivative of the strains (the bending moments) of the structure at $x = a_1, a_2, \cdots, a_N$ are measured. A and CAC^{-1} becomes positive definite self-adjoint operators on H and the above conditions are satisfied.

General Structures The above conditions are satisfied when A is a self-adjoint positive definite operator on the Hilbert space H, $C = A^\alpha$ ($0 \le \alpha \le 1$ is a constant) and $D = \delta A$ where δ is a positive constant, since $CAC^{-1} = A$. If $C = A^\alpha$ ($0 \le \alpha \le 1$ is a constant), the observed data correspond to time derivatives of the elastic displacements (velocity) of the structure when $\alpha = 0$, to the time derivative of the strains when $\alpha = 1/2$, and to the shear forces when $\alpha = 3/4$.

3 Static Output Feedback Control of Flexible Structures

Consider a static output feedback control

$$f(t,x) = G(x)u_{obs}(t) \quad (15)$$

where the feedback gain $G(x)$ is

$$G(x) = -\tilde{g}\begin{pmatrix} 0, \cdots, 0 \\ (C^{-1}c_1)(x), \cdots, (C^{-1}c_N)(x) \end{pmatrix} \quad (16)$$

where $\tilde{g}$ is a positive constant.

By this static output feedback, we obtain the following closed loop system.

$$\begin{aligned} &\frac{\partial}{\partial t}\begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix} \quad (17) \\ &= \begin{pmatrix} u_2(t,x) \\ -A_x u_1(t,x) - D_x u_2(t,x) \end{pmatrix} \\ &- \begin{pmatrix} 0 \\ \tilde{g}\sum_{k=1}^N (C^{-1}c_k)(x)(c_k(\cdot), Cu_2(t,\cdot))_H \end{pmatrix} \end{aligned}$$

Introduce a new variable $y_i(t) = Cu_i(t) (i = 1,2)$ and $y(t) = (y_1(t), y_2(t))^T$, the closed loop system Eq.(17) is rewritten as follows.

$$\begin{aligned} &\frac{\partial}{\partial t}\begin{pmatrix} y_1(t,\cdot) \\ y_2(t,\cdot) \end{pmatrix} \quad (18) \\ &= \begin{pmatrix} y_2(t,\cdot) \\ -\tilde{A}y_1(t,\cdot) - \tilde{D}y_2(t,\cdot) \end{pmatrix} \\ &- \begin{pmatrix} 0 \\ \tilde{g}\sum_{k=1}^N c_k(\cdot)(c_k(\cdot), y_2(t,\cdot))_H \end{pmatrix} \end{aligned}$$

where $\tilde{A} = CAC^{-1}$ and $\tilde{D} = CDC^{-1}$.

By defining an operator $\tilde{C} : \mathcal{D}(\tilde{A}^{1/2}) \times H \to R^N$ as

$$\tilde{C}y = [(c_k, y_2)_H; k \downarrow 1, 2, \cdots, N] \quad (19)$$

for $y = (y_1, y_2)^T \in \mathcal{D}(\tilde{A}^{1/2}) \times H$, the transformed closed loop equation Eq.(18) can be rewritten as

$$\frac{d}{dt}y(t) = \left(\tilde{A} - \tilde{g}\tilde{C}^*\tilde{C}\right)y(t) \quad (20)$$

The operator $\tilde{A} - \tilde{g}\tilde{C}^*\tilde{C}$ in the transformed closed loop system Eq.(20) becomes dissipative on the Hilbert space $\mathcal{D}(\tilde{A}^{1/2}) \times H$.

Note The inner product $\langle \cdot, \cdot \rangle$ in the Hilbert space $\mathcal{D}\left(\tilde{A}^{1/2}\right) \times H$ is defined as follows:

$$\langle u, v \rangle = \left(\tilde{A}^{1/2}u_1, \tilde{A}^{1/2}v_1\right)_H + (u_2, v_2)_H \quad (21)$$

for $u = (u_1, u_2)^T, v = (v_1, v_2)^T \in \mathcal{D}\left(\tilde{A}^{1/2}\right) \times H$

Proof

$$\begin{aligned} &\langle \left(\tilde{A} - \tilde{g}\tilde{C}^*\tilde{C}\right)y, y \rangle \\ &= -\left(\tilde{D}y_2, y_2\right)_H - \tilde{g}\left(\tilde{C}y, \tilde{C}y\right)_{R^N} \quad (22) \\ &\le 0 \quad (23) \end{aligned}$$

Q.E.D.

Theorem 1 *The operator $\tilde{A}-\tilde{g}\tilde{C}^*\tilde{C}$ generates an analytic semigroup on the Hilbert space $\mathcal{D}(\tilde{A}^{1/2})\times H$, and the evolution equation Eq.(20) has an unique solution.*

Proof Since $\tilde{A}$ generates an analytic semigroup and $-\tilde{g}\tilde{C}^*\tilde{C}$ is bounded, $\tilde{A}-\tilde{g}\tilde{C}^*\tilde{C}$ also generates an analytic semigroup. Q.E.D.

Corollary 2 *$u(t)=(u_1(t),u_2(t))^T$ where $u_i(t)=C^{-1}y_i(t)$ satisfies the closed loop system Eq.(17).*

Proof This corollary is easily proved based on the fact that $y(t)=(y_1(t),y_2(t))^T$ satisfies the transformed closed loop system Eq.(18). Q.E.D.

4 Asymptotic Stability of the Closed Loop System

The following results about the asymptotical stability were proved in [1].

We show that this system Eq.(18) is asymptotically stable in the Hilbert space $\mathcal{D}\left(\tilde{A}^{1/2}\right)\times H$. (This equation looks like the equation in [4])

Theorem 3 *The solution $y(t)$ of the transformed closed loop system Eq.(18) is asymptotically stable in the Hilbert space $\mathcal{D}\left(\tilde{A}^{1/2}\right)\times H$.*

Proof For the solution $y(t)=(y_1(t),y_2(t))^T$ of the closed system Eq.(18), we have

$$\begin{aligned}\frac{d}{dt}\langle y(t),y(t)\rangle &= -2\left(\tilde{D}y_2(t),y_2(t)\right)_H \quad (24)\\ &\quad -2\tilde{g}\left(\tilde{C}y(t),\tilde{C}y(t)\right)_H\\ &\leq 0 \quad (25)\end{aligned}$$

Therefore, for $\frac{d}{dt}\langle y(t),y(t)\rangle=0$, it is necessary that $y_2(t)=0$, since $\tilde{D}$ is a positive definite operator.

This leads to the conclusion that $dy_2/dt=0$ and $y_1(t)=0$ from Eq.(18). Thus, $\langle y(t),y(t)\rangle$tends to 0 as $t\to\infty$. Q.E.D.

Corollary 4 *The solution $u(t)$ of the closed loop system Eq.(17) is asymptotically stable in the Hilbert space $H\times H$.*

Proof By the previous theorem, we have

$$\lim_{t\to\infty}\|y_1(t)\|_{\mathcal{D}\left(\tilde{A}^{1/2}\right)}=0 \quad (26)$$

and

$$\lim_{t\to\infty}\|y_2(t)\|_H=0 \quad (27)$$

Since $u_i(t)=C^{-1}y_i(t)$ $(i=1,2)$ and C^{-1} is bounded on the Hilbert space H, we have

$$\lim_{t\to\infty}\|u_1(t)\|_H=\lim_{t\to\infty}\|u_2(t)\|_H=0 \quad (28)$$

Q.E.D.

5 Exponential Stability of the Closed Loop System

We show that this system Eq.(20) is exponentially stable in the Hilbert space $\mathcal{D}\left(\tilde{A}^{1/2}\right)\times H$. The following proof of the exponential stability is based on the energy multiplier method [12][14].

Theorem 5 *The transformed closed loop system Eq.(20) is exponentially stable in the Hilbert space $\mathcal{D}\left(\tilde{A}^{1/2}\right)\times H$.*

Proof

Let

$$\begin{aligned}E(t) &= \frac{1}{2}\langle y(t),y(t)\rangle \quad (29)\\ &= \frac{1}{2}\left(y_2(t),y_2(t)\right)_H \quad (30)\\ &\quad +\frac{1}{2}\left(\tilde{A}^{1/2}y_1(t),\tilde{A}^{1/2}y_1(t)\right)_H\end{aligned}$$

be an energy function for the transformed closed loop system Eq.(18).

By the equivalenct property of L^p-stability and exponential stability for a strongly continuous semigroup system[15], it sufficies to proved that

$$\int_0^\infty E(t)^2\,dt<\infty \quad (31)$$

Note that $E(t)$ is weakly monotone decreasing as a function of t by Eq.(25).

Let $0<\epsilon<1$ and define

$$V(t)=2(1-\epsilon)\,t\,E(t)+(y_2(t),y_1(t))_H \quad (32)$$

Since

$$\begin{aligned}&|2\left(y_2(t),y_1(t)\right)_H|\\ \leq\ &(y_2(t),y_2(t))_H+(y_1(t),y_1(t))_H \quad (33)\\ \leq\ &(y_2(t),y_2(t))_H \quad (34)\\ &+Con_1\left(\tilde{A}^{1/2}y_1(t),\tilde{A}^{1/2}y_1(t)\right)_H\\ \leq\ &(1+Con_1)E(t) \quad (35)\end{aligned}$$

for some positive constant Con_1, there exists a positive constant Con_2 such that

$$
\begin{aligned}
& \{2(1-\varepsilon)t - Con_2\} E(t) & \\
\leq\; & V(t) & (36) \\
\leq\; & \{2(1-\varepsilon)t + Con_2\} E(t) & (37)
\end{aligned}
$$

Thus, we have

$$E(t) \leq \frac{V(t)}{2(1-\varepsilon)t - Con_2} \qquad (38)$$

Obviously $V(t)$ becomes positive for $t > T_1$ where

$$2(1-\varepsilon)T_1 - Con_2 = 0 \qquad (39)$$

Considering the time derivative of $V(t)$, we obtain

$$
\begin{aligned}
& \frac{dV}{dt} & \\
=\; & 2(1-\varepsilon)E(t) + 2(1-\varepsilon)t\,\frac{dE}{dt} & (40) \\
& + (dy_2/dt, y_1(t))_H + (y_2(t), y_2(t))_H & \\
=\; & (1-\varepsilon)(y_2(t), y_2(t))_H & (41) \\
& +(1-\varepsilon)\left(\tilde{A}^{1/2}y_1(t), \tilde{A}^{1/2}y_1(t)\right)_H & \\
& -2(1-\varepsilon)t\left(\tilde{D}\,y_2(t), y_2(t)\right)_H & \\
& -\left(\tilde{D}\,y_2(t) + \tilde{A}y_1(t), y_1(t)\right)_H & \\
& + (y_2(t), y_2(t))_H & \\
=\; & (2-\varepsilon)(y_2(t), y_2(t))_H & (42) \\
& -\varepsilon\left(\tilde{A}^{1/2}y_1(t), \tilde{A}^{1/2}y_1(t)\right)_H & \\
& -2(1-\varepsilon)t\left(\tilde{D}y_2(t), y_2(t)\right)_H & \\
& -\left(\tilde{D}y_2(t), y_1(t)\right)_H & \\
\leq\; & (2-\varepsilon)Con_3\left(\tilde{D}^{1/2}y_2(t), \tilde{D}^{1/2}y_2(t)\right)_H & (43) \\
& -\varepsilon\left(\tilde{A}^{1/2}y_1(t), \tilde{A}^{1/2}y_1(t)\right)_H & \\
& -2(1-\varepsilon)t\left(\tilde{D}y_2(t), y_2(t)\right)_H & \\
& -\left(\tilde{D}^{1/2}y_2(t), \tilde{D}^{1/2}y_1(t)\right)_H &
\end{aligned}
$$

for some positive constant Con_3.

For an arbitrary constant $a > 0$,

$$
\begin{aligned}
& \left|-2\left(\tilde{D}y_2(t), y_1(t)\right)_H\right| & \\
=\; & \left|-2\left(\tilde{D}^{1/2}y_2(t), \tilde{D}^{1/2}y_1(t)\right)_H\right| & (44) \\
\leq\; & a^2\left(\tilde{D}^{1/2}y_2(t), \tilde{D}^{1/2}y_2(t)\right)_H & (45) \\
& +\frac{1}{a^2}\left(\tilde{D}^{1/2}y_1(t), \tilde{D}^{1/2}y_1(t)\right)_H &
\end{aligned}
$$

Consequently

$$
\begin{aligned}
& \frac{dV}{dt} & \\
\leq\; & (2-\varepsilon)Con_3\left(\tilde{D}^{1/2}y_2(t), \tilde{D}^{1/2}y_2(t)\right)_H & (46) \\
& -\varepsilon\left(\tilde{A}^{1/2}y_1(t), \tilde{A}^{1/2}y_1(t)\right)_H & \\
& -2(1-\varepsilon)t\left(\tilde{D}^{1/2}y_2(t), \tilde{D}^{1/2}y_2(t)\right)_H & \\
& +\frac{a^2}{2}\left(\tilde{D}^{1/2}y_2(t), \tilde{D}^{1/2}y_2(t)\right)_H & \\
& +\frac{1}{2\,a^2}\left(\tilde{D}^{1/2}y_1(t), \tilde{D}^{1/2}y_1(t)\right)_H & \\
=\; & \left\{(2-\varepsilon)Con_3 - 2(1-\varepsilon)t + \frac{a^2}{2}\right\} & (47) \\
& \left(\tilde{D}^{1/2}y_2(t), \tilde{D}^{1/2}y_2(t)\right)_H & \\
& -\varepsilon\left(\tilde{A}^{1/2}y_1(t), \tilde{A}^{1/2}y_1(t)\right)_H & \\
& +\frac{1}{2\,a^2}\left(\tilde{D}^{1/2}y_1(t), \tilde{D}^{1/2}y_1(t)\right)_H &
\end{aligned}
$$

Since

$$
\begin{aligned}
& \frac{1}{2\,a^2}\left(\tilde{D}y_1(t), y_1(t)\right)_H & \\
\leq\; & \frac{\alpha\,Con_4 + \beta}{2\,a^2}\left(\tilde{A}y_1(t), y_1(t)\right)_H & (48)
\end{aligned}
$$

for some positive constant Con_4 and a is arbitrary and can be chosen such that

$$-\varepsilon + \frac{\alpha\,Con_4 + \beta}{2\,a^2} < 0 \qquad (49)$$

we have

$$
\begin{aligned}
& \frac{dV}{dt} & \\
\leq\; & \left\{(2-\varepsilon)Con_3 - 2(1-\varepsilon)t + \frac{a^2}{2}\right\} & (50) \\
& \left(\tilde{D}^{1/2}y_2(t), \tilde{D}^{1/2}y_2(t)\right)_H &
\end{aligned}
$$

Therefore

$$\frac{dV}{dt} \leq 0 \quad \text{for } \forall t > T_2 \qquad (51)$$

where T_2 is defined as follows:

$$(2-\varepsilon)Con_3 - 2(1-\varepsilon)T_2 + \frac{a^2}{2} = 0 \qquad (52)$$

Since $E(t)$ is weakly monotone decreasing as a function of $t > 0$ and $V(t)$ is also weakly monotone decreasing as a function of $t > T_2$, $E(t)$ can be estimated as

$$
\begin{aligned}
E(t) &< \frac{V(T)}{2(1-\varepsilon)t - Con_2} & (53) \\
&< \frac{2(1-\varepsilon)T + Con_2}{2(1-\varepsilon)t - Con_2}E(0) & (54)
\end{aligned}
$$

where $T = max\{T_1, T_2\}$. Thus it follows that

$$\int_T^{\infty} E(t)^2\, dt < \infty \qquad (55)$$

Q.E.D.

6 Conclusions

In this paper, we consider the static output feedback control for general flexible structures. The sufficient conditions (see **CONDITIONS**) of system structures and outputs(observation mechanisms) are clarified for the the static output feedback control to be effective (the closed loop system becomes exponentially stable).

These **CONDITIONS** are satisfied by the wide class of flexible structures and obsevation mechanisms. It is shown that not only strain but also other outputs are effective in the stabilization of flexible structures by the static output feedback control, and this control method is applicable for many types of flexible structures including cantilevers and free ends beams and other structures who has general boundary conditions.

The key idea of the proof of the closed loop exponential stability is the energy multiplier method. Even if the parameters of the structure are unkown, the closed loop system becomes exponentially stable.

References

[1] M.Hattori, S.Tadokoro and T.Takamori, "A note on the mathematical essence of the static output feedback control of flexible structures", Proc. of the 35 th IEEE Conference on Decision and Control, 1996

[2] R.V.Gressang and G.B.Lamont, "Observers for Systems Characterized by Semigroups", IEEE Trans. Automat. Contr. , vol.AC-20, pp.523-528, 1975.

[3] Y.Sakawa and T.Matsushita, "Feedback Stabilization of a Class of Distributed Systems and Construction of a State Estimator" IEEE Trans. Automat. Contr. , vol.AC-20, pp.748-753, 1975

[4] Y.Sakawa,"Observability and Feedback Stabilization of Distributed Systems of Hyperbolic Type", SICE Trans. vol.12 no.3,pp.251-256,1976(in Japanese)

[5] T.Nambu,"Feedback Stabilization for Distributed Parameter Systems of Parabolic Type", Journal of Differential Equations, Vol.33, No.2, pp.167-188, 1979

[6] T.Nambu,"Feedback Stabilization for Distributed Parameter Systems of Parabolic Type, 2", Archive for Rational Mechanics and Analysis, Vol.79, No.3, pp.241-259, 1982

[7] M.J.Balas, "Direct Velocity Feedback Control of Large Space Structures", Journal of Guidance and Control, Vol.2, No.3, pp.252-253, 1979

[8] S.M.Joshi, "Robustness Properties of Collocated Controllers for Flexible Strucrures", Journal of Guidance and Control, Vol.9, No.1, pp.85-9, 1989

[9] T.Kida, I.Yamaguchi, S.Ueno and K.Hiraoka, "DVFB Control of Large Space Structures (Analytical and Experimental Study on Its Stability)" SICE Trans. Vol.25, No.12, pp.1324-1330 (1989)

[10] K.Koujitani, M.Ikeda and T.Kida, "Optimal Control of Large Space Structures by Collocated Feedback", SICE Trans. Vol.25, No.8, pp.882-888, 1989 (in Japanese)

[11] Z.H.Luo, "Direct Strain Feedback Control of Flexible Robot Arms : New Theoretical and Experimental Results", IEEE Trans. Automat. Contr. vol.38, pp.1610-1622, 1993

[12] Z.H.Luo and B.Guo, "Further Theoretical Results on Direct Strain Feedback Control of Flexible Robot Arms", IEEE Trans. on Automatic Control, Vol.40, No.4, pp.747-751 (1995)

[13] Z.H.Luo, N.Kitamura and B.Guo, "Shear Force Feedback Control of Flexible Robot Arms", (Submitted to IEEE Trans. Automatic Control)

[14] G.Chen, M.C.Delfour, A.M.Krall and G.Payre, "Modeling, stabilization and control of serially connected beams", SIAM J. Contr. Optim. Vol.25, pp.526-546 (1987)

[15] A.Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations", New York: Springer-Verlag (1983)

$L^p - L^q$ asymptotic behaviours of solutions to the perturbed Schrödinger equations

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1 Introduction

In this paper, we consider the asymptotic behaviour of solutions to the Schrödinger equation :

$$\begin{aligned} iu_t &= Hu \\ u|_{t=0} &= \psi \end{aligned}$$

where $H = H_0 + V$ is the Hamiltonian. H_0 is an n dimensional Laplacian i.e. $H_0 = -(\partial/\partial x_1)^2 - \ldots - (\partial/\partial x_n)^2$. V is a multiplication operator associated with a measurable function satisfying the following conditions.

