第１７回
発展方程式若手セミナー
報告集

１９９５年１２月

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序

第１７回発展方程式若手セミナーは、平成７年度科学研究費補助金（総合研究Ａ）の援助のもとに、平成７年８月２日より８月５日までの４日間、岐阜市のぎふ長良川ハイツで約６０名が参加して行われました。本年は、特別講演に二宮広和先生（東京工業大学理学部）をお招きし、「拡散の効果について」という題目で講演をして頂いた他、２０件を越える一般講演、及び大学院生による short communications が行われました。

本報告集は、特別講演と一般講演の講演内容を元にまとめたものです。

１７回を迎えた今回のセミナーは、発展方程式のみならず，非線形微分方程式全般に亘る多彩なテーマによる講演が連日行われ、活発な討論・情報交換が行われました。この報告集が参加された方や関連分野の研究者の方々の研究に貢献できますよう願っております。

このセミナーの準備・運営に当たり多くの方々に協力していただきました。特に、大阪大学 井川満先生、宮崎大学 山田直記先生、神戸商船大学 丸尾健二先生、広島修道大学 角谷敦先生、姫路工業大学 山本本吉孝先生、及び神戸大学大学院生諸氏に厚く御礼申し上げます。また、二宮広和先生には、渡米前の忙しい時期にもかかわらず、快く特別講演をお引き受けいただきました。この場をお借り致しまして、深く感謝申し上げる次第です。最後に来年度の幹事は、上智大学の平田均先生です。所属変更等は速やかにご連絡下さいますようおねがい申し上げます。

平成７年１２月

第１７回発展方程式若手セミナー幹事

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反応拡散方程式における拡散の効果

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発展方程式若手セミナー（1995.8.2〜1995.8.5）

1. 序

非線型発展方程式の解の挙動は興味深い問題であり，数多くの研究がなされている．大域的安定な定常解が存在するような単純な場合も多く知られているが，大抵の場合はそれらの解の挙動が複雑となる．その原因の多くは非線形性にあるが，拡散の効果も大きいことを忘れてはいけない．本講演では拡散が解の挙動にどのような影響を与えていくかについて説明していきたい．特に反応拡散系に限って考察していく．

一言に拡散の効果といっても，領域の有界・非有界や領域の形などの情報も入ってくる．領域の形に依存する研究も近年盛んになりつつあるが，ここではもっと基本的な拡散の有無のもたらす違いに注目する．つまり，

$$u_t = f(u) \tag{1.1}$$

と

$$\begin{cases} u_t = D\Delta u + f(u) & (x \in \Omega, t > 0), \\ \dfrac{\partial u}{\partial n} = 0 & (x \in \partial\Omega, t > 0) \end{cases} \tag{1.2}$$

の解の挙動の違いについて調べていく．ここで $u \in \mathbb{R}^m$, f は $\mathbb{R}^m$ から $\mathbb{R}^m$ への滑らかな関数，

$$D = \mathrm{diag}(d_1, d_2, \cdots, d_m