Conditions of V
V is a real valued measurable function on $\boldsymbol{R}^n$, $n \geq 3$. For some $\sigma > 2(n-2)/(n-1)$, $\mathcal{F}(< \cdot >^\sigma V) \in L^{\frac{n-1}{n-2}}(\boldsymbol{R}^n)$ and

$$\|\mathcal{F}(< \cdot >^\sigma V)\|_{L^{\frac{n-1}{n-2}}} << 1. \tag{1}$$

In the above condition, $\mathcal{F}$ is the Fourier transform, $< \cdot >$ is maltiplication operator associated with $< x >= (1 + |x|^2)^{1/2}$, $L^p(\boldsymbol{R}^n)$ is the functional space which consists of measurable functions on $\boldsymbol{R}^n$ whose p th power is integrable,

These conditions assure the selfadjointness of H with the domain $\mathcal{D}(H) = W^{2,2}(\boldsymbol{R}^n)$, the Sobolev space (cf.Agmon [1]). Thus there exists a solution operator e^{-itH} of the initial value problem. We consider the wave operator $W_\pm$ defined as follows.

$$W_+ = s - \lim_{t \to +\infty} e^{itH} e^{-itH_0} \mathrm{in} L^2(\boldsymbol{R}^n), \tag{2}$$

$$W_- = s - \lim_{t \to -\infty} e^{itH} e^{-itH_0} \mathrm{in} L^2(\boldsymbol{R}^n). \tag{3}$$

$W_\pm$ have important properties (cf. Agmon [1] or Kuroda [7]):

1. $Range W_+ = Range W_- = L^2_{ac}(H)$ (completeness),

2. $W_\pm$ are partial isometries,

3. $W_{\pm}e^{-itH_0} = e^{-itH}W_{\pm}$ (intertwining property).

Note that $L^2_{ac}(H)$ is the absolutely continuous subspace for H.

Our main theorem is

Theorem 1 *let* $1 < p \leq 2,\ 2 \leq q < \infty, 1/p+1/q = 1$. *Then for any* $\phi \in L^2(\boldsymbol{R}^n)\cap L^p(\boldsymbol{R}^n)$,

$$\lim_{t\to\pm\infty} |t|^{n(1/p-1/2)}\|e^{-itH}W^{\pm}\phi - e^{-itH_0}\phi\|_{L^q} = 0.$$

It is easy to obtain the next corollary from Theorem 1.

Corollary 2 *Let* p, q *as in Theorem 1. Then for any* $\psi \in L^2_{ac}(H) \cap L^p(\boldsymbol{R}^n)$,

$$\lim_{t\to\pm\infty} |t|^{n(1/p-1/2)}\|e^{-itH}\psi - e^{-itH_0}W^*_{\pm}\psi\|_{L^q} = 0.$$

Yajima [12] showed that $W_{\pm}$ are extended as operators in $\mathcal{B}(L^p), (1 \leq p \leq \infty)$. One can see the outline of the proof in next section. Thus, by using the intertwining property and $L^p - L^q$ estimate of e^{-itH_0}, we have $e^{-itH}W_{\pm}\phi, e^{-itH}W^*_{\pm}\psi \in L^q$.

The $L^p - L^q$ estimate of e^{-itH} and its time decay was investigated by sevral autors [2] [3] [4]. One expects that $\|e^{-itH}P_{ac}\phi\|_\infty = O(|t|^{-n/2})$ as $t \to \pm\infty$, where P_{ac} is the projection onto $L^2_{ac}(H)$. However, It is known that $\|e^{-itH}P_{ac}\phi\|_\infty$ decays at worse order as $t \to \pm\infty$ when 0 is an eigenvalue or resonance for H, where 0 is called a resonance for H when the equation $-\Delta u + Vu = 0$ has a nontrivial distributional solution such that $< x >^{-\gamma} u$ belongs to $L^2(\boldsymbol{R}^n)$ for any $\gamma > 0$.

It is interesting that, in Corollary 2, $\|e^{-itH}\psi - e^{-itH_0}W^*_{\pm}\psi\|$ decays at smaller order than $-n(1/p-1/2)$. It seems that $e^{-itH_0}W^*_{\pm}\psi$ approximates $e^{-itH}\psi$ so nicely at $t = \pm\infty$.

2 L^p boundedness of the wave operators $W_{\pm}$

In this section, we see the survey of the proof for key lemma, which was given by Yajima [12]. The statement is

Lemma 1 (Yajima) *Let* $1 \leq p \leq \infty, n \geq 3$. *Then* $W_{\pm}$ *can be extended as operators on* $\mathcal{B}(L^p)$. *There exist positive constants* C_1, C_2 *such that*

$$\|W_{\pm}\|_{\mathcal{B}(L^p)} \leq \frac{C_1}{1 - C_2\|\mathcal{F}(< \cdot >^{\sigma} V)\|_{L^{\frac{n-1}{n-2}}}}. \tag{4}$$

Remark In his paper [12], the $W^{k,p}$ boundedness of $W_{\pm}$ are given for some large potential V. In this case, we have to impose some spectral conditions on H, i.e. *0 is neither an eigenvalue nor resonance for* H. Since the number of pages is limited, we only show the proof for the small potential.

proof of Lemma 1 (outline)
Let $R_0(z) = (H_0 - zId)^{-1}$, $R(z) = (H - zId)^{-1}$. We decompose V as $V = A \times B^*$,

$(A=|V|^{1/2}, B=|V|^{1/2}sgn(V))$. Then, for $A, B \in L^n(\boldsymbol{R}^n), f \in L^2(\boldsymbol{R}^n)$, we have $AR_0(\lambda \pm i\epsilon)f, AR(\lambda \pm i\epsilon)f \in L^2(R^+, L^2(R^n), d\lambda)$. And as $\epsilon \to +0$, they converge to $AR_0(\lambda \pm i0)f, AR(\lambda \pm i0)f$ in $L^2(R^+, L^2(R^n), d\lambda)$, respectively. Moreover,

$$\begin{aligned}
\sup_{\epsilon>0}\int_0^\infty \|AR_0(\lambda \pm i\epsilon)f\|_{L^2}^2 d\lambda &= \int_0^\infty \|AR_0(\lambda \pm i0)f\|_{L^2}^2 d\lambda && (5)\\
&\le C\|f\|_{L^2}^2\|A\|_{L^n}^2, && (6)\\
\sup_{\epsilon>0}\int_0^\infty \|AR(\lambda \pm i\epsilon)f\|_{L^2}^2 d\lambda &= \int_0^\infty \|AR(\lambda \pm i0)f\|_{L^2}^2 d\lambda && (7)\\
&\le C\|f\|_{L^2}^2\|A\|_{L^n}^2. && (8)\\
\|AR_0(z)B^*\|_{B(L^2)} &\le C\|A\|_{L^n}\|B\|_{L^n}, \text{for} z \in \boldsymbol{C}\backslash[0,\infty). && (9)
\end{aligned}$$

Note that, in (9), C is independent of $z \in \boldsymbol{C}\backslash[0,\infty)$ The same holds for $BR_0(\lambda \pm i\epsilon)f, BR(\lambda \pm i\epsilon)f$. These results are often seen in the analytic perturbation theorem (cf. Reed - Simon [11], Kato [5], Kato - Yajima [6], Kuroda [8]).
We have the stationary representation formula of $W_\pm$, i.e. for $f, g \in L^2(\boldsymbol{R}^n)$,

$$(W_\pm f, g) = (f,g) - \frac{1}{2\pi i}\int_0^\infty (A\{R_0(\lambda \pm i0) - R_0(\lambda \mp i0)\}f, BR(\lambda \pm i0)g)d\lambda. \qquad (10)$$

This result is due to Duhamel's principle for e^{-itH} and Stone's formula in the spectral theory.

From now on, we consider only W_+ since the proof is similar for W_-. By the repeated use of the resolvent equation, we obtain

$$R(z) = \sum_{m=0}^{N-1}(-1)^m\{R_0(z)V\}^m R_0(z) + (-1)^N\{R_0(z)V\}^N R(z). \qquad (11)$$

We substitute this expansion to (10) and define the operator W_m as follows.

$$\begin{aligned}
&(W_m f, g)\\
&= \tfrac{(-1)^m}{2\pi i}\textstyle\int_0^\infty (A\{R_0(\lambda + i0) - R_0(\lambda - i0)\}f, B\{R_0(\lambda + i0)V\}^m R_0(\lambda + i0)g)d\lambda \quad (12)
\end{aligned}$$

If $\|V\|_{L^{n/2}} << 1$, then we obtain $W_+ = Id + \sum_{m=0}^\infty W_m$ in $\mathcal{B}(L^2)$, which follows from (5), (6), (7), (8), (9).

Formally, we have

$$W_m f = \frac{(-1)^m}{2\pi i}\int_0^\infty \{R_0(\lambda - i0)V\}^{m+1}\{R_0(\lambda + i0) - R_0(\lambda - i0)\}f d\lambda.$$

W_m is more easily handled than W_+, since it is directly calculated by using Fourier and inverse Fourier transform.

Lemma 2 *There exist constants C_1, C_2 such that for any $1 \le p \le \infty$, $f \in L^2 \cap L^p$,*

$$\|W_m f\|_{L^p} \le C_1 (C_2 \|\mathcal{F}(< \cdot >^\sigma V)\|_{L^{\frac{n-1}{n-2}}})^{m+1} \|f\|_{L^p} \tag{13}$$

Lemma 2 follows from the integal operator representation :

$$W_m f(x) = \int_{[0,\infty)^m \times I \times \Sigma^{m+1}} \hat{K}_m(t_1, \ldots t_m, \tau, \omega_1, \ldots, \omega_{m+1}) f(\bar{x} + \rho) dt_1 \ldots dt_m d\tau d\omega_1 \ldots d\omega_{m+1} \tag{14}$$

where Σ is the surface of the unit sphere in $\boldsymbol{R}^n$, $\bar{y} = y - 2(y \cdot \omega_{m+1})$ for any $y \in \boldsymbol{R}^n$, $\rho = t_1 \bar{\omega_1} + \ldots + t_m \bar{\omega_m} - \tau \omega_{m+1}$, $I = (-\infty, 2\omega_{m+1} \cdot (x + t_1 \omega_1 + \ldots + t_m \omega_m))$ is the range of integral variable τ, $K_m(k_1, \ldots, k_{m+1}) = i^{m+1} (2\pi)^{-(m+1)n/2} \Pi_{j=1}^{m+1} \mathcal{F}V(k_j - k_{j-1}), k_0 = 0$ and

$$\hat{K}_m = \int_{[0,\infty)^{m+1}} e^{-i \sum_{j=1}^{m+1} t_j s_j /2} (s_1 \ldots s_{m+1})^{m_1} K(s_1 \omega_1, \ldots, s_{m+1} \omega_{m+1}) ds_1 \ldots ds_{m+1}. \tag{15}$$

Thus, in order to prove Lemma 2, we have to show that

Lemma 3 *Let $\sigma > 2(n-2)/(n-1)$. Then there exists a constant C_2 such that*

$$\|\hat{K}_m\|_{L^1([0,\infty)^{m+1} \times \Sigma^{m+1})} \le (C_2 \|\mathcal{F}(< \cdot >^\sigma V)\|_{L^{\frac{n-1}{n-2}}})^{m+1}. \tag{16}$$

Lemma 3 can be proved as follows.

$$\begin{aligned} \|\hat{K}_m\|_{L^1([0,\infty)^{m+1} \times \Sigma^{m+1})} &\le C^{m+1} \|(\Pi_{j=1}^{m+1} < t_j >^{\sigma/2}) \hat{K}_m\|_{L^1(\Sigma^{m+1}, L^{n-1}([0,\infty)^{m+1}))}, \\ &\le C^{m+1} \|\hat{V}\|_{H^\sigma_{\frac{n-1}{n-2}}}^{m+1}, \\ &= C^{m+1} \|\mathcal{F}(< \cdot >^\sigma V)\|_{L^{\frac{n-1}{n-2}}}^{m+1}. \end{aligned}$$

Hence we completes the proof of Lemma 1. □

By the standard density argument, we obtein Theorem 1.

3 Some conjectures

In Theorem 1, we do not state any results for the case $p = 1, q = \infty$. Of course, it is easy to show that the defference $e^{-itH} W_\pm \phi - e^{-itH_0} \phi$ decays at the same order as $|t|^{-n/2}$, since each term decays at the order. However, does the defference decay at smaller order? I think it will be proved negatively if $\hat{\phi}(0) \ne 0$, i.e. the case where the state ϕ posseses 0 in momentum component.

References

[1] Agmon, S., *Spectral properties of Schrödinger operators and scattering theory*, Ann. Scuola Norm. Sup. Pisa, Ser. IV, **2** (1975), 151 - 218.

[2] Jensen, A. and T. Kato, *Spectral properties of Schrödinger operators and time - decay of the wave functions*, Duke Math. J. **46** (1979), 583 - 611.

[3] Jensen, A., *Spectral properties of Schrödinger operators and time - decay of the wave functions. Results in $L^2(R^m)$, $m \geq 5$*, Duke Math. J. **47** (1980), 57 - 80.

[4] Journe, J. L., A. Soffer and C. D. Sogge, *Decay estimates for Schrödinger operators*, Comm. Pure. Apple. Math. **44** (1991), 573 - 604.

[5] Kato, T., *Wave operators and similarity for some non - selfadjoint operators*, Math. Ann. **162** (1966), 258 - 279.

[6] Kato, T. and K. Yajima, *Some examples of smooth operators and the associated smoothing effect*, Reviews in Math. Phys. **1** (1989), 481 - 496.

[7] Kuroda, S. T., *Scattering theory for differential operators, I and II*, J. Math. Soc. Japan **25** (1972), 75 - 104 and 222 - 234.

[8] Kuroda, S. T., *An itroduction to scattering theory*, Lecture Notes Series **51** Aarhus University (1978), Aarhus, Denmark.

[9] Reed, M. and B. Simon, *Methods of Modern Mathematical Pysics I, Functional Analysis*, Academic Press, New York - San Francisco - London (1975).

[10] Reed, M. and B. Simon, *Methods of Modern Mathematical Pysics II, Fourier Analysis, Self-adjointness*, Academic Press, New York - San Francisco - London (1975).

[11] Reed, M. and B. Simon, *Methods of Modern Mathematical Pysics III, Scattering Theory*, Academic Press, New York - San Francisco- London (1979).

[12] Yajima, K., *The $W^{k,p}$ - continuity of wave operators for Schrödinger operators*, Proc. Japan. Acad. **69** (1993), 94 - 98.

Traveling Wave Solutions for Some Quasilinear Diffusion Equations

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1 Introduction

We are concerned with the following quasilinear diffusion equation which appears in mathematical biology:

$$u_t = \varphi(u)_{xx} + f(u), \qquad x \in \mathbf{R}, \quad t > 0, \tag{1.1}$$

where φ satisfies

$$\text{(A1)} \qquad \begin{cases} \varphi \in C^1[0,1] \cap C^2(0,1], \quad \varphi(0) = 0, \\ \varphi'(u) > 0 \quad \text{for} \quad u \in (0,1] \end{cases}$$

and f satisfies

$$\text{(A2)} \qquad \begin{cases} f \in C^2[0,1], \quad f(0) = f(\alpha) = f(1) = 0, \\ f'(0) < 0, \quad f'(1) < 0, \\ f(u) < 0 \quad \text{for} \quad u \in (0,\alpha), \\ f(u) > 0 \quad \text{for} \quad u \in (\alpha, 1) \end{cases}$$

with some $\alpha \in (0,1)$.

We are interested in traveling wave solutions for (1.1); that is a solution of the form $u(x,t) = q(x - ct)$.

Definition. *A function $u(x,t) = q(z)$ with $z = x - ct$ is called a traveling wave solution for (1.1) if $q(z)$ satisfies*

$$\{\varphi(q)\}'' + cq' + f(q) = 0 \quad \text{for} \quad z \in (-\infty, z^*) \tag{1.2}$$

and

$$\begin{cases} \lim_{z \to -\infty} q(z) = 1, \\ q'(z) < 0 \quad \textit{for} \quad z \in (-\infty, z^*), \\ \lim_{z \to z^*} q(z) = 0 \quad \textit{and} \quad \lim_{z \to z^*} \varphi(q(z))' = 0, \\ q(z) \equiv 0 \quad \textit{for} \quad z \in (z^*, +\infty) \quad \textit{if} \quad z^* < +\infty. \end{cases}$$

Here c denotes the velocity of the traveling wave.

2 Results

The sign of velocity c of traveling wave solutions q is given by following method. Multiply (1.2) by $\varphi(q)' = \varphi'(q)q'$ and integrate the resulting expression over $-\infty$ to z^*; then we have

$$c = \left(\int_0^1 \varphi'(u) f(u) du \right) \bigg/ \left(\int_{-\infty}^{z^*} \varphi'(q(z)) \{q'(z)\}^2 dz \right).$$

If we set $s(\varphi, f) = \int_0^1 \varphi'(u)f(u)du$, then we see that

$$s(\varphi, f) \gtreqless 0 \Longleftrightarrow c \gtreqless 0. \tag{2.1}$$

Our main results are divided into two cases; (a) $\varphi'(0) \neq 0$ (non-degenerate) and (b) $\varphi'(0) = 0$ (degenerate).

Theorem 1 (non-degenerate case). *Let $\varphi'(0) > 0$. If φ and f satisfy (A1) and (A2) respectively, then there exist a unique number c and a unique (except for translation) traveling wave solution $q(z)$, $z = x - ct$, of (1.1) with $z^* = +\infty$. Moreover, q has the following properties:*
(i) $C_1 e^{\lambda_1 z} \leq 1 - q(z)$, $|q'(z)|$, $|q''(z)| \leq C_2 e^{\lambda_1 z}$ *as* $z \to -\infty$,
(ii) $C_3 e^{\lambda_2 z} \leq q(z)$, $|q'(z)|$, $|q''(z)| \leq C_4 e^{\lambda_2 z}$ *as* $z \to +\infty$,
where

$$\lambda_1 = \frac{-c + \sqrt{c^2 - 4\varphi'(1)f'(1)}}{2\varphi'(1)} > 0$$

and

$$\lambda_2 = \frac{-c - \sqrt{c^2 - 4\varphi'(0)f'(0)}}{2\varphi'(0)} < 0,$$

and C_1, C_2, C_3 and C_4 are some positive numbers.

Theorem 2 (degenerate case). *In addition to (A1) and (A2), assume*

$$0 < \liminf_{u \to 0} \frac{\varphi(u)}{u^{1+k}} \leq \limsup_{u \to 0} \frac{\varphi(u)}{u^{1+k}} < +\infty. \tag{2.2}$$

Then there exist a unique number c and a unique (except for translation) traveling wave solution $q(z)$, $z = x - ct$, of (1.1). Moreover, q has the following properties:
(i) *there exist some positive numbers C_1 and C_2,*

$$C_1 e^{\lambda z} \leq 1 - q(z), \quad |q'(z)|, \quad |q''(z)| \leq C_2 e^{\lambda z} \qquad \text{as} \quad z \to -\infty,$$

where

$$\lambda = \frac{-c + \sqrt{c^2 - 4\varphi'(1)f'(1)}}{2\varphi'(1)} > 0,$$

(ii) *if $s(\varphi, f) > 0$, then $z^* < +\infty$ and*

$$D_1(z^* - z)^{1/k} \leq q(z) \leq D_2(z^* - z)^{1/k} \quad \text{for} \quad z^* - \delta \leq z \leq z^*$$

with some positive constants D_1, D_2 and δ,
(iii) *if $s(\varphi, f) = 0$ then $z^* < +\infty$ and*

$$K_1(z^* - z)^{2/k} \leq q(z) \leq K_2(z^* - z)^{2/k} \quad \text{for} \quad z^* - \delta \leq z \leq z^*$$

with some positive constants K_1, K_2 and δ,
(iv) *if $s(\varphi, f) < 0$, then $z^* = +\infty$ and*

$$q(z) \geq C e^{-\mu z} \quad \text{as} \quad z \to +\infty,$$

with some positive constants C and μ.

Remark. For semilinear parabolic equation, the corresponding result to Theorem 1 has been established by Aronson-Weinberger [1] (Theorem 4.2) and Fife-McLeod [2] (Theorem 2.4). In non-degenerate case we get the same result as linear diffusion. For the special case $\varphi(u) = u^m$, the similar result to Theorem 2 has been shown by Hosono[3] (Theorem 1).

3 Sketch of Outline of Proof

Our strategy of the proof is to transform (1.2) into the system of the first order ordinary differential equations and study the behavior of trajectories by the standard phase plane analysis.

If we define a new unknown function $p = \varphi'(q)q'$, then (1.2) is written as

$$\begin{cases} \varphi'(q)q' &= p, \\ \varphi'(q)p' &= -cp - \varphi'(q)f(q). \end{cases} \tag{3.1}$$

It is convenient to introduce a new variable τ by

$$\frac{dz}{d\tau} = \varphi'(q(z)) \tag{3.2}$$

and rewrite (3.1) as

$$\begin{cases} \dot{q} &= p, \\ \dot{p} &= -cp - \varphi'(q)f(q), \end{cases} \tag{3.3}$$

where "$\cdot$" denotes $d/d\tau$. In q-p plane, (3.3) has three equilibrium points

$$P_0 = (0,0), \quad P_1 = (1,0), \quad P_2 = (\alpha, 0),$$

so that we have only to show the existence and uniqueness of a trajectory which connects P_1 with P_0 and lies in the region $D = \{(q,p); 0 \le q \le 1, \quad p \le 0\}$.

Let S_c be a trajectory starting from P_1 and lying in D. If $(q,p) \in S_c$, then p can be represented as a function of q and it satisfies

$$\frac{dp}{dq} = -c - \frac{\varphi'(q)f(q)}{p}. \tag{3.4}$$

Since S_c connects with P_1, we have to add the boundary condition

$$p|_{q=1} = 0$$

to (3.4).

We will show two basic properties of S_c.

Lemma 1. *For $i = 1, 2$, let $p = p_i(q)$ be two trajectories of (3.3) corresponding to c_i and let p_i satisfy $p_i(1) = 0$. If $p_i(q) < 0 (i = 1, 2)$ for $q_1 \le q < 1$, then*

$$c_1 \gtreqless c_2 \Longleftrightarrow p_1(q) \gtreqless p_2(q) \quad \text{for} \quad q \in [q_1, 1).$$

Lemma 2. *Let $s(\varphi, f) \ne 0$. Then*
(i) *there exists $\underline{c}$, depending on φ and f, such that S_c crosses the negative part of p-axis for every $c < \underline{c}$,*
(ii) *there exists $\bar{c}$, depending on φ and f, such that S_c intersects the segment $\{(q,p); 0 < q \le \alpha, p = 0\}$ for every $c > \bar{c}$.*

¿From Lemmas 1 and 2 we get following lemma for the existence and uniqueness of the trajectory S_c.

Lemma 3. *Let* $c^* = \sup\{c \in [\underline{c}, \infty); \text{there exists } \mu > 0 \text{ such that } (0, -\mu) \in S_c\}$. *Then* S_{c^*} *is a unique trajectory of (3.3) which connects* P_1 *with* P_0.

These lemmas are shown essentially by the same idea as Fife-McLeod[2] and Hosono[3].

¿From Lemma 3 the existence and uniqueness of the traveling wave solution is shown, then we will give the aymptotic properties. In what follows, we write c instead of c^*.

Proof of (i) of Theorems 1 and 2. Let $S_c = \{(q(\tau), p(\tau)); \tau \in \mathbf{R}\}$ and note that $(q(\tau), p(\tau))$ approaches P_1 as $\tau \to -\infty$. The linearization of (3.3) at P_1 implies that S_c has a slope

$$\lambda_1 := \frac{-c + \sqrt{c^2 - 4\varphi'(1)f'(1)}}{2}.$$

¿From this fact we get

$$A_1 e^{\lambda_1 \tau} \le 1 - q(\tau) \le A_2 e^{\lambda_1 \tau} \quad \text{for} \quad \tau < \tau_0, \tag{3.5}$$

where A_1, $A_2 > 0$ are some constants and $\tau_0 < 0$ is a sufficiently small number. Applying (3.5) to (3.2) we have

$$\frac{dz}{d\tau} = \varphi'(q) = \varphi'(1) + a(\tau)$$

with $|a(\tau)| \le Ce^{\lambda_1 \tau}$ as $\tau \to -\infty$. Thus

$$|z(\tau) - (z_0 + \varphi'(1)\tau)| \to 0 \quad \text{as} \quad \tau \to -\infty \tag{3.6}$$

with some z_0. From (3.5) and (3.6) it follows that

$$B_1 e^{\mu_1 z} \le 1 - q(z) \le B_2 e^{\mu_1 z} \quad \text{as} \quad z \to -\infty, \tag{3.7}$$

where B_1, $B_2 > 0$ and $\mu_1 = \lambda_1/\varphi'(1)$. Moreover, we can show an asymptotic property for q' and q''.

Proof of (ii) of Theorem 1. When $\varphi'(0) \neq 0$, we have that P_0 is saddle point by the linearization (3.3) at P_0. Therefore, (ii) of Theorem 1 is shown by the same way as $z \to -\infty$.

Proof of (ii), (iii) and (iv) of Theorem 2. Since $\varphi'(0) = 0$, the linearized matrix at P_0 has two eigenvalues 0 and $-c$. The proof is devided into three cases.

(a) Case $s(\varphi, f) > 0$. For $(q, p) \in S_c$ we have from (3.4)

$$\frac{dp}{dq} = -c - \frac{\varphi'(q)f(q)}{p} \le -c < 0$$

near P_0. This inequality shows $p \le -cq$ for small $q > 0$; so that it follows from (3.3) that

$$q(\tau) \le Ce^{-c\tau} \quad \text{for} \quad \tau > \tau_2 \tag{3.8}$$

with some constant C and sufficiently large τ_2. If we set

$$\tilde{p}(q) = -\frac{\varphi'(q)f(q)}{p(q)},$$

it follows from (A1) and (A2) that

$$|\tilde{p}(q)| \leq \frac{\varphi(q)|f(q)|}{cq} \leq \tilde{C}q^k \quad \text{near} \quad q = 0$$

with some $\tilde{C} > 0$. By (3.4)

$$p(q) = \int_0^q (-c + \tilde{p}(s))ds = -cq + \tilde{Q}(q),$$

where $|\tilde{Q}(q)| \leq \tilde{C}q^{k+1}/(k+1)$ for small $q > 0$. Then we see that

$$\dot{q}(\tau) = p(\tau) = -cq(\tau) + \tilde{Q}(q(\tau)) = q(\tau)(-c + \tilde{Q}_1(q(\tau))),$$

with $|\tilde{Q}_1(q)| \leq \tilde{C}q^k/(k+1)$. Since $\int_{\tau_2}^{\tau} \tilde{Q}_1(s)ds$ is bounded (see (3.8)), it follows that

$$B_1 e^{-c\tau} \leq q(\tau) \leq B_2 e^{-c\tau}, \quad \tau > \tau_2 \tag{3.9}$$

with some B_1, $B_2 > 0$. Application of (3.9) to (3.2) implies

$$C_1 e^{-kc\tau} \leq \frac{dz}{d\tau} = \varphi'(q) \leq C_2 e^{-kc\tau} \quad \text{as} \quad \tau \to +\infty, \tag{3.10}$$

with some C_1, $C_2 > 0$. Then we see that there exists some $z^* < +\infty$ such that

$$\frac{C_1}{kc} e^{-kc\tau} \leq z^* - z(\tau) \leq \frac{C_2}{kc} e^{-kc\tau},$$

which together with (3.9) yields

$$D_1(z^* - z)^{1/k} \leq q(z) \leq D_2(z^* - z)^{1/k} \quad \text{as} \quad z \to z^*$$

with some D_1, $D_2 > 0$.

(b) Case $s(\varphi, f) < 0$. In this case S_c connects P_0 tangentially to the q-axis and stays above the curve $p = -\frac{1}{c}\varphi'(q)f(q)$ in a neighborhood of P_0. ¿From (A1) and (A2) there exists a positive number C_0 such that

$$-\frac{1}{c}\varphi'(q)f(q) \geq -C_0 q^{1+k}, \quad \text{as} \quad q \downarrow 0.$$

Then it is found that

$$0 \geq \dot{q}(\tau) = p(\tau) \geq -\frac{1}{c}\varphi'(q(\tau))f(q(\tau)) \geq -C_0 q(\tau)^{1+k} \quad \text{for} \quad \tau > \tau_0,$$

where τ_0 is a sufficiently large number. Solving this inequality yields

$$q^k(\tau) \geq \frac{1}{C_0 k\tau + C_1}, \tag{3.11}$$

where C_1 is a constant. Applying (3.11) to (3.2) we get

$$\frac{dz}{d\tau} = \varphi'(q) \geq \frac{C_2}{C_0 k\tau + C_1},$$

with some C_2. It is shown that

$$z \geq z_0 + C_2 \log \frac{C_0 k\tau + C_1}{C_0 k\tau_0 + C_1} \tag{3.12}$$

with some z_0. Then we have $z \to +\infty$ as $\tau \to +\infty$. Moreover, it is found from (3.11) and (3.12) that

$$q(z) \geq Ae^{-z/(C_2 k)},$$

with some constant A.

(c) Case $s(\varphi, f) = 0$. Since vanishes, it follows from (3.4) that

$$\frac{1}{2}\{p(q)\}^2 + F(q) = 0,$$

where $F(q) = \int_0^q \varphi'(s)f(s)ds$. Because $p(q) \leq 0$ for $q \in (0,1)$, we see

$$p = -\sqrt{-2F(q)}. \tag{3.13}$$

This is the required trajectory. Apply $p = \varphi'(q)q'$ to (3.13) and regard z as a function of q; then we find

$$\frac{dz}{dq} = -\frac{\varphi'(q)}{\sqrt{-2F(q)}}.$$

¿From (2.2) we see that

$$-C_1 q^{k/2-1} \leq \frac{dz}{dq} = -\frac{\varphi'(q)}{\sqrt{-2F(q)}} \leq -C_2 q^{k/2-1} \quad \text{near} \quad q = 0$$

with some C_1, $C_2 > 0$. Then there exists some $z^* < +\infty$ such that

$$\frac{2}{k} C_2 q^{k/2} \leq z^* - z \leq \frac{2}{k} C_1 q^{k/2}.$$

Therefore, we get

$$K_1 (z^* - z)^{2/k} \leq q(z) \leq K_2 (z^* - z)^{2/k} \quad \text{as} \quad z \to z^*$$

with some K_1, $K_2 > 0$.

4 Numerical simulation

We will give some pictures(Fig.1-3). Numerical simulation is carried out for $\varphi(u) = u^2 + u^3$ and $f(u) = u(u-\alpha)(1-u)$. Each figure exhibit the phase plane for (3.3) and the time-evolution of the solution for (1.1) with initial value

$$u(x,0) = \begin{cases} 1 & x < -1, \\ 0.5(1-x) & -1 \leq x < 1, \\ 0 & x \geq 1. \end{cases}$$

References

[1] D.G.Aronson and H.F.Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve propagation* in Partial Differential Equations and Related Topics, ed. J.A.Goldstein. Lecture Notes in Mathematics **446**, 5–49 New York: Springer 1975.

[2] P.C.Fife and J.B.McLeod, *The approach of solutions of nonlinear equations to traveling front solutions*, Arch. Rational Mech. Anal., **65**, 335–361, 1977.

[3] Y.Hosono, *Traveling wave solutions for some density dependent diffusion equations*, Japan J. Appl. Math., **3**, 163–196, 1986.

[4] F.Sánchez-Garduño and P.K.Maini, *Traveling wave phenomena in some degenerate reaction-diffusion equations*, J. Differential Equations, **117**, 281–319, 1995.

[5] P.Hartman *Ordinary Differential Equations*, Wiley, New York, 1973.

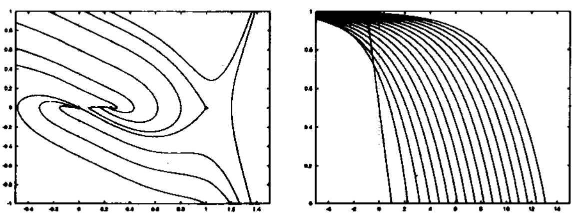

Fig.1
$\alpha = 1/4$, $c = 0.634902$

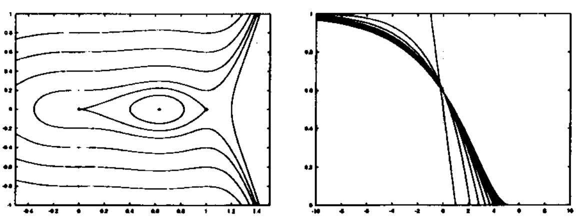

Fig.2
$\alpha = 12/19$, $c = 0$

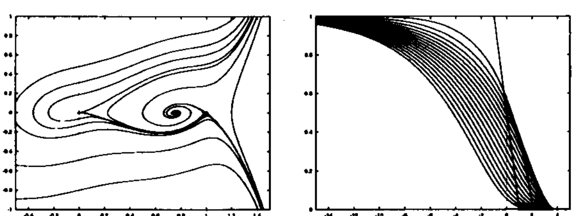

Fig.3
$\alpha = 3/4$, $c = -0.307201$

Eigenvalue problem for some quasilinear elliptic equations in unbounded domains.

橋本 貴宏 (愛媛大・理)

1 Introduction

本稿は早稲田大学の大谷光春教授との共同研究によるものである. Ω を滑らかな境界 $\partial\Omega$ を持つ $\mathbf{R}^N$ の領域, $1 < p, q < \infty$ とし, 次の Emden-Fowler 型方程式について考える.

$$(\mathrm{P}) \begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) &= |u|^{q-2}u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

(P) の正値解に関しては, [5], [8], [3] の結果より, $\Omega = \mathbf{R}^N$, $\Omega = \Omega_0$ star-shaped bounded, そして $\Omega = \mathbf{R}^N \setminus \overline{\Omega_0}$ の三つの問題には, 次の表で示すような相補的な関係がある

領域 \ 指数	$1 < q < p^*$	$q = p^*$	$q > p^*$
内部	正値解あり	正値解なし	非自明解なし
$\mathbf{R}^N$	非自明解なし	正値解あり	非自明解なし
外部	非自明解なし	正値解なし	

ここに p^* は Sobolev 指数 : $p^* = Np/(N-p)$ (for $p < N$); $p^* = \infty$ (for $p \geq N$) である. また u が非自明解であるとは $u \in \{u \in L^q(\Omega); |\nabla u| \in L^p(\Omega), u|_{\partial\Omega} = 0\}$ かつ超関数の意味で (P) を満たすことを言う.

それでは, それ以外の非有界領域の場合はどうなるのであろうか. Ω が柱状領域, すなわち $\Omega := \Omega_d \times \mathbf{R}^{N-d}$ with Ω_d bounded. の形の領域の時には, [7] と [5] の結果により次のようになる.

(1) もし, $p < q < p^*$ ならば (P) は正値解をもつ.

(2) Ω_d が star-shaped であるとする.

(a) もし, $q = p^*$ ならば (P) は正値解を持たない.

(b) もし, $q > p^*$ ならば (P) は非自明解を持たない.

つまり, 内部問題や外部問題では, p^* が critical exponent であったが, 柱状領域の場合では, $q = p$ も critical exponent であることが予想される. Ω が有界領域の場合

は p-Laplacian に対しても第一固有関数の存在や, 第一固有値の単純性に関する結果があるが, Ω が非有界領域の場合は, $\Omega = \mathbf{R}^N$ ([2]) のみ研究されている. 本稿では, $q = p$ で, さらに右辺に重みをつけた,

$$(\mathrm{E})\begin{cases} -\Delta_p u = \lambda a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

ここに, $\lambda > 0$, $a(x) \geq 0$; 有界関数, Ω: 柱状領域, の場合の正値解について考える. 更に, $q < p$ の場合や柱状領域以外の領域への応用についても触れる.

2 Main results

今回得られた結果を以下に述べる.

Theorem 1 $\lambda_1 = \inf_{v \in C_0^\infty(\Omega)} R(v)$; $R(v) = \dfrac{\|\nabla v\|_p^p}{\|a^{1/p} v\|_p^p}$ とおく.

(1) $a \in L^\alpha(\Omega) \cap L^\infty(\Omega)$ *with* $\alpha > \max(\frac{N}{p}, 1)$ とする. この時, *(E)* は $\lambda = \lambda_1$ のとき, そしてそのときのみ正値解 $u \in W_0^{1,p}(\Omega)$ をもつ. さらに, λ_1 は *simple.*

(2) $x_1 \in \mathbf{R}^{N-d} \setminus \{0\}$ が存在して, $a(x + x_1) \leq a(x)$ *for a.e.* $x \in \Omega$. ならば, (E) は任意の $\lambda > 0$ に対して正値解を持たない.

3 Existence of the first eigenfunction

まず最初に, 第一固有関数の存在を証明する. 非線形抽象的固有値問題に対しては, 以下の結果が成り立つ (see Berger [1, p. 335]).

Theorem 2 X を回帰的 *Banach* 空間とし, A, B を X 上の C^1-級汎関数とする. このとき, 以下の *3* つを仮定する.

(1) A は X 上, 弱下半連続 かつ *coercive.*

(2) $u_n \rightharpoonup u$ ならば $B(u_n) \to B(u)$

(3) $B'(u) = 0$ ならば $u = 0$.

このとき, $\lambda_1 = \inf_{B(u)=1} A(u)$ とすると $\exists u \in X$ *s.t.* $A'(u) = \lambda_1 B'(u)$.

Theorem 1 の証明には, 次の不等式が必要である.

Lemma 3 $\alpha > \max(\frac{N}{p}, 1)$, $C > 0$ が存在して, $a \in L^\alpha(\Omega)$ ならば

$$\int_\Omega a(x)|u|^p dx \leq C\|a\|_\alpha \int_\Omega |\nabla u|^p dx.$$

proof $a \in L^\alpha$ に対しては

$$\int_\Omega a(x)|u|^p dx \le \|a\|_\alpha \left(\int_\Omega |u|^{p\cdot\alpha/\alpha-1}\right)^{\alpha-1/\alpha}$$

が成り立ち, $p < N$ ならば, $\theta \in (0,1)$ が存在して

$$\left(\int_\Omega |u|^{p\cdot\alpha/\alpha-1}\right)^{\alpha-1/\alpha} \le \|u\|_p^{1-\theta}\|u\|_{p^*}^{\theta} \le C\|\nabla u\|_p.$$

$p = N$ $(p > N)$ のときは, 任意の r に対し $L^r \subset W_0^{1,p}$ $(L^\infty \subset W_0^{1,p})$ なので,

$$\left(\int_\Omega |u|^{p\cdot\alpha/\alpha-1}\right)^{\alpha-1/\alpha} \le C\|\nabla u\|_p.$$

ゆえに, $\int_\Omega a(x)|u|^p dx \le C\|a\|_\alpha \int_\Omega |\nabla u|^p dx.$

Proof of Theorem 1 (existence)
$A(u) = \int_\Omega |\nabla u|^p dx$, $B(u) = \int_\Omega a(x)|u|^p dx$ とし, $X = W_0^{1,p}(\Omega)$ とする.
(1) $\|u\|_X = \left(\int_\Omega |\nabla u|^p dx\right)^{1/p}$ であるので, $A(u)$ は weakly-l.s.c. かつ coercive である.
(2) $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ とする.

$$I_n := \int_{|x|\le R} a(x)(|u_n|^p - |u|^p)dx, J_n := \int_{|x|\ge R} a(x)(|u_n|^p - |u|^p)dx,$$

とおくと, $B(u_n) - B(u) = I_n + J_n$.

$$|J_n| \le C\|a\|_{L^\alpha(|x|\ge R)}(\|\nabla u_n\|_p + \|\nabla u\|_p)$$

となるが, $\{u_n\}$ は $W_0^{1,p}(\Omega)$ で有界であり, $a \in L^\alpha(\Omega)$ なので, 任意の $\varepsilon > 0$ に対し, R_0 が存在して,

$$|J_n| \le \frac{\varepsilon}{2} \quad \forall R > R_0, \forall n \in \mathbb{N}.$$

また, $\Omega_R = \Omega \cap B_R$ とすると, u_n は $L^\beta(\Omega_R)$ $(\beta < p^*)$ で u に強収束するので, 自然数 N が存在して, $n \ge N$ に対して

$$|I_n| \le \|a\|_\infty \||u_n|^p - |u|^p\|_{L^1(\Omega_R)} < \frac{\varepsilon}{2}.$$

よって, $B(u_n) \to B(u)$.
(3) $\langle B'(u), u\rangle = pB(u)$ であり $a(x) > 0$ なので, $B(u) = 0$ ならば $u = 0$. よって, Theorem 2 の仮定をみたすので, $\exists u \in W_0^{1,p}(\Omega)$ s.t. $\Delta_p u = \lambda_1 |u|^{p-2}u$. □

4 Generalized eigenvalue problem

上の定理の中で, 固有値の単純性については, 柱状領域以外の領域に対しても成り立つ. 更に, $q < p$ の問題も扱えるように, 以下のような一般化された固有値問題を考える

$$(\mathrm{GE})\begin{cases} -\Delta_p u = \lambda \|a^{1/q}u\|_{L^q}^{p-q} a(x)|u|^{q-2}u & \text{in } \Omega, \quad \lambda > 0, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Proposition 4 次のような "一般化された *Rayleigh* 商" を考える. $R_q(v) := \dfrac{\|\nabla v\|_p^p}{\|a^{1/q}v\|_q^p}$ そして $\lambda_1 := \inf_{v \in C_0^\infty(\Omega)} R_q(v)$ とおく. この時以下が成り立つ.

(1) もし, $\lambda \neq \lambda_1$ ならば *(GE)* は正値解を持たない.
(2) もし, $\lambda = \lambda_1$ ならば *(GE)* の固有値 λ_1 は *simple*.

Proof 1. $\lambda < \lambda_1$ のときは (GE) の両辺に u をかけると

$$\begin{aligned} \int_\Omega |\nabla u|^p dx &= \lambda \|a^{1/q}u\|_{L^q}^{p-q} \int_\Omega a(x)u^q dx \\ &= \lambda \|a^{1/q}u\|_{L^q}^p \\ &\le \lambda_1 \|a^{1/q}u\|_{L^q}^p. \end{aligned}$$

λ_1 の定義に矛盾.

$\lambda > \lambda_1$ の場合は$\Omega_\varepsilon \subset \Omega$ を dist $(\Omega_\varepsilon, \partial\Omega) \le \varepsilon$, $\Omega_\epsilon \subset B_{1/\varepsilon}$ であるような滑らかな領域であると定義する. $\sup_{C_0^\infty(\Omega_\varepsilon)} R_q(v) := \dfrac{1}{\lambda_1^\varepsilon}$ とおくと, $\lim_{\varepsilon\to 0} \lambda_1^\varepsilon = \lambda_1$. であるから, $\varepsilon_0 > 0$ が存在して, $\lambda_1 < \lambda_1^{\varepsilon_0} < \lambda$.

$\lambda_1^{\varepsilon_0}$ に対応する positive sol. ([6] により存在が言える) を u_0 とおくと

$$-\Delta_p u_0 = \lambda_1^{\varepsilon_0} \|a^{1/q}u_0\|_q^{p-q} a \cdot (u_0)^{q-1} \text{ in } \Omega_{\varepsilon_0}.$$

v を (E) の正値解とすると, [6] と Harnack の不等式 [9] より一般性を失うことなく,

$$\begin{aligned} u_0 &\le v \quad \text{in } \Omega_{\varepsilon_0}, \\ \|a^{1/q}u_0\|_q^{p-q} &\le \|a^{1/q}v\|_q^{p-q} \end{aligned}$$

とできる.

$$\begin{aligned} -\Delta_p u_0 &= \lambda_1^{\varepsilon_0} \|a^{1/q}u_0\|_q^{p-q} a \cdot (u_0)^{p-1} \\ &\le \lambda_1^{\varepsilon_0} a v^{p-1} = \lambda_1 a \cdot (\eta v)^{p-1} = -\Delta_p(\eta v) \end{aligned}$$

where $\eta = (\lambda_1^{\varepsilon_0}/\lambda_1)^{1/(p-1)}$ であり, また, $0 = u_0|_{\partial\Omega_{\varepsilon_0}} < \eta v|_{\partial\Omega_{\varepsilon_0}}$ なので p–Laplacian の比較定理より

$$u_0 \le \eta v \text{ in } \Omega_{\varepsilon_0}.$$

k 回繰り返すと, $0 < u_0 \le \eta^k v$ for all $k \in \mathbb{N}$. $k \to \infty$ とすると $u_0 = 0$ となり矛盾.

3. $\lambda = \lambda_1$ のとき,

$$J_q^{\lambda_1}(v) = \int_\Omega |\nabla v|^p dx - \lambda_1 (\int_\Omega a v^q dx)^{p/q}$$

とおけば

$$J_q^{\lambda_1}(u) \ge 0 \quad \text{for all} \quad u \in W_0^{1,p}(\Omega)$$
$$J_q^{\lambda_1}(u) = 0 \iff u\text{: sol. of (E)}$$

である. u, v を (GE) の 2 つの正値解とする.
$M(t,x) = \max(u(x), tv(x))$, $m(t,x) = \min(u(x), tv(x))$ とおくと M, m は (GE) の $W_0^{1,p}(\Omega)$ 解になる.
a.e. $x_0 \in \Omega$ に対し, $t_0 = \dfrac{u(x_0)}{v(x_0)}$ とする. e: unit vector in $\mathbf{R}^N$ に対し

$$\begin{cases} u(x_0 + he) - u(x_0) \le M(t_0, x_0 + he) - M(t_0, x_0) \\ t_0 v(x_0 + he) - t_0 v(x_0) \le M(t_0, x_0 + he) - M(t_0, x_0) \quad \text{for all } h \in \mathbf{R} \end{cases}$$

$h \to \pm 0$ とすることにより, $\nabla u(x_0) = t_0 \nabla v(x_0)$ for all $x_0 \in \Omega$. よって$\left(\dfrac{u}{v}\right)(x) = t_0$ for all $x \in \Omega$. □

Proof of theorem 1 (simplicity & nonexistence)
まず, u が (E) の解ならば, $q = p$ とすれば (GE) の解になるので, Proposition 4 よりすぐに, $\lambda = \lambda_1$ のときのみに解をもち, かつ λ_1 は simple であることがわかる.

u を (GE) の正値解であるとすると, $J_{\lambda_1}(u) = 0$ である. 関数 u_1 を, $u_1(x) := u(x + x_1)$ で定義すると,

$$\begin{aligned} 0 &\le J_{\lambda_1}(u) = \int_\Omega |\nabla u_1|^p dx - \lambda_1 \int_\Omega a(x) u_1^p dx \\ &\le \int_\Omega |\nabla u_1|^p dx - \lambda_1 \int_\Omega a(x + x_1) u(x + x_1)^p dx \\ &\le \int_\Omega |\nabla u(x)|^p dx - \lambda_1 \int_\Omega a(x) u(x)^p dx = 0. \end{aligned}$$

従って, u_1 も (GE) の正値解となる. Proposition 4 より $t > 0$ が存在して, $u(x) = t \cdot u_1(x)$. $u_1(x)$ は $u(x)$ の平行移動なので, $t = 1$. すなわち,

$$u(x) = u(x + x_1) \text{ for all } x \in \Omega$$

となるが, $u \in L^p(\Omega)$ かつ Ω は x_1 方向に非有界なので, $u \equiv 0$ □

Remark Proposition 4 より, $q < p$: 'sub-principal case' での非存在定理も導ける.

Theorem 5

$$\begin{cases} -\Delta_p u = a(x)|u|^{q-2}u & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases}$$

は, *Theorem 1* の *(2)* の条件を満たすか, あるいは以下の条件

$$m > 0 \text{ が存在して } m \le a(x) \ for\ a.e.\ x \in \Omega.$$

を満たすならば正値解を持たない.

参考文献

[1] M. S. BERGER, "Nonlineality and Functional Analysis", Academic Press, 1977.

[2] J. FLECKINGER, R. F. MANÁSEVICH, N. M. STAVRAKAKIS & F. DE. THÉLIN, Principal eigenvalues for some quasilinear elliptic equations on $\mathbf{R}^N$, *preprint.*

[3] T. HASHIMOTO & M.ÔTANI, Pohozaev type inequalities for some quasilinear elliptic equations, To appear.

[4] T. HASHIMOTO & M.ÔTANI, Nonexistence of positive solutions for some quasilinear elliptic equations in strip-like domains, *Continuous and Discrete Dynamical Systems, to appear.*

[5] M. ÔTANI, Existence and nonexistence of nontrivial solutions of some nonlinear degenerate elliptic equations, *J.Funct.Anal.* **76** (1988), 140-159.

[6] M. ÔTANI & T. TESHIMA, On the first eigenvalue of some quasilinear elliptic equations, *Proc. Japan. Acad.* **64** (1988), 8-10.

[7] I. SCHINDLER, Quasilinear elliptic boundary-value problems on unbounded cylinders and a related mountain pass lemma, *Arch. Rational Mech. Anal.*, **120** (1992), 363–374.

[8] G. TALENTI, Best constant in Sobolev inequality, *Annali di Mat. Pura e Appl.*, **110** (1976), 353–372.

[9] N. S. TRUDINGER, On Harnack type inequalities and their application to quasilinear elliptic equations, *Comm. Pure Appl. Math.*, **20** (1967), 721-747.

Existence of the singular ground state with maximal intensity

Tokushi Sato (Tohoku University)

In this talk, we consider singular ground states of the scalar field equation in $\mathbf{R}^n$ with space dimension $n \geq 2$. For $p > 1$ we call u a *ground state* of the scalar field equation if $u \in C^2(\mathbf{R}^n)$ and u satisfies

$$(\mathrm{P})_0 \qquad \begin{cases} -\Delta u + u = u^p, \quad u > 0 \quad \text{in } \mathbf{R}^n, \\ u(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases}$$

It is known that $(\mathrm{P})_0$ has a solution if and only if $1 < p < (n+2)/(n-2)$. (We agree that $(n+2)/(n-2) = n/(n-2) = \infty$ for $n = 2$.) For simplicity, we assume that u attains its maximum at the origin for a solution u to $(\mathrm{P})_0$. Next we call u a *singular ground state* of the scalar field equation if $u \in C^2(\mathbf{R}^n \setminus \{0\})$ and u satisfies

$$(\mathrm{P}) \qquad \begin{cases} -\Delta u + u = u^p, \quad u > 0 \quad \text{in } \mathbf{R}^n \setminus \{0\}, \\ u(x) \to \infty \quad \text{as } x \to 0, \quad u(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases}$$

Note that any solution to (P) or $(\mathrm{P})_0$ is radially symmetric ([3]). Concerning this problem, Ni–Serrin [8] showed that (P) has no solution if $n \geq 3$ and $p \geq (n+2)/(n-2)$. Recently, existence results of solutions to (P) for $1 < p < (n+2)/(n-2)$ are proved by several authors (e.g. [4,9]).

In the following, we only consider the case where $1 < p < n/(n-2)$. Then the behavior of the singularity of any solution at the origin must be

$$u(x) \sim \kappa E(x) \quad \text{as } x \to 0$$

for some constant $\kappa > 0$ depending on u, and we call κ the *intensity* of the singularity. Here E is the fundamental solution of $-\Delta$ in $\mathbf{R}^n$, i.e.

$$E(x) := \begin{cases} \dfrac{1}{(n-2)n\omega_n} \dfrac{1}{|x|^{n-2}} & \text{if } n \geq 3, \\ \dfrac{1}{2\pi} \log \dfrac{1}{|x|} & \text{if } n = 2 \end{cases}$$

(ω_n denotes the volume of a unit ball in $\mathbf{R}^n$). Thus we consider the problem

$$(\mathrm{P})_\kappa \qquad \begin{cases} -\Delta u + u = u^p, \quad u > 0 \quad \text{in } \mathbf{R}^n \setminus \{0\}, \\ u(x) \sim \kappa E(x) \quad \text{as } x \to 0, \quad u(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases}$$

instead of (P). Any solution $u \in C^2(\mathbf{R}^n \setminus \{0\})$ to $(\mathrm{P})_\kappa$ satisfies

$$-\Delta u + u = u^p + \kappa \delta_0 \quad \text{in } \mathcal{D}'(\mathbf{R}^n), \quad u \geq \kappa E_1 \quad \text{in } \mathbf{R}^n \setminus \{0\},$$

where E_1 is the fundamental solution of $-\Delta + 1$ in $\mathbf{R}^n$, i.e.

$$E_1(x) := \frac{1}{(2\pi)^{n/2}} \frac{1}{|x|^{(n-2)/2}} K_{(n-2)/2}(|x|)$$

(K_ν denotes the modified Bessel function of order ν). Note that E_1 satisfies

$$-\Delta E_1 + E_1 = \delta_0 \quad \text{in } \mathcal{D}'(\mathbf{R}^n)$$

and

$$E_1(x) \sim E(x) \quad \text{as } x \to 0, \quad E_1(x) \sim c_n \frac{e^{-|x|}}{|x|^{(n-1)/2}} \quad \text{as } |x| \to \infty.$$

Concerning this problem, we know the following fact.

Fact. (i) There exists $\kappa^* > 0$ such that problem $(\mathrm{P})_\kappa$ has a solution for $0 < \kappa < \kappa^*$ and has no solution for $\kappa > \kappa^*$.
(ii) Problem $(\mathrm{P})_\kappa$ has at least two solutions for $0 < \kappa \ll 1$.

Now we consider the existence of a solution to $(\mathrm{P})_{\kappa^*}$ where

$$\kappa^* := \sup\{\, \kappa > 0 \mid (\mathrm{P})_\kappa \text{ has a solution } \}$$

is called the *maximal intensity*. Our main result is the following.

Theorem. Let $n \geq 2$ and $1 < p < n/(n-2)$.
(A) There exists a unique solution $u_1 \in C^2(\mathbf{R}^n \setminus \{0\})$ to $(\mathrm{P})_{\kappa^*}$.
(B) Problem $(\mathrm{P})_\kappa$ has at least two solutions for $0 < \kappa^* - \kappa \ll 1$ near u_1 in an appropriate sense.

In the following, we describe the outline of the proof of Theorem. From two propositions below, we have part (A) of Theorem.

Proposition 1. Let $n \geq 2$ and $1 < p < n/(n-2)$. Assume that $u_1 \in C^2(\mathbf{R}^n \setminus \{0\})$ is a solution to $(\mathrm{P})_{\kappa_1}$ and the linearized problem

$$(\mathrm{L}\,; u_1) \qquad \begin{cases} -\Delta\varphi + \varphi = p u_1^{p-1}\varphi, \quad \varphi > 0 \quad \text{in } \mathbf{R}^n \setminus \{0\}, \\ \varphi(0) = 1, \quad \varphi(x) \to 0 \quad \text{as } |x| \to \infty \end{cases}$$

has a radial solution $\varphi_1 \in C^2_*(\mathbf{R}^n \setminus \{0\})$. Then $\kappa_1 = \kappa^*$ and a solution to $(\mathrm{P})_{\kappa_1}$ is unique. Here

$$C^2_*(\mathbf{R}^n \setminus \{0\}) := C^2(\mathbf{R}^n \setminus \{0\}) \cap C(\mathbf{R}^n).$$

Proposition 2. Let $n \geq 2$ and $1 < p < n/(n-2)$. Then there exists $(u_1, \varphi_1\,; \kappa_1)$ which satisfies the assumption of Proposition 1.

In order to prove Proposition 1 we use the properties below. Let $(u_1, \varphi_1\,; \kappa_1)$ be a solution in the sence of Proposition 1 and set

$$u_1 - \kappa_1 E_1 = \zeta^\nu v_1, \quad \varphi_1 = \zeta^\nu \psi_1,$$

where $0 < \nu < 1$ and $\zeta \in C^\infty(\mathbf{R}^n)$ is a radial function which is nonincreasing in $r = |x|$ and satisfies

$$\zeta(x) = \begin{cases} 1 & \text{for } 0 \le |x| \ll 1, \\ E_1(x) & \text{for } |x| \gg 1. \end{cases}$$

Then $(v_1, \psi_1 ; \kappa_1)$ satisfies

$$v_1 = V[v_1 ; \kappa_1] \ge 0, \quad \psi_1 = \Psi[v_1 ; \kappa_1]\psi_1 > 0,$$

where

$$V[v ; \kappa] := \zeta^{-\nu} \cdot E_1 * [(\zeta^\nu v + \kappa E_1)_+^p], \quad \Psi[v ; \kappa]\psi := \zeta^{-\nu} \cdot E_1 * [p(\zeta^\nu v + \kappa E_1)_+^{p-1} \zeta^\nu \psi].$$

Note that $\Psi[v_1 ; \kappa_1] : L^q(\mathbf{R}^n) \to L^q(\mathbf{R}^n)$ is a compact operator if $p < q < n/(n-2)$. Positivity of ψ_1 yields that

$$\operatorname{Ker}(I - \Psi[v_1 ; \kappa_1]) = [\psi_1] \; (\subset L^q(\mathbf{R}^n)),$$

and we can see that

$$(I - \Psi[v_1 ; \kappa_1])(L^q(\mathbf{R}^n)) = [\psi_1^*]^\perp, \quad \psi_1^* := p u_1^{p-1} \varphi_1 \zeta^\nu$$

by using Fredholm's alternative. Proposition 1 follows from this fact and the covexity of the nonlinearity function.

In order to prove Proposition 2 we introduce a parameter $\tau \in [0,1]$ and consider

$$(\mathrm{P}_\tau)_\kappa \qquad \begin{cases} -\Delta u + u = u^p - (1-\tau)(\kappa E_1)^p, \quad u \ge \kappa E_1 \quad \text{in } \mathbf{R}^n \setminus \{0\}, \\ u(x) \sim \kappa E(x) \quad \text{as } x \to 0, \quad u(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases}$$

Now we set up the following problem.

Definition. For $\tau \in [0,1]$, $(u, \varphi ; \kappa)$ is a solution to (Q_τ) if
(i) $\kappa > 0$,
(ii) $u \in C^2(\mathbf{R}^n \setminus \{0\})$ is a radial solution to $(\mathrm{P}_\tau)_\kappa$,
(iii) $\varphi \in C_*^2(\mathbf{R}^n \setminus \{0\})$ is a radial solution to $(\mathrm{L} ; u)$.
We set

$$T := \{ \tau \in [0,1] \mid (\mathrm{Q}_\tau) \text{ has a solution } \}.$$

We claim that $T = [0,1]$ which is equivalent to that T is nonempty, closed and open in $[0,1]$. We divide its proof into three steps.

Step 1 $[\, 0 \in T \,]$.
First note that $u_0 = \kappa_0 E_1$ is a solution to $(\mathrm{P}_0)_{\kappa_0}$ for all $\kappa_0 > 0$. So we claim that $(\mathrm{L} ; u_0)$ has a radial solution for some $\kappa_0 > 0$. To do this, we consider the minimizing problem

$$\inf \left\{ \frac{\|\nabla\varphi\|_2^2 + \|\varphi\|_2^2}{\|E_1^{(p-1)/2}\varphi\|_2^2} \;\middle|\; \varphi \in W^{1,2}(\mathbf{R}^n) \setminus \{0\} \right\} \; (\equiv \overline{\lambda}).$$

By the standard argument we can see that $\overline{\lambda} > 0$ and there exists a minimizer $\varphi_0 \in W^{1,2}(\mathbf{R}^n) \setminus \{0\}$. Furthermore, we see that $\varphi_0 \in C_*^2(\mathbf{R}^n \setminus \{0\})$, φ_0 is radial and satisfies

$$\begin{cases} -\Delta\varphi_0 + \varphi_0 = \overline{\lambda} E_1^{p-1}\varphi_0 & \text{in } \mathbf{R}^n \setminus \{0\}, \\ \varphi_0(0) = 1, \quad \varphi_0(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

(by normalization of the value at the origin). For $\kappa_0 > 0$ such that $\overline{\lambda} = p\kappa_0^{p-1}$, φ_0 is a solution to $(\mathrm{L}\,; u_0)$ and $(u_0, \varphi_0\,; \kappa_0)$ is a solution to (Q_0).

Step 2 [Closedness of T].

For $\tau \in T$ we denote a solution to (Q_τ) by $(u_\tau, \varphi_\tau\,; \kappa_\tau)$ and set

$$u_\tau - \kappa_\tau E_1 = w_\tau = z_\tau E_1, \quad \varphi_\tau = y_\tau E_1.$$

Then we see that z_τ and y_τ are increasing in r, while u_τ and φ_τ are decreasing in r. Moreover, $\{\kappa_\tau\}_{\tau \in T}$ is decreasing in τ and hence $0 < \kappa_\tau \le \kappa_0$. For $0 < \nu < 1$ we multiply ζ^ν both sides of

$$-\Delta w_\tau + w_\tau = u_\tau^p - (1-\tau)(\kappa_\tau E_1)^p \quad \text{in } \mathcal{D}'(\mathbf{R}^n)$$

and integrate on $\mathbf{R}^n$. Then we have

$$(*) \qquad \int_{\mathbf{R}^n} u_\tau^p \zeta^\nu dx \le M_\nu$$

for some $M_\nu > 0$, by making use of integration by part and Young's inequality. From the integral representation of solutions we can see that $\{z_\tau\}_{\tau \in T}$ and $\{\varphi_\tau\}_{\tau \in T}$ are locally uniformly bounded and locally equi-continuous in $\mathbf{R}^n$.

Now we assume $\{\tau_j\}_{j=1}^\infty \subset T$, $\tau_j \to \tau$ as $j \to \infty$. By the Ascoli–Arzelà theorem there exist a subsequence $\{j_i\}_{i=1}^\infty$, radial functions $z, \varphi \in C(\mathbf{R}^n)$ and $\kappa \ge 0$ such that

$$z_{\tau_{j_i}} \to z, \quad \varphi_{\tau_{j_i}} \to \varphi \quad \text{locally uniformly in } \mathbf{R}^n, \quad \kappa_{\tau_{j_i}} \to \kappa \quad \text{as } i \to \infty.$$

Set $u - \kappa E_1 = w = zE_1$ and $\varphi = yE_1$. Then we have

$$z(0) = 0, \quad \varphi(0) = 1, \quad u \ge \kappa E_1, \quad \varphi \ge 0 \quad \text{in } \mathbf{R}^n \setminus \{0\}.$$

Furthermore, z and y are nondecreasing in r and hence $\varphi > 0$ in $\mathbf{R}^n$. Since u and φ are nonincreasing in r, there exist $\gamma, \tilde{\gamma} \ge 0$ such that

$$u(x) \to \gamma, \quad \varphi(x) \to \tilde{\gamma} \quad \text{as } |x| \to \infty.$$

From $(\mathrm{P}_{\tau_j})_{\kappa_{\tau_j}}$, $(\mathrm{L}\,; u_{\tau_j})$ and $(*)$ we have

$$-\Delta w + w = u^p - (1-\tau)(\kappa E_1)^p, \quad -\Delta\varphi + \varphi = pu^{p-1}\varphi \quad \text{in } \mathbf{R}^n \setminus \{0\} \ \ (\text{in } \mathcal{D}'(\mathbf{R}^n))$$

and $u \not\equiv 0$. As $|x| \to \infty$, we have $\gamma = 0, 1$ and $\tilde{\gamma} = 0$.

If $\gamma = 1$, then $pu^{p-1} - 1 \ge p - 1 > 0$ in $\mathbf{R}^n$ and $\varphi(x) = \varphi(|x|)$ satisfies

$$\varphi'' + \frac{n-1}{r}\varphi' + (pu^{p-1} - 1)\varphi = 0 \quad \text{for } r > 0$$

and hence φ is oscillating, which is a contradiction. Therefore, $\gamma = 0$ holds true.

Finally we note that $(\mathrm{L}\,;u)$ has no nontrivial radial solution if $u \in C^2(\mathbf{R}^n)$ is a ground state. From this fact we can easily see that $\kappa > 0$ and hence $(u,\varphi\,;\kappa)$ is a solution to (Q_τ). Therefore, T is closed.

Step 3 [Openness of T].
Assume $\tau_0 \in T$ and set

$$\begin{cases} \Lambda_{\tau_0} := \left\{ \xi \in X(\mathbf{R}^n)_{\mathrm{r}} \;\middle|\; \int_{\mathbf{R}^n} p u_{\tau_0}^{p-1} \varphi_{\tau_0} \cdot \xi E_1 \, dx = 0 \right\}, \\ \Sigma_{\tau_0} := \left\{ \xi \in X(\mathbf{R}^n)_{\mathrm{r}} \;\middle|\; \int_{\mathbf{R}^n} p(p-1) u_{\tau_0}^{p-2} \varphi_{\tau_0}^2 \cdot \xi E_1 \, dx = 1 \right\}, \end{cases}$$

where

$$X(\mathbf{R}^n)_{\mathrm{r}} := \{ \xi \in C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) \mid \xi \text{ is radial and } \xi(0) = 0 \}.$$

We introduce a small parameter ε and solve (Q_τ) for $|\tau_0 - \tau| \ll 1$ in the form below :

(i) $$(z, y\,; \kappa, \tau) = (\varepsilon(y_0 + \varepsilon\xi), y_0 + \varepsilon\eta\,; \kappa_0 - \varepsilon\rho, \varepsilon^2\sigma) \quad \text{for } 0 < \varepsilon \ll 1,$$

where $(\xi, \eta\,; \rho, \sigma) \in \Lambda_0^2 \times \mathbf{R}^2$ if $\tau_0 = 0$,

(ii) $$(z, y\,; \kappa, \tau) = (z_{\tau_0} + \varepsilon\xi, y_{\tau_0} + \varepsilon\eta\,; \kappa_{\tau_0} - \varepsilon\rho, \tau_0 + \varepsilon\sigma) \quad \text{for } 0 < |\varepsilon| \ll 1,$$

where $(\xi, \eta\,; \rho, \sigma) \in (\Sigma_{\tau_0} \times \Lambda_{\tau_0}) \times \mathbf{R}^2$ if $\tau_0 > 0$.
To do this, we use the contraction mapping principle repeatedly.

By the three steps above we can conclude $T = [0,1]$ and part (A) of Theorem is established. In order to prove part (B) we introduce a small parameter ε and solve $(\mathrm{P})_\kappa$ for $0 < \kappa_1 - \kappa \ll 1$ in the form

$$(z\,; \kappa) = (z_1 + \varepsilon(y_1 + \varepsilon\xi)\,; \kappa_1 - \varepsilon^2\rho) \quad \text{for } 0 < |\varepsilon| \ll 1,$$

where $(\xi\,; \rho) \in \Lambda_1 \times \mathbf{R}$. This completes proof of Theorem.

References

[1] H. Berestycki, P. L. Lions and L. A. Peletier, *An ODE approach to the existence of positive solutions for semilinear problems in* $\mathbf{R}^N$, Indiana Univ. Math. J. **30** (1981) 141–157.

[2] C.-C. Chen and C.-S. Lin, *Uniquness of the ground state solutions of* $\Delta u + f(u) = 0$ *in* $\mathbf{R}^n$, $n \geq 3$, Comm. Partial Differential Equations **16** (1991) 1549–1572.

[3] B. Gidas, W.-M. Ni and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbf{R}^n$, Advances in Math. Studies **7A** (1981) 369–402.

[4] R. A. Johnson, X. Pan and Y. Yi, *Singular solutions of the elliptic equation* $\Delta u - u + u^p = 0$, (preprint).

[5] J. P. Keener and H. B. Keller, *Perturbed bifurcation theory*, Arch. Rational Mech. Anal. **50** (1973) 159–175.

[6] J. P. Keener and H. B. Keller, *Positive solutions of convex nonlinear eigenvalue problems*, J. Differential Equations **16** (1974) 103–125.

[7] P. L. Lions, *Isolated singularities in semilinear probles*, J. Differential Equations **38** (1980) 441–450.

[8] W.-M. Ni and J. Serrin, *Nonexistence theorems for singular solutions of quasilinear partial differential equations*, Comm. Pure Appl. Math. **39** (1986) 379–399.

[9] T. Sato, *Positive solutions with weak isolated singularities to some semilinear elliptic equations*, Tôhoku Math. J. **47** (1995) 55–80.

Global Existence for Systems of Wave Equations with Different Speeds

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以下の内容は上見　練太郎（北海道大学理学部）との共同研究に基づく．

1　序

次の連立波動方程式の初期値問題を考える．

$$
(1.1)\qquad \begin{cases} \partial_t^2 u^i - c_i^2 \Delta u^i = F_i(\partial u, \partial^2 u) & \text{in} \quad [0,\infty)\times \mathbf{R}^2, \\ u^i(0,\cdot) = \varepsilon f^i,\ \partial_t u^i(0,\cdot) = \varepsilon g^i & \text{in} \quad \mathbf{R}^2 \quad (i = 1, 2, \ldots, m). \end{cases}
$$

ただし，

$$
\partial u = {}^t(\partial u^1, \ldots, \partial u^m),\ \partial u^i = {}^t(\partial_0 u^i, \partial_1 u^i, \partial_2 u^i),
$$

$$
\partial_0 = \partial_t = \partial/\partial t,\ \partial_1 = \partial/\partial x_1,\ \partial_2 = \partial/\partial x_2,
$$

$$
c_i > 0,\ \varepsilon > 0
$$

である．また，f^i, g^iは台がコンパクトな C^∞級関数であるとし，

$$
F_i(\partial u, \partial^2 u) = \sum_{j=1}^{m} \sum_{\alpha,\beta=0}^{2} A_{ij}^{\alpha\beta}(\partial u)\partial_\alpha \partial_\beta u^j + B_i(\partial u) \quad (i = 1, 2, \ldots, m),
$$

$A_{ij}^{\alpha\beta}$, B_iは原点の近傍で C^∞級で，

$$
(1.2)\qquad \begin{aligned} &A_{ij}^{\alpha\beta} = A_{ij}^{\beta\alpha} = A_{ji}^{\alpha\beta}, \\ &\left|A_{ij}^{\alpha\beta}(\partial u)\right| \le M\,|\partial u|^2,\ |B_i(\partial u)| \le M\,|\partial u|^3, \end{aligned}
$$

とする．我々は，small data に対して初期値問題 (1.1) が滑らかな大域解をもつかどうかに関心を持つ．

まず，単独方程式 ($m=1$) の場合を考える．上のように非線形項 $F_1(\partial u, \partial^2 u)$ が未知関数について 3 次のオーダーであるという仮定のみでは，たとえ small data であっても大域解が存在するとは限らない ($F_1 = (\partial_t u)^3$ や $|\nabla u|^2 \Delta u$ など有限時間で爆発する例あり)．ε を small parameter とすると解の life-span $T(\varepsilon)$ は $C\exp(C\varepsilon^{-2})$ 以上であり ([2])，ε^{-1} について多項式オーダーである場合と区別して almost global に存在するという．空間次元が 3 のときには F_1 が未知関数について 2 次の場合に almost global に解が存在する．そして almost global に解が存在する場合にはさらに null condition という条件を課すことにより，大域解の存在が示されていた．

$B_1 = 0$ とし，$\partial u = v$ とおいて (1.1) ($m=1$) を 1 階双曲型方程式になおす．さらに，plane wave solution, すなわち $v(t,x) = w(t,s), s = \sum_{k=1}^{2} \zeta_k x_k$ という形の解が満たす方程式

$$
(1.3)\qquad \partial_t w + a(w)\partial_s w = 0 \quad (\,a(w)\ \text{は 3 次正方行列})
$$

を考える．F. John と J. Shatah は (1.3) の解の life-span に関する考察から null condition を次のようにとらえた ([1])．すなわち，(1.3) に対し初期値 $w(0,s) = \varepsilon\phi(s)$ を与えると，解の life-span は $C\varepsilon^{-1}$ 程度なのだが，どんな $\zeta = (\zeta_1, \zeta_2) \neq 0$ に対してもこのような解の life-span が $C\varepsilon^{-3}$ 以上になるという条件から null condition が導ける，というものである．

我々は，この考察をスピード c_i が相異なる連立方程式 (1.1) に適用し，スピードの異なる連立方程式に対する null condition というべき条件を導いた．さらに，その null condition をみたすあるクラスに対し，大域解の存在定理を得た．

2 Null Condition

ここでは上で説明した John と Shatah の考察に基づく null condition の導出の概略を述べる．(1.1) において，c_i はすべて相異なるとする．すなわち，

$$c_i \neq c_j \quad (i \neq j). \tag{2.1}$$

また，$B_i = 0$ とする．すると，(1.1) は

$$\sum_{j=1}^{m} \sum_{\alpha,\beta=0}^{2} a_{ij}^{\alpha\beta}(\partial u)\partial_\alpha \partial_\beta u^j = 0 \quad (i = 1, \ldots, m)$$

とかける．ここで，$\partial u = v$ とおいて 1 階化すると，

$$\sum_{\alpha=0}^{2} a^\alpha(v)\partial_\alpha v = 0. \tag{2.2}$$

ただし

$$v = {}^t(v^1, \ldots, v^m),\ v^i = {}^t(v_0^i, v_1^i, v_2^i),\ v_\alpha^i = \partial_\alpha u^i, \tag{2.3}$$

$$a^\alpha(v) = \left(A_{ij}^\alpha(v)\,;\, i, j = 1, \ldots, m\right),$$

$$A_{ij}^0 = \begin{pmatrix} a_{ij}^{00} & & \mathbf{0} \\ & \delta_{ij} & \\ \mathbf{0} & & \delta_{ij} \end{pmatrix},$$

$$A_{ij}^1 = \begin{pmatrix} 2a_{ij}^{10} & a_{ij}^{11} & a_{ij}^{12} \\ -\delta_{ij} & & \\ 0 & & \mathbf{0} \end{pmatrix},\ A_{ij}^2 = \begin{pmatrix} 2a_{ij}^{20} & a_{ij}^{21} & a_{ij}^{22} \\ 0 & & \mathbf{0} \\ -\delta_{ij} & & \end{pmatrix}. \tag{2.4}$$

方程式 (2.2) の plane wave solution とは，

$$v(t, x) = w(t, s),\ s = \sum_{k=1}^{2} \zeta_k x_k \quad (\zeta \in \mathbf{R}^2 \setminus \{0\}) \tag{2.5}$$

という形の解のことである．(2.5) と (2.2) より w は

$$\partial_t w + a(w)\partial_s w = 0, \quad a(w) = a^0(w)^{-1}\sum_{i=1}^{2}\zeta_i a^i(w) \tag{2.6}$$

を満たす．(2.6) を初期条件

$$w(0,s) = \varepsilon\phi(s), \quad \phi \in C_0^\infty(\mathbf{R}) \tag{2.7}$$

のもとで考える．初期値問題 (2.6)-(2.7) の解の life-span を $T(\varepsilon)$ とする．

$$\begin{aligned} &\det\left(\lambda a^0(w) - \sum_{i=1}^{2}\zeta_i a^i(w)\right) = \lambda^m P(\lambda), \\ &P(\lambda) = \det(p_{ij}),\ p_{ij} = a_{ij}^{00}\lambda^2 - 2\lambda\sum_{k=1}^{2}a_{ij}^{k0}\zeta_k + \sum_{k,l=1}^{2}a_{ij}^{kl}\zeta_k\zeta_l \quad (i,j=1,\dots,m) \end{aligned} \tag{2.8}$$

より，$a(w)$ の 0 でない固有値は $2m$ 個ある．これを $\lambda_i^\pm(w)$ $(i=1,\dots,m)$, 対応する固有ベクトルを $\xi_i^\pm(w)$ $(i=1,\dots,m)$ とする．ここで，$\lambda_i^\pm(w)$ は $\lambda_i^\pm(0) = \pm c_i|\zeta|$ となるものである．Li Ta-tsien, Kong De-xing, Zhou Yi [4] によると，任意の $\zeta \in \mathbf{R}^2\setminus\{0\}$ に対し，$T(\varepsilon) \geq C\varepsilon^{-3}$ となるための条件は，

$$\begin{aligned} &\sum_{j=1}^{m}\sum_{\alpha=0}^{2}\left.\frac{\partial\lambda_i^\pm}{\partial w_\alpha^j}\right|_{w=0}\left(\xi_i^\pm(0)\right)_\alpha^j = 0, \\ &\sum_{j,k=1}^{m}\sum_{\alpha,\beta=0}^{2}\left.\frac{\partial^2\lambda_i^\pm}{\partial w_\alpha^j\partial w_\beta^k}\right|_{w=0}\left(\xi_i^\pm(0)\right)_\alpha^j\left(\xi_i^\pm(0)\right)_\beta^k = 0 \end{aligned} \qquad (i=1,\dots,m,\ \zeta\in\mathbf{R}^2\setminus\{0\}) \tag{2.9}$$

が成り立つことである．(2.8) および (2.1) より，

$$\begin{aligned} &\left.\frac{\partial\lambda_i^\pm}{\partial w_\alpha^j}\right|_{w=0} = 0, \\ &\left.\frac{\partial^2\lambda_i^\pm}{\partial w_\alpha^j\partial w_\beta^k}\right|_{w=0} = \frac{\mp c_i|\zeta|}{2}\sum_{\gamma,\delta=0}^{2}\left.\frac{\partial^2 a_{ii}^{\gamma\delta}}{\partial w_\alpha^j\partial w_\beta^k}\right|_{w=0}\left(\xi_i^\pm(0)\right)_\gamma^i\left(\xi_i^\pm(0)\right)_\delta^i. \end{aligned} \tag{2.10}$$

また，任意の ζ に対し，

$$\left\{\left(\xi_i^\pm(0)\right)_0^i\right\}^2 - c_i^2\sum_{j=1}^{2}\left\{\left(\xi_i^\pm(0)\right)_j^i\right\}^2 = 0. \tag{2.11}$$

よって (2.9)-(2.11) より，次の条件が得られる．

$$\sum_{\alpha,\beta,\gamma,\delta=0}^{2} \left.\frac{\partial^2 a_{ii}^{\gamma\delta}}{\partial w_\alpha^i \partial w_\beta^i}\right|_{w=0} X_\alpha^i X_\beta^i X_\gamma^i X_\delta^i = 0 \tag{2.12}$$

for any $X^i = (X_0^i, X_1^i, X_2^i)$ such that $(X_0^i)^2 - c_i^2 \sum_{j=1}^2 (X_j^i)^2 = 0$.

3 大域解の存在

我々は，(2.12) を満足する $F_i(\partial u, \partial^2 u)$ として，次のようなものを考えた．

$$\begin{aligned} \left.\frac{\partial^2 A_{ii}^{\alpha\beta}}{\partial(\partial_\gamma u^i)\partial(\partial_\delta u^i)}\right|_{\partial u=0} &= 0 \quad \begin{pmatrix} i = 1, \ldots, m \\ \alpha, \beta, \gamma, \delta = 0, 1, 2 \end{pmatrix}, \\ \left.\frac{\partial^3 B_i}{\partial(\partial_\alpha u^i)\partial(\partial_\beta u^i)\partial(\partial_\gamma u^i)}\right|_{\partial u=0} &= 0 \quad \begin{pmatrix} i = 1, \ldots, m \\ \alpha, \beta, \gamma = 0, 1, 2 \end{pmatrix}. \end{aligned} \tag{3.1}$$

すなわち，(2.12) において $X_\alpha^i X_\beta^i X_\gamma^i X_\delta^i$ の係数が全て 0 になる場合である．

定理 (1.2), (2.1), (3.1) を仮定する．このとき十分小さな正数 ε_0 をとれば任意の $\varepsilon\,(0 < \varepsilon < \varepsilon_0)$ に対し初期値問題 (1.1) は $[0, \infty) \times \mathbf{R}^2$ において C^∞級の解をただ一つ持つ．

M. Kovalyov は [3] において，$A_{ij}^{\alpha\beta} = 0$ かつ

$$\left.\frac{\partial^3 B_i}{\partial(\partial_\alpha u^j)\partial(\partial_\beta u^j)\partial(\partial_\gamma u^j)}\right|_{\partial u=0} = 0 \quad \begin{pmatrix} i, j = 1, \ldots, m \\ \alpha, \beta, \gamma = 0, 1, 2 \end{pmatrix}$$

のときに滑らかな大域解が存在することを示した．上の定理はその拡張である．

証明では Kovalyov [2] の解の表示を用い，導関数の各点評価を行う．具体的には ∂u^i について $1/(|x|+1)^{1/2-\gamma}(|x|+t+1)^\gamma(||x|-c_i t|+1)^{1/2}\,(0 < \gamma < 1/2)$ の decay を出し，ウェイト付きの H^s ノルムを用いて評価する．条件 (3.1) より，F_iとしては $F_i(\partial u, \partial^2 u) = \partial_\alpha u^j \partial_\beta u^k \partial_\gamma \partial_\delta u^l\ (j \neq i)$ のように j, k, l が全て i にならないようなものを考えている．従って，$F_i(\partial u, \partial^2 u)$ を i 番目の light cone $\{\,|x| = c_i t\,\}$ の近くで評価する際に，F_i に $\partial u^j\ (j \neq i)$ が含まれていることから，$c_i \neq c_j$ により decay の悪さが補われている．

この各点評価とエネルギー評価を組み合わせ，局所解の life-span が ∞ であることを示すことにより定理の証明を得る．

参考文献

[1] F. John, Nonlinear wave equations, Formation of singularities, Pitcher lectures in the Math. Sci., Lehigh Univ., American Math. Soc., 1990

[2] M. Kovalyov, Long-time behaviour of solutions of a system on non-linear wave equations, Comm. in PDE., **12** (1987), 471-501.

[3] M. Kovalyov, Resonance-type behaviour in a system of nonlinear wave equations, J. Differential Equations **77** (1989), 73-83.

[4] Li Ta-tsien, Kong De-xing and Zhou Yi, Global classical solutions for quasilinear non-strictly hyperbolic systems, preprint.

THE LIFESPAN OF SOLUTIONS TO QUASILINEAR HYPERBOLIC SYSTEMS IN THE CRITICAL CASE

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In this paper, we consider the existence and blowing up of classical solutions to the following quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \quad (x,t) \in \mathbb{R} \times (0, T(\varepsilon)), \tag{1}$$

$$u(x,0) = \varepsilon\varphi(x), \qquad x \in \mathbb{R}, \tag{2}$$

where $T(\varepsilon)$ stands for the lifespan of C^1 solutions to the Cauchy problem (1.1) and (1.2), $u = {}^t(u_1, u_2)$ and $A(u) = (a_{ij}(u))$ is a 2×2 matrics. We assume that $a_{ij}(u)(i,j = 1,2)$ is C^∞ function of u and $A(u)$ has 2 distinct real eigenvalues $\lambda_1(u) < \lambda_2(u)$ in a neighbourhood of $u = 0$. This assumption means the system (1.1) is strictly hyperbolic. As you see in below $|u|$ stays small as long as classical solutions exist. Thus we only have to make assumption on $A(u)$ near $u = 0$. We let $l_i(u)$ and $r_i(u)$ be left and right eigenvectors corresponding to $\lambda_i(u)$ respectively, *i.e.*,

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad \text{and} \quad A(u)r_i(u) = \lambda_i(u)r_i(u). \tag{3}$$

We note that $\lambda_i(u)$, $l_i(u)$ and $r_i(u)$ are C^∞ functions of u similarly to $a_{ij}(u)$. Without loss of generality, we may assume that

$$l_i(u)r_i(u) = \delta_{ij} \quad (i,j = 1,2) \text{ and } {}^t r_i(u)r_i(u) = 1 \quad (i = 1,2), \tag{4}$$

where δ_{ij} means Kronecker's delta. We also assume that $\varphi(x)$ belongs to $C^1(\mathbb{R})^2$ and satisfies

$$\sup_{x\in\mathbb{R}}\{(1+|x|)^{1+\mu}(|\varphi(x)|+|\varphi'(x)|)\}<\infty, \tag{5}$$

for some constant $\mu > 0$.

In [3] Li Ta-tsien, Zhou Yi and Kong De-xing obtained an upper and lower bound of the lifespan $T(\varepsilon)$ for general systems not only 2×2 systems. To state their results we need the following notation.

Definition. In u-space we construct a curve defined by a solution for the following initial value problem;

$$\begin{aligned}\frac{du(s)}{ds} &= r_i(u(s)) \qquad \text{for small} \quad s, \quad i=1,2,\\ u(0) &= 0.\end{aligned}$$

We call the curve *i-th characteristic trajectory passing through* $u = 0$ and denote $\Gamma_i(0)$.

Then they proved the following. When the system (1) is weakly linearly degenerated, namely when each eigenvalue λ_i is constant along i-th characteristic trajectory, C^1 solutions exist time globally. On the other hand, when the system (1) is not weakly linearly degenerated, if the following value;

$$\alpha = \min\left\{\alpha_j \geqq 0 \,\middle|\, \left.\frac{d^{\alpha_j}\lambda_j(u(s))}{ds^{\alpha_j}}\right|_{s=0} \neq 0 \right\}$$

is finite, they showed

$$c\varepsilon^{-\alpha-1} \leqq T(\varepsilon) \leqq C\varepsilon^{-\alpha-1}$$

for some constant c and C and sufficiently small ε. Then one may be interested in the case the system (1) is not weakly linaerly degenerated and $\alpha = \infty$, which is called *critical case*. The critical case is also studied in scalar case in [3]. They considered the Cauchy problem

$$\frac{\partial u}{\partial t} + \lambda(u)\frac{\partial u}{\partial x} = 0, \quad (x,t) \in \mathbb{R}\times(0,T(\varepsilon)),$$

$$u(x,0) = \varepsilon\varphi(x), \qquad x \in \mathbb{R},$$

where $\lambda'(u) = -\exp(-a(|u|))$ and $a(s) > 0$ tends to infinity monotoneously at origin, for example $1/s^p$ or $(\log s)^2$. They assume $\varphi(x)$ satisfies the condition (5). Then they proved that

$$c_1 \exp(a(c_2\varepsilon)) \leqq T(\varepsilon) \leqq C_1 \exp(a(C_2\varepsilon))$$

for some constant c_1, c_2, C_1 and C_2 and sufficiently small ε.

In this paper we obtain an upper and lower bound of the lifespan $T(\varepsilon)$ analogous to above inequality in critical case for 2×2 system (1) and (2). Precisely speaking, we will prove

Theorem. *Let $u = u^i(s)$ be points on the i-th characteristic trajectory passing through $u = 0$. We assume that $\lambda_i(u^i(s)) \in C^\infty$ and for some $i_0 \in \{1,2\}$, $l_{i_0}(0)\varphi(x) \not\equiv 0$,*

$$-\frac{d\lambda_{i_0}}{ds}(u^{i_0}(s)) = F(s) \equiv \begin{cases} \exp\left(-\frac{1}{a(|s|)}\right) & 0 < |s| \leqq M, \\ 0 & s = 0, \end{cases} \tag{6}$$

and if $i \neq i_0$,

$$\left|\frac{d\lambda_i}{ds}(u^i(s))\right| \leqq F(s) \qquad \text{for} \qquad |s| \leqq M \tag{7}$$

for some $M > 0$, where the function $a(\cdot)$ satisfies

$$a(\cdot) \in C^\infty[0,M], \tag{8}$$

$$a(0) = 0, \tag{9}$$

$$a(\cdot) \text{ increases in } [0,M] \text{ monotonely,} \tag{10}$$

for any A, B and $\mu > 0$ there exists an $\varepsilon_0 > 0$ such that

$$\frac{a(A\varepsilon + B\varepsilon^2)}{a(A\varepsilon - B\varepsilon^2)} \leqq 1 + \mu \qquad \text{for } 0 < \varepsilon \leqq \varepsilon_0. \tag{11}$$

Then there exist an $\varepsilon_1 > 0$ and constants c_1, c_2, C_1, C_2 such that

$$c_1 \exp\left(\frac{1}{a(c_2\varepsilon)}\right) \leqq T(\varepsilon) \leqq C_1 \exp\left(\frac{1}{a(C_2\varepsilon)}\right) \tag{12}$$

holds for $0 < \varepsilon \leqq \varepsilon_1$.

The method of the proof of Theorem follows the one used in [3] essentially. To prove the blowing up part of Theorem we construct an ordinary differential equation with respect to $|\partial u/\partial x|$ using *a priori* estimates of some kinds of L^∞ norms of u and $\partial u/\partial x$. Solving the ordinary differential equation we find $|\partial u/\partial x|$ goes infinity in finite time. Li Ta-tsien, Zhou Yi and Kong De-xing use an weighted L^∞ norm$V_\infty^c(T)$ defined below which is not bounded time globally in general. In noncritical case his solution blows up while an *a priori* estimate of $V_\infty^c(T)$ holds. But in critical case our solution might blow up over the time $V_\infty^c(T)$ stays small. However, fortunately, in 2×2 case we find that $V_\infty^c(T)$ is bounded time globally. This is the reason why we restrict ourselves in 2×2 case.

Typical examples of $a(\cdot)$ satisfying the assumptions (8)-(11), espetially (11), are

$$(A) \qquad a(s) = s^p, \qquad p > 0,$$

$$(B) \qquad a(s) = \frac{1}{\{\log(\frac{1}{s})\}^p}, \qquad p > 0,$$

$$(C) \qquad a(s) = \exp\left(-\frac{1}{s^p}\right), \qquad 0 < p < 1.$$

We verify the above in the end of this paper.

In [5], Li Ta-tsien, Zhou Yi and Kong De-xing treat a non strictly hyperbolic system and obtain a result similar to the one in [3]. They assume

$$\lambda_1(0) = \lambda_2(0) = \cdots = \lambda_p(0) < \lambda_{p+1}(0) < \cdots < \lambda_n(0).$$

The proof of the result goes almost same line with the strictly hyperbolic case. However, the normalized transformation defined in below does not always exist in non strictly hyperbolic case. To realize the existence of the normalized transfomation, they also assume that the i-th characteristic is weakly linearly degenerated for $i = 1, 2, \cdots, p$. Thus if we consider the non strictly hyperbolic case in 2×2 systems, we need that all characteristics are weakly linearly degenerated. This assumption implies that the solution exists time globally and there is no meaning of considering the critical case.

We comment on a scalar equation

$$\frac{\partial u}{\partial x} + \lambda(u)\frac{\partial u}{\partial x} = 0.$$

To obtain the estimate of the lifespan $T(\varepsilon)$ Li Ta-tsien, Zhou Yi and Kong De-xing some tight assumptions on $\lambda(u)$ like (8)-(11). However, if you are concerned in blowing up only, these assumption are unneccesary. In fact, if $\lambda(\cdot)$ is not constant and the initial value $\varphi(\cdot)$ changes its sign, classical solutions do not exist globally, see Fritz John [1].

Now we prove that (A), (B) and (C) above satisfy the assumption (11).

(A) $a(s) = s^p$ for $p > 0$.

By mean value theorem, we have

$$\begin{aligned} a(A\varepsilon + B\varepsilon^2) - a(A\varepsilon - B\varepsilon^2) =& (A\varepsilon + B\varepsilon^2)^p - (A\varepsilon - B\varepsilon^2)^p \\ =& 2Bp\varepsilon^2(A\varepsilon + B\theta\varepsilon^2)^{p-1} \end{aligned}$$

for some $\theta \in [-1, 1]$. Then it follows that

$$\begin{aligned} \frac{a(A\varepsilon + B\varepsilon^2)}{a(A\varepsilon - B\varepsilon^2)} =& 1 + 2Bp\varepsilon^2 \frac{(A\varepsilon + B\theta\varepsilon^2)^{p-1}}{(A\varepsilon - B\varepsilon^2)^2} \\ \leqq& 1 + 2Bp\varepsilon^2 \frac{(2A\varepsilon)^{p-1}}{(\frac{1}{2}A\varepsilon)^p} \\ \leqq& 1 + 2Bp\frac{2}{A}4^p\varepsilon \\ =& 1 + \frac{Bp4^{p+1}}{A}\varepsilon, \end{aligned}$$

where we assume $p \geqq 1$.Thus if we take $\varepsilon_0 < A\mu/Bp4^p$, (11) holds. The case $0 < p < 1$ goes the same line to the above.

(B) $a(s) = \frac{1}{(\log \frac{1}{s})^p}$ for $p > 0$.

We take ε_0 so small that $2A\varepsilon_0 < 1$ *a priori*. Similarly to the above we have

$$\begin{aligned} a(A\varepsilon + B\varepsilon^2) - a(A\varepsilon - B\varepsilon^2) =& \frac{1}{(\log \frac{1}{A\varepsilon + B\varepsilon^2})^p} - \frac{1}{(\log \frac{1}{A\varepsilon - B\varepsilon^2})^p} \\ =& 2Bp\varepsilon^2 \frac{1}{(\log \frac{1}{A\varepsilon + B\theta\varepsilon^2})^{p+1}} \left(\frac{1}{A\varepsilon + B\theta\varepsilon^2}\right) \end{aligned}$$

for some $\theta \in [-1, 1]$. Then we find that

$$\begin{aligned}\frac{a(A\varepsilon + B\varepsilon^2)}{a(A\varepsilon - B\varepsilon^2)} &= 1 + 2Bp\varepsilon^2 \left(\frac{\log \frac{1}{A\varepsilon - B\varepsilon^2}}{\log \frac{1}{A\varepsilon + B\theta\varepsilon^2}}\right)^p \frac{1}{\log \frac{1}{A\varepsilon + B\theta\varepsilon^2}} \frac{1}{A\varepsilon + B\theta\varepsilon^2} \\ &\leqq 1 + 2Bp\varepsilon^2 \left(\frac{\log \frac{2}{A\varepsilon}}{\log \frac{1}{2A\varepsilon}}\right)^p \frac{1}{\log \frac{2}{A\varepsilon}} \frac{2}{A\varepsilon} \\ &\leqq 1 + \frac{Bp2^{2+p}}{A}\varepsilon.\end{aligned}$$

Therefore we should put $\varepsilon_0 < A\mu/Bp2^{2+p}$ to make $a(s)$ satisfy (11).

(C) $a(s) = \exp\left(-\frac{1}{s^p}\right)$ for $0 < p < 1$.
The argument similar to the above implies

$$\begin{aligned}&a(A\varepsilon + B\varepsilon^2) - a(A\varepsilon - B\varepsilon^2) \\ &= \exp\left(-\frac{1}{(A\varepsilon + B\varepsilon^2)^p}\right) - \exp\left(-\frac{1}{(A\varepsilon - B\varepsilon^2)^p}\right) \\ &= 2Bp\varepsilon^2 \frac{1}{(A\varepsilon + B\theta\varepsilon^2)^{1+p}} \exp\left(-\frac{1}{(A\varepsilon + B\theta\varepsilon^2)^p}\right)\end{aligned}$$

for some $\theta \in [-1, 1]$. Since $p < 1$, we obtain

$$\begin{aligned}&\frac{a(A\varepsilon + B\varepsilon^2)}{a(A\varepsilon - B\varepsilon^2)} \\ &= 1 + 2Bp\varepsilon^2 \frac{1}{(A\varepsilon + B\theta\varepsilon^2)^{1+p}} \exp\left(\frac{1}{(A\varepsilon - B\varepsilon^2)^p} - \frac{1}{(A\varepsilon + B\theta\varepsilon^2)^p}\right) \\ &\leqq 1 + 2Bp\varepsilon^2 \frac{1}{(A\varepsilon - B\varepsilon^2)^{1+p}} \exp\left(\frac{1}{(A\varepsilon - B\varepsilon^2)^p} - \frac{1}{(A\varepsilon + B\varepsilon^2)^p}\right) \\ &= 1 + 2Bp\varepsilon^2 \frac{1}{(A\varepsilon - B\varepsilon^2)^{1+p}} \exp\left(\frac{(A\varepsilon + B\varepsilon^2)^p - (A\varepsilon - B\varepsilon^2)^p}{(A^2\varepsilon^2 - B^2\varepsilon^4)^p}\right) \\ &\leqq 1 + 2Bp\varepsilon^2 \left(\frac{2}{A\varepsilon}\right)^{1+p} \exp\left(\frac{c_2\varepsilon^{1+p}}{c_1\varepsilon^{2p}}\right) \\ &\leqq 1 + 2Bp\left(\frac{2}{A}\right)^{1+p} \varepsilon^{1-p} \exp\left(\frac{c_2}{c_1}\varepsilon^{1-p}\right).\end{aligned}$$

Thus if we take ε_0 satisfying

$$2Bp\left(\frac{2}{A}\right)^{1+p}\varepsilon_0^{1-p}\exp\left(\frac{c_2}{c_1}\varepsilon_0^{1-p}\right) \leqq \mu,$$

we know that (11) holds.

REFERENCES

[1] F. John, *Nonlinear wave equations, formations of singularities*, Pitcher Lectuers in the Math. Sci., Amer. Math. Soc., 1989.

[2] F. John, *Formation of singularities in one-dimensional nonlinear wave properga-tion*, Comm. Pure Appli. Math. **27** (1974), 377-405.

[3] Li Ta-tsien, Zhou Yi and Kong De-xing, *Global classical solutions for general quasilinear hyperbolic systems with dacay initial data*, to appear, Jour. Math. Pure and Appli..

[4] Li Ta-tsien, Zhou Yi and Kong De-xing, *Weak linear degeneracy and global classical solutions for general quasilinear hyperbolic systems*, Comm. in P.D.E. **19** (1994), 1263-1317.

[5] Li Ta-tsien, Zhou Yi and Kong De-xing, *Global classical solutions for quasilinear non-strictly hyperbolic systems*, to appear.

[6] Li Ta-tsien and Yu Wen-ci, *Boundary Value Problems for Quasilinear Hyperbolic Systems*, DukeUniversity Mathematics Series V (1985).

非線型波動方程式のソボレフ空間 $\dot{H}^{s+\frac{1}{2}}$での大域解の構成について *

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この報告には,非線形波動方程式の大域解の構成を Besov 空間を用いて行なう過程について書かれています。証明は紙面の都合上省略します。私のとった大域解の構成法は、基本解の評価とソボレフの不等式から,Besov 空間を使うことによってかなり機械的に行なえるのが特徴です.

発展方程式のセミナーでは新しい結果として発表しましたが、その後, 9 5年の Lindblad, Sogge [3] の論文の中に一つの結果として私の発表したものと同じ結果が記載されていました。その論文をまだ深く読んではいないのですが、方法として Besov空間は使っていないようです。 Besov 空間を使う利点は,誰もがそれを使えば簡単にこの問題を直観的に解けることにあります。特に,非線形項の評価が容易になることです。今回報告する結果は、既に得られている結果ではありましたが,それを容易にするものです。

講演の後に,何名かの方から Reference を E-mailで送るように頼まれました。遅くなりましたがその方々全員にお送りしましたが、本当に送れているのかわかりません。もし、送られてこなかった方がいらっしゃいましたら,この場を借りてお詫び致します。

最後に、現在は inhomogeneous Besov space を使って,局所解の構成を研究しています。何か参考になることがありましたら,是非お教え下さい。

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1 はじめに

非線型波動方程式の大域解を、ソボレフ空間 $\dot{H}^{s+\frac{1}{2}}$ （$s+\frac{1}{2}$は便宜上のものです） で構成しようというのが目標です。

よく知られているように、非線型波動方程式は、

$$V(t)=F^{-1}\cos t|\cdot|F,\ U(t)=F^{-1}\frac{\sin t|\cdot|}{|\cdot|}F,\ \Gamma h(t)=\int_0^t U(t-\tau)h(\tau)d\tau$$

とおけば、

*supported by JSPS

$$(NLW) \quad u(t) = V(t)\phi + U(t)\psi + \Gamma(f(u))(t),\ u(0) = \phi \in \dot{H}^{s+\frac{1}{2}}\ ,\ \ u_t(0) = \psi \in \dot{H}^{s-\frac{1}{2}}$$

と書けます。ここで、$f(u)$ については、簡単のため、$f(u) = |u|^{p-1}u$ つまり、シングルパワーと考えていただいて結構です。示すべきことは $\Phi(u)$ を

$$\Phi(u) := \ V\phi + U\psi + \Gamma f(u)$$

とおいた時に、Φ がある完備距離空間上の時間に依らない縮小写像であることを示すことです。
問題となるのはこの完備距離空間の見つけ方になるのですが、これに対して次の二つの定理が非常に有用となります。

Theorem 1 (due to Pecher,Theorem 1.1) *Assume* $2 \le q < \infty$, $\frac{1}{2} - \frac{\nu}{n+1} \le \frac{1}{q} \le \frac{1}{2} - \frac{\nu}{2n}$. *For* $t \neq 0$, *the following estimate holds.*

$$\|I^{-\nu}V(t)g\|_q \le c|t|^{\nu - n(q-2)/q}\|g\|_{q'} \quad (\frac{1}{q'} = 1 - \frac{1}{q}) \ .$$

(上の定理は、今後 $\nu = 1$ の場合しか使っていません。)

Theorem 2 (due to Hardy ,Littlewood,Pólya, *Inequalities p290* **)** *Suppose that* $1 < p < q < \infty$, $\alpha = \frac{1}{p} - \frac{1}{q}$ *and h is non-negative in* $(0,T)$, *then the following inequality holds.*

$$\| \int_0^t (t-\tau)^{\alpha-1}h(\tau)d\tau; L_t^q(0,T)\| \le c\|h; L^p(0,T)\|$$

この二つの定理を使うと、Γ について次の挙動がわかります。

Proposition 1 *Let* q, r *be* $\frac{1}{2} - \frac{1}{n+1} \le \frac{1}{q} \le \frac{1}{2} - \frac{1}{2n}$, $\frac{1}{r} = \frac{n-1}{2} - \frac{n}{q}$ *.Then the following inequality holds.*

$$\|\Gamma h; L^r(0,T;L^q)\| \le c\|h; L^{r'}(0,T;L^{q'})\|$$

Here c is independent of T.

ここで、

$$l_0 = \{(\frac{1}{q}, \frac{1}{r}) \mid \frac{1}{2} - \frac{1}{n+1} \le \frac{1}{q} \le \frac{1}{2n}, \frac{1}{r} = \frac{n-1}{2} - \frac{n}{q}\}$$
$$l_0' = \{(\frac{1}{q'}, \frac{1}{r'}) \mid \frac{1}{q} + \frac{1}{q'} = 1, \frac{1}{r} + \frac{1}{r'} = 1, (\frac{1}{q}, \frac{1}{r}) \in l_0\}$$

とおくと、上の**Proposition**は、Γ が l_0'上の任意の点を l_0 上のその共役な点に、時間に関係なく移すことをいっていますが、空間の指数 q は 2 に取れないので、Hilbert 空間解 ($\dot{H}^{s+\frac{1}{2}}$) を構成することができません。
そこで、次のように上の line を修正して線形評価を得ます。この時、大域解の構成については、有界定数が時間によらないことが必要なので、修正した分を空間の微分指数にまわして処理しています。

2 線形評価

Proposition 2 (homogeneous Besov space version) *For any s ($s \in \mathbf{R}$), θ, θ_1 ($0 \le \theta, \theta_1 \le 1$), V, U, Γ are bounded operators on $\dot{H}^{s+\frac{1}{2}}, \dot{H}^{s-\frac{1}{2}}, L^{r'_\theta}(0,T;\dot{B}^{s+s'_{\theta_1}}_{q'_{\theta_1}})$ to $L^{r_\theta}(0,T;\dot{B}^{s+s_\theta}_{q_\theta})$ respectively and its bounded constant is independent of T and $V\phi, U\psi, \Gamma h \in C([0,T];\dot{H}^{s+\frac{1}{2}})$ for any $\phi \in \dot{H}^{s+\frac{1}{2}}$, $\psi \in \dot{H}^{s-\frac{1}{2}}$, $h \in L^{r'_{\theta_1}}(0,T;\dot{B}^{s+s'_{\theta_1}}_{q'_{\theta_1}})$ respectively.*

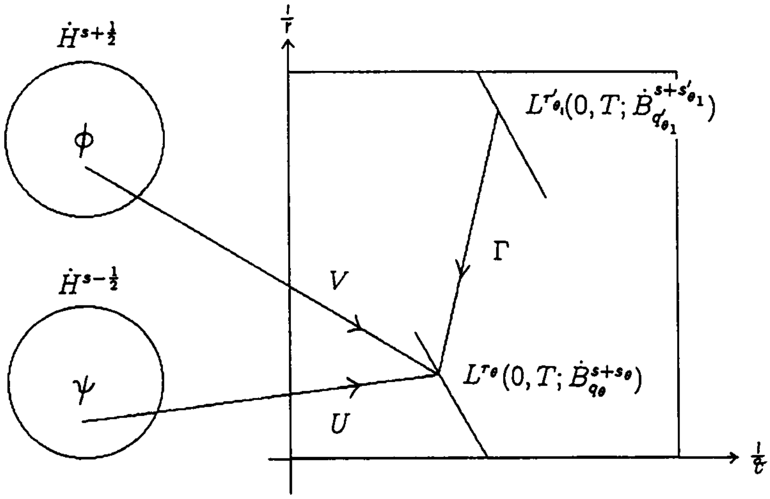
ここで, $\frac{1}{q_\theta}, \frac{1}{r_\theta}, s_\theta$ は $\frac{1}{q}$ と $\frac{1}{2}$, $\frac{1}{r}$ と 0, s と 0 とを θ と $1-\theta$ に内分する点です。θ_1についても同様です。
以上で線形評価は終了です。ここで、

$$l = \{(\frac{1}{q}, \frac{1}{r}) \mid \frac{1}{2} - \frac{1}{n+1} \le \frac{1}{q} \le \frac{1}{2}, \frac{1}{r} = -\frac{n-1}{2}\frac{1}{q} + \frac{n-1}{4}\}$$

$$l' = \{(\frac{1}{q'}, \frac{1}{r'}) \mid \frac{1}{q} + \frac{1}{q'} = 1, \frac{1}{r} + \frac{1}{r'} = 1, (\frac{1}{q}, \frac{1}{r}) \in l\}$$

とおくと、上の**Proposition**は、Γ が l'上の任意の点を l 上の任意の点に時間に関係なく移すことを言っています。
この **Proposition** により,$\Phi(u)$ において、ϕ, ψ は V, U によって l 上の任意の点に移されることがわかりました。このことから、Φ を縮小写像にするには、完備距離空間を次のようにすればよいとの推測が立ちます。

$$\begin{aligned}\mathcal{X}(T,R) &= \{u \in L^{r_\theta}(0,T;\dot{B}^{s+s_\theta}_{q_\theta}) \mid \|u; L^{r_\theta}(0,T;\dot{B}^{s+s_\theta}_{q_\theta})\| \le R\} \\ d(u,v) &= \|u-v\|_{\mathcal{X}}\ , \quad \|\cdot\|_{\mathcal{X}} = \|\cdot; L^{r_\theta}(0,T;\dot{B}^{s+s_\theta}_{q_\theta})\|.\end{aligned}$$



3 非線形項の評価

Γ は l' を l に移しますから、次の問題は θ, θ_1 をうまく調整することによって、$f(u)$ を l' 上の点に乗せれるかが問題となります。ここで、非線形項の評価が問題になり、その評価を簡単に行うためにBesov空間を使います。

Definition 1 *We say $f \in \Omega_{s,p}$, if f satisfy $f \in C^{[s]}(\mathbf{R}^2, \mathbf{R}), f(0) = \ldots = f^{([s])}(0) = 0$,*

$$|f^{([s])}(z) - f^{([s])}(w)| \leq \begin{cases} c(|z|^{p-[s]-1} + |w|^{p-[s]-1})|z-w| & (p \geq [s]+1)\ , \\ c|z-w|^{p-[s]} & (p < [s]+1)\ . \end{cases}$$

Theorem 3 $(0 < s)$ *Let s, p, f be $0 < s,\ 1 \leq p,\ s < p,\ f \in \Omega_{s,p}$ and l, q, r be $1 \leq l \leq \infty, 2 \leq q, r < \infty$.*
Under these assumptions, if l, q, r, p satisfy $\frac{1}{l} = \frac{p-1}{q} + \frac{1}{r}$, then

$$\|f(u)\|_{\dot{B}^s_l} \leq c\|u\|^{p-1}_{\dot{B}^0_q}\|u\|_{\dot{B}^s_r} \quad .$$

この定理を証明するには次のノルムの別表現が有用です(簡易版です)。

$$\forall s > 0, \quad \|u; \dot{B}^s_q\| \sim \{\int_0^\infty (t^{[s]-s} \sup_{|y|<t} \|\partial^{[s]}u(\cdot) - \partial^{[s]}u(\cdot + y)\|_q)^2 \frac{dt}{t}\}^{\frac{1}{2}}$$

以上で、非線形項の評価は終了です。

4 完備距離空間

次に行うことは、Φ が $(\mathcal{X}, d)$ 上の縮小写像であるかを調べることです。線形評価と非線型項評価から次の計算ができます。

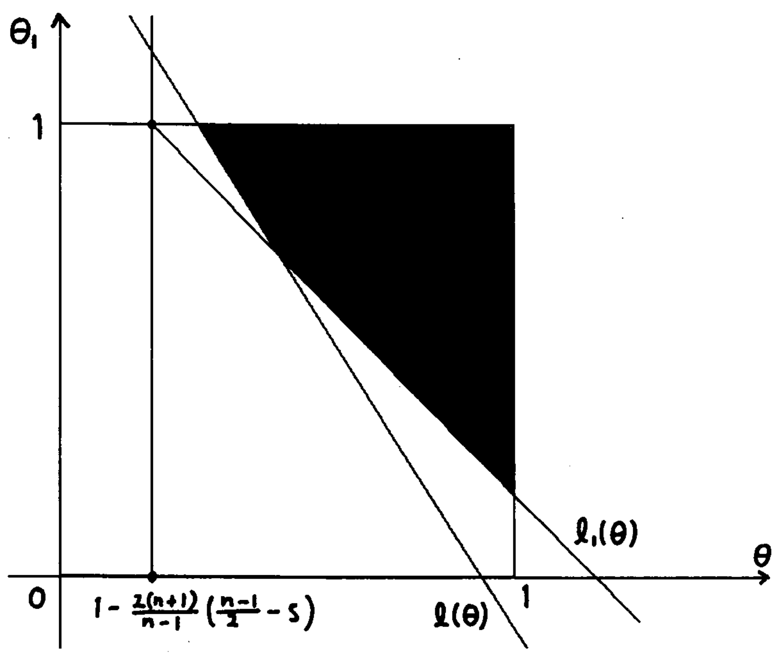
$$\begin{aligned}
\|\Phi(u)\|_{\mathcal{X}} &= \|V\phi + U\psi + \Gamma(f(u)); L^{r_\theta}(0,T; \dot{B}^{s+s_\theta}_{q_\theta})\| \\
&\leq c\|\phi\|_{\dot{H}^{s+\frac{1}{2}}} + c\|\psi\|_{\dot{H}^{s-\frac{1}{2}}} + c\|f(u); L^{r'_{\theta_1}}(0,T; \dot{B}^{s+s'_{\theta_1}}_{q'_{\theta_1}})\| \\
&\leq c\|\phi\|_{\dot{H}^{s+\frac{1}{2}}} + c\|\psi\|_{\dot{H}^{s-\frac{1}{2}}} + c\|\|u(t)\|^p_{\dot{B}^{s+s'_{\theta_1}}_{\tilde{l}_2}}; L^{r'_{\theta_1}}_t(0,T)\| \\
&\leq c\|\phi\|_{\dot{H}^{s+\frac{1}{2}}} + c\|\psi\|_{\dot{H}^{s-\frac{1}{2}}} + cT^{\frac{1}{r'_{\theta_1}} - \frac{p}{r_\theta}}\|u\|^p_{\mathcal{X}} \\
d(\Phi(u), \Phi(v)) &= \|\Gamma(f(u) - f(v)); L^{r_\theta}(0,T; \dot{B}^{s+s_\theta}_{q_\theta})\| \\
&\leq c\|f(u) - f(v); L^{r'_{\theta_1}}(0,T; \dot{B}^{s+s'_{\theta_1}}_{q'_{\theta_1}})\|
\end{aligned}$$

$$
\begin{aligned}
&\leq c\|(\|u(t)\|^{p-1}_{\dot{B}^{s+s'_{\theta_1}}_{\tilde{l}_2}} + \|v(t)\|^{p-1}_{\dot{B}^{s+s'_{\theta_1}}_{\tilde{l}_2}})\|u-v\|_{\dot{B}^{s+s'_{\theta_1}}_{\tilde{l}_2}} ; L_t^{r'_{\theta_1}}(0,T)\| \\
&\leq cT^{\frac{1}{r'_{\theta_1}}-\frac{p}{r_\theta}}(\|u\|^{p-1}_{\mathcal{X}} + \|v\|^{p-1}_{\mathcal{X}})d(u,v)
\end{aligned}
$$

以上の計算から、Φ が $(\mathcal{X},d)$ 上の縮小写像であるためには、θ,θ_1 が次の条件を満たせばよいことがわかります。

$$
\begin{aligned}
&l(\theta) := (\frac{n-1}{2}-s)^{-1}\{-(\frac{n+3}{2}-s)\theta + \frac{4s}{n-1}\} \\
&l_1(\theta) := -\frac{n-1}{2n}\theta + \frac{-n(n-3)+2(n+1)s}{2n} \\
&\theta > 1 - \frac{2(n+1)}{n-1}(\frac{n-1}{2}-s),\ \theta_1 < 2s,\ \theta_1 \geq l(\theta),\ \theta_1 \geq l_1(\theta) \\
&p = (\frac{1}{q_\theta} - \frac{s+s_\theta}{n})^{-1}(\frac{1}{q'_{\theta_1}} - \frac{s+s'_{\theta_1}}{n})
\end{aligned}
$$

これを図に表示したものから、特に $(\theta,\theta_1) = (l^{-1}(1),1)$（$l$ 上では T の指数は 0 となります）の場合には、次の二つの Corollary が得られることになります。



5 結果

Corollary 1 $(\theta_1 = 1)$ *Let* n, s, p, f *be* $n \geq 3$, $\frac{1}{2} < s < \frac{n-1}{2}$, $1 + (\frac{n-1}{2} - s)^{-1} \leq p \leq 1 + 2(\frac{n-1}{2} - s)^{-1}$, $2 \leq p, s + \frac{1}{2} < p$, $f \in \Omega_{s,p}$, *Under these assumptions, for any* $\phi \in \dot{H}^{s+\frac{1}{2}}$, $\psi \in \dot{H}^{s-\frac{1}{2}}$, (NLW) *has a local solution in* $C([0,T]; \dot{H}^{s+\frac{1}{2}})$.
And moreover if $p = 1 + 2(\frac{n-1}{2} - s)^{-1}$ *and* $\|\phi\|_{\dot{H}^{s+\frac{1}{2}}}$, $\|\psi\|_{\dot{H}^{s-\frac{1}{2}}}$ *are sufficiently small, then* (NLW) *has a global solution in* $L^\infty(0,\infty; \dot{H}^{s+\frac{1}{2}}) \cap C([0,\infty); \dot{H}^{s+\frac{1}{2}})$.

Corollary 2 *Let* n, p *be* $3 \leq n \leq 7$, $\max(2, 1 + \frac{4}{n-2}) < p < \infty$.
If s, f *satisfy* $s = \frac{n-1}{2} - \frac{2}{p-1}$, $f \in \Omega_{s,p}$, *then for any* ϕ, $\psi \in \mathcal{S}$ *such that* $\|\phi\|_{\dot{H}^{s+\frac{1}{2}}}$, $\|\psi\|_{\dot{H}^{s-\frac{1}{2}}}$ *are sufficiently small* (NLW) *has a global solution in* $L^\infty(0,\infty; \dot{H}^{s+\frac{1}{2}}) \cap C([0,\infty); \dot{H}^{s+\frac{1}{2}})$.

最初の Corollary は、$\dot{H}^{s+\frac{1}{2}}$ 解で大域解を求めるときに使い、2番目のものは非常に大きいに p 対して大域解を求めるときに使えます。

以上、大雑把に大域解の構成について述べましたが、もう少し細かく計算すると、局所解も構成できます。しかしこれも Lindblad,Sogge [3] によって得られている結果の一つでした。

最後に注意として、Besov 空間について述べると、Besov 空間はソボレフ空間と補間関係で結ばれているわけですから、ソボレフ空間だけを用いて行うこともできると思います。実際、加藤先生の論文 [5] ではソボレフ空間を用いた方が適当であるとも書いています。私自身はどちらがどう良いのかまだわかってはいません (知っている方がいらっしゃいましたら、教えてください)。

参考文献

[1] J.Bergh and J.Löfström, *Interpolation Spaces*, Springer,Berlin,(1976).

[2] H.Pecher, ***Local Solutions of Semilinear Wave Equations in*** H^{s+1}, Mathematical Methods in the Applied Sciences, Vol 19, pp.145-170,(1996).

[3] H.Lindblad and C.D. Sogge, ***On Existence and Scattering with Minimal Regularity for Semilinear Wave Equaions***, JOURNAL OF FUNCTIONAL ANALYSIS **130** ,pp.357-426,(1995).

[4] T.Kato, ***Nonlinear Schrödinger equations***, Lecture Notes in Physics, Springer,Berlin, Vol.345,pp.218-263 ,(1989).

[5] T.Kato, *ON NONLINEAR SCHRÖDINGER EQUATIONS, II. H^s-SOLUTIONS AND UNCONDITIONAL WELL-POSEDNESS*, JURNAL D'ANALYSE MATHÉMATIQUE,Vol.67,pp.281-306,(1995).

[6] T.Cazenave and F.B.Weissler *THE CAUCHY PROBLEM FOR THE CRITICAL NONLINEAR SCHRÖDINGER EQUATION IN* H^s, Nonlinear Analysis,Theory,Methods & Applications,Vol.14, No.10,pp.807-836 ,(1990).

抽象的放物型方程式の anti-periodic 解の一意性について

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1 Introduction 実 Hilbert 空間 H に於いて, 時間に依存する非線形作用素 $\partial\varphi^t$ に支配される次の非線形抽象発展方程式

$$\text{(E)} \qquad \frac{du}{dt}(t) + \partial\varphi^t(u(t)) \ni f(t), \quad t \in [0,T].$$

の anti-periodic 問題 $u(0) = -u(T)$ の解の存在について得られた結果について報告する. ここで $\partial\varphi^t(\cdot)$ は時間依存性を有する下半連続凸関数 $\varphi^t : H \to (-\infty, +\infty]$ の劣微分作用素である. 劣微分 $\partial\varphi$ とは適正 $(\varphi \not\equiv +\infty)$ 下半連続凸関数 $\varphi : H \to (-\infty, +\infty]$ に対して

Definition $f \in \partial\varphi(u) \Leftrightarrow \varphi(v) - \varphi(u) \geq (f, v-u) \quad {}^{\forall}v \in H$

で定義される 非線形作用素で一般的には多価であるが, Fréchet 微分の一般化になっている. Anti-periodic 問題とは, Anti-periodic 関数の定義を

Definition $u(t)$ is T-anti-periodic $\Leftrightarrow \quad u(t+T) = -u(t) \quad$ a.e. $t \in \mathbf{R}$

とするとき, 方程式 (E) の anti-periodic な解を求める問題をいう.

Remark $u(t)$ is T-anti-periodic
$\Rightarrow \ u(t)$ is $2T$-periodic, and $\displaystyle\int_t^{t+2T} u(\tau)d\tau = 0, \ {}^{\forall}t \in \mathbf{R}$
Example $u(t) = \sin(\frac{\pi}{T}t)$ is T-anti-periodic

Anti-periodic 問題の特徴としては, coerciveness と evenness の定義を

Definition $\varphi^t(\cdot)$ is coercive
$\Leftrightarrow \displaystyle\liminf_{|u|\to+\infty} \frac{\varphi^t(u)}{|u|} = l(t), \text{ with } \int_0^T l(t)dt = +\infty.$
Definition $\varphi^t(\cdot)$ is even $\Leftrightarrow \varphi^t(-u) = \varphi^t(u) \quad {}^{\forall}u \in H \quad {}^{\forall}t \in [0,T].$

とする時, periodic 問題では, 解の存在を示すのに, $\varphi^t(\cdot)$ の coerciveness

を仮定することが本質的に必要なのに対し, anti-periodic 問題では, 必ずしも$\varphi^t(\cdot)$ が coercive でなくても, $\varphi^t(\cdot)$ が even であることを仮定するだけで, 解の存在を示せる事にある.

2 Cauchy problem, periodic problem, anti-periodic problem で今まで得られている結果の概説

Cauchy Problem に関しては, 次のφ^t の t に関する滑らかさの条件

$$
\begin{aligned}
(\mathrm{A}.\ \varphi^t)\ :\ & {}^{\exists}m_1, m_2 > 0 \ \text{ such that }\ {}^{\forall}t_0 \in [0,T],\ {}^{\forall}x_0 \in D(\partial\varphi^{t_0}), \\
& {}^{\exists}x(t) : [0,T] \to H \ \text{ such that} \\
& (1)\quad |x(t) - x_0| \le m_1|t - t_0|(\varphi^{t_0}(x_0) + 1)^{1/2} \qquad {}^{\forall}t \in [0,T], \\
& (2)\quad \varphi^t(x(t)) - \varphi^{t_0}(x_0) \le m_2|t - t_0|(\varphi^{t_0}(x_0) + 1) \quad {}^{\forall}t \in [0,T].
\end{aligned}
$$

のもとで, Kenmochi [2], [3], Yamada [9], Ôtani [8] などによって解の存在が示されている.

Periodic Problem に関しては, $(\mathrm{A}.\ \varphi^t)$ + 「$\varphi^0 = \varphi^T$」 + 「φ^t の coerciveness 」のもとで, Nagai [5], Kenmochi [2], Yamada [10] などによって解の存在が示されている.

Anti-periodic Problem に関しては, $(\mathrm{A}.\ \varphi^t)$ + 「$\varphi^0 = \varphi^T$」 + 「φ^t の evenness 」のもとで, 最初, Okochi [6] によって $\varphi^t \equiv \varphi$, の場合について解の存在が示され, φ^tが t に依存する場合については, Okochi [7] により, ある条件の下で解の存在が示されている.

3 主結果 我々の結果は, [7] で仮定された条件が無くても, 本質的に $(\mathrm{A}.\ \varphi^t)$ + 「$\varphi^0 = \varphi^T$」 + 「φ^t:evenness 」だけで, (E) の anti-periodic 解の存在を示すことが出来たというものである.

Theorem 1 $(\mathrm{A}.\ \varphi^t)$ のもとで, $\varphi^0 = \varphi^T$, φ^t : even ならば, (E) の anti-periodic 解が存在する. このときさらに $u \in W^{1,2}(0,T;H)$ かつ$\varphi^t(u(t))$ は $[0,T]$ で絶対連続である.

Anti-Periodic 解の一意性は, 周期解の一意性を保証する条件よりも弱い条件のもとで保証される. すなわち任意の 2 つの周期解の差が常に定ベクトルであれば, Anti-Periodic 解は一意的である. この為の十分条件は例

えば, 次の定理で示される (c.f. Kenmochi–Ôtani [4]).

Theorem 2 次の (0), (i), (ii) のいずれかの条件を満たすとき, (E) の解は一意的である.

(0) $\varphi^t(u) = \varphi(u)\ \ ^\forall t \in [0,T]\ \ ^\forall u \in H$.

(i) ある $t_0 \in [0,T]$ に対してφ^{t_0}が $D(\varphi^{t_0})$ 上で狭義凸である.

(ii) (E) のすべての解 u に対して$-\frac{du}{dt}(t) = (\partial\varphi^t(u(t)) - f(t))^0$ a.e. $t \in [0,T]$ が成立する.

ここで, 方程式 (E) に対し次のような解の class を導入する.

$\mathcal{F}^{2\tau} \overset{\text{def}}{=} \Big\{u \in C(\mathbf{R}; H) \text{ such that } u(t+2\tau) = u(t),\ ^\forall t \in \mathbf{R}\Big\}$,
$\mathcal{F}^{\tau}_A \overset{\text{def}}{=} \Big\{u \in C(\mathbf{R}; H) \text{ such that } u(t+\tau) = -u(t),\ ^\forall t \in \mathbf{R}\Big\}$,
$\mathcal{F}^{\tau} \overset{\text{def}}{=} \Big\{u \in C(\mathbf{R}; H) \text{ such that } u(t+\tau) = u(t),\ ^\forall t \in \mathbf{R}\Big\}$,
$\mathcal{P}^{2\tau} \overset{\text{def}}{=} \Big\{u : \text{ solution of (E) such that } u(t+2\tau) = u(t),\ ^\forall t \in \mathbf{R}\Big\}$,
$\mathcal{P}^{\tau}_A \overset{\text{def}}{=} \Big\{u : \text{ solution of (E) such that } u(t+\tau) = -u(t),\ ^\forall t \in \mathbf{R}\Big\}$,
$\mathcal{P}^{\tau} \overset{\text{def}}{=} \Big\{u : \text{ solution of (E) such that } u(t+\tau) = u(t),\ ^\forall t \in \mathbf{R}\Big\}$,
$\mathcal{F}^{\tau}_A(t) \overset{\text{def}}{=} \{u(t) : u \in \mathcal{F}^{\tau}_A\}, \quad \mathcal{F}^{\tau}(t) \overset{\text{def}}{=} \{u(t) : u \in \mathcal{F}^{\tau}\}, \quad ^\forall t \in \mathbf{R}$.
$\mathcal{P}^{\tau}_A(t) \overset{\text{def}}{=} \{u(t) : u \in \mathcal{P}^{\tau}_A\}, \quad \mathcal{P}^{\tau}(t) \overset{\text{def}}{=} \{u(t) : u \in \mathcal{P}^{\tau}\}, \quad ^\forall t \in \mathbf{R}$.

Remark 1

(i) In general, Anti–periodic solution is not unique.

(ii) $\mathcal{P}^T_A \subset \mathcal{P}^{2T}$.

(iii) $\mathcal{P}^T_A \neq \phi \Longleftrightarrow \mathcal{P}^{2T} \neq \phi$.

$\mathcal{P}^{2T}$, $\mathcal{P}^T_A$, $\mathcal{P}^T$ に対して次の定理が成立する.

Theorem 3 $\mathcal{P}^{2T} = \mathcal{P}^T_A \oplus \mathcal{F}^T$, i.e.,
$\forall u \in \mathcal{P}^{2T},\ \exists w \in \mathcal{P}^T_A,\ \exists v \in \mathcal{F}^T,$ such that $u = w + v$,
$w(t) = \mathrm{Proj}_{\mathcal{P}^T_A(t)} u(t) = \frac{u(t) - u(t+T)}{2}, \quad ^\forall u \in \mathcal{P}^{2T}, \quad ^\forall t \in \mathbf{R}$,
$\exists u^* \in \mathcal{P}^{2T}$, such that $w = \frac{u+u^*}{2}$.

したがって, 次の $\mathcal{P}^{2T}$ の一意分解が成立する.

Corollary 2 $^\forall u \in \mathcal{P}^{2T},\ ^\exists/w \in \mathcal{P}^T_A,\ ^\exists/v$: T-periodic $\in C([0,T]; H)$

such that $u = w + v$.

$\mathcal{P}^{2T} = \mathcal{P}_A^T$となる為の必要十分条件として次の系が成立する.

Corollary 3 $\mathcal{P}^{2T} = \mathcal{P}_A^T \iff \mathcal{P}_A^T(0)$ has an interior point in $\mathcal{P}_{2T}(0)$.

T-anti-periodic 解の一意性の必要十分条件としては次の系が成立する.

Corollary 4 $\mathcal{P}_A^T = \{w_0\} \iff {}^\forall u_1, u_2 \in \mathcal{P}_{2T}\ \ u_1 - u_2$ is T-periodic.

4 Theorem 1 の証明の概略. まず最初に,

Lemma 0 (A. φ^t), φ^t is even ならば, 一般性を失わずに, $\varphi^t(u) \geq 0$, $\varphi^t(0) \leq C$, ${}^\forall u \in H$, ${}^\forall t \in [0,T]$ とすることが出来る.

が成り立つので, φ^tはそのようなものとする. そして, 次の近似方程式を導入する.

$$(\mathrm{E})_\varepsilon \begin{cases} \dfrac{du_\varepsilon}{dt}(t) + \partial\varphi^t(u_\varepsilon(t)) + \varepsilon u_\varepsilon(t) \ni f(t), \\ u_\varepsilon(0) = -u_\varepsilon(T) \end{cases}$$

$\partial\varphi^t(u) + \varepsilon u$ に対応する functional ϕ_ε^t $(\partial\phi_\varepsilon^t(u) = \partial\varphi^t(u) + \varepsilon u)$ は coercive であるので, $(\mathrm{E})_\varepsilon$の Cauchy 問題の解 $v_\varepsilon(t)$ は t に関して有界になる. したがって, periodic 問題と同様にして, $(\mathrm{E})_\varepsilon$の anti-Periodic 問題の解の存在がいえる. その一意性はϕ_ε^tが strictly convex である事より保証される.

アプリオリ評価 **(I)** : まず, $(\mathrm{E})_\varepsilon$ に $\frac{du_\varepsilon}{dt}$ を掛ける. そして, 次の

Lemma 1 (A. φ^t) $\Rightarrow$

$$|\frac{d}{dt}\varphi^t(u(t)) - (\partial\varphi^t(u(t)), \frac{du}{dt}(t))|$$
$$\leq m_1|\partial\varphi^t(u(t))|(\varphi^t(u(t)) + 1)^{1/2} + m_2(\varphi^t(u(t)) + 1), \quad {}^\forall t \in [0,T]$$

と, $\varphi^t(u_\varepsilon(t))$, $|u_\varepsilon(t)|^2$ が T-periodic である事に注意して, $[0,T]$ で積分すると, 次の式を得る.

$$(\mathrm{A})\quad \frac{1}{2}\int_0^T |\frac{du_\varepsilon}{dt}(t)|^2 dt \leq C\Big(\int_0^T |f(t)|^2 dt + \int_0^T (\varphi^t(u_\varepsilon(t)) + 1)dt\Big) + \varepsilon^2 \int_0^T |u_\varepsilon(t)|^2 dt$$

アプリオリ評価 **(II)** : $(\mathrm{E})_\varepsilon$に u_εを掛ける. そして, ${}^\forall\delta > 0$ を固定

する. $\varphi^t(0) \le C,\ \forall t \in [0,T]$ に注意して, $[0,T]$ で積分すると, 次の式を得る.

$$\text{(B)} \quad \int_0^T \varphi^t(u_\varepsilon(t))dt + 2\varepsilon \int_0^T |u_\varepsilon(t)|^2 dt \le \delta \int_0^T |u_\varepsilon(t)|^2 dt + C_\delta \int_0^T (|f(t)|^2+1)dt$$

ここで, Haraux [1] によって示された次の不等式

Lemma 2 $\quad u(0) = -u(T),\ \ u \in W^{1,2}(0,T;H)$
$\Rightarrow |u|_{L^\infty(0,T;H)} \le T^{1/2}|\frac{du}{dt}|_{L^2(0,T;H)}$

を用いて, (A) と (B) を辺々をたせば, $0 < \varepsilon \le 1$ に対し,

$$(\frac{1}{2} - \delta T^2) \int_0^T |\frac{du_\varepsilon}{dt}(t)|^2 dt \le C'_\delta \int_0^T (|f(t)|^2 + 1)dt$$

を得る. よって, $\left\{\frac{du_\varepsilon}{dt}\right\}$ は $L^2(0,T;H)$ で有界となり, もう一度 Lemma 2 を用いれば, $\{u_\varepsilon\}$ は $L^\infty(0,T;H)$ で有界となる. したがって, 適当な部分列を引き抜けば, 以下を得る.

$$u_n \rightharpoonup u \quad \text{weakly in } L^2(0,T;H)$$

$$\frac{du_n}{dt} \rightharpoonup \frac{du}{dt} \quad \text{weakly in } L^2(0,T;H)$$

ここで, Ascoli-Arzela の定理を用いて証明される次の Lemma

Lemma 3 $\quad u_n \rightharpoonup u \ \ \text{weakly in } L^p(0,T;H),\ p > 1$
$\frac{du_n}{dt} \rightharpoonup \frac{du}{dt} \ \ \text{weakly in } L^q(0,T;H),\ q > 1$
$\Rightarrow u_n \to u \quad \text{in } C([0,T];H_w)$
($C([0,T];H_w)$ is equipped with uniform convergence topology)

を用いれば,

$$u_n \to u \quad \text{in } C([0,T];H_w)$$

を得る. $u_n(t)$ は T-anti-periodic であるので, $(u_n(0)+u_n(T),v)_H = 0 \quad {}^\forall v \in H$ となる. よって, $(u(0)+u(T),v)_H = 0 \quad {}^\forall v \in H$ となるので, $u(0) = -u(T)$ を得る. $u(t)$ が (E) の解である事を見るには, まず$\partial\varphi^t(u_\varepsilon)$ の定義に $(\text{E})_\varepsilon$ を代入すれば, ${}^\forall w \in L^2(0,T;H)$ に対し,

$$\int_0^T \varphi^t(w(t))dt - \int_0^T \varphi^t(u_n(t))dt \ge \int_0^T (f(t) - \frac{du_n}{dt}(t) - \varepsilon u_n(t), w(t) - u_n(t))dt$$

となる. そこで,

$$\varepsilon u_\varepsilon \to 0 \text{ strongly in } L^2(0,T;H)$$

$$\int_0^T (\frac{du_\varepsilon}{dt}(t), u_\varepsilon(t))dt \quad = \tfrac{1}{2}|u_\varepsilon(T)|^2 - \tfrac{1}{2}|u_\varepsilon(0)|^2 \quad = 0$$

$$\int_0^T (\frac{du}{dt}(t), u(t))dt \quad = \tfrac{1}{2}|u(T)|^2 - \tfrac{1}{2}|u(0)|^2 \quad = 0$$

$$\Psi(u) = \int_0^T \varphi^t(u(t))dt \text{ is weakly semicontinuous on } L^2(0,T;H)$$

に注意して, $\varepsilon \searrow +0$ とすれば,

$$\int_0^T \varphi^t(w(t))dt - \int_0^T \varphi^t(u(t))dt \geq \int_0^T (f(t) - \frac{du}{dt}(t), w(t) - u(t))dt$$

となる. したがって $\partial\varphi^t(u)$ の定義より, u は (E) の解となる.

[*Q. E. D.*]

参考文献

[1] Haraux, A. , Anti-periodic solutions of some nonlinear evolution equations, Manuscripta Math. , **63**, (1989), 479–505.

[2] Kenmochi, N. , Some nonlinear parabolic variational inequalities, Israel J. Math. , **22** (1975), 304-331.

[3] Kenmochi, N. , Solvability of nonlinear equations with time-dependent constrains and applications, Bull. Fac. Education Chiba Univ. , **30** (1981), 1-87.

[4] Kenmochi, N. , and Ôtani, M. , Asymptotic behavior of periodic systems generated by time-dependent subdifferential operators, Funkcial Ekvac, **29** (1986), 219–236.

[5] Nagai, T. , Periodic solutions for certain time-dependent parabolic variational inequalities, Hiroshima Math. J. , **5** (1975), 537–549.

[6] Okochi, H. , On the existance of periodic solutions to nonlinear abstract parabolic equations, J. Math. Soc. Japan, **40** (1988), 541–533.

[7] Okochi, H. , On the existence of anti-periodic solutions to nonlinear parabolic equations in noncylindrical domains, Nonlinear Anal. Theory, Methods and Applications, **14** No. 9 (1990), 771-783.

[8] Ôtani, M. , Nonlinear evolution equations with time-dependent constrains, Adv. Math. Sci. Appl. , **3** (1993/94), 383–399.

[9] Yamada, Y. , On evolution equations generated by subdifferential operators, J. Fac. Sci. Uni. Tokyo, **23** (1976), 491-515.

[10] Yamada, Y. , Periodic solutions of certain nonlinear parabolic differential equations in domains with periodically moving boundaries, Nagoya Math. J. , **70** (1978), 111-123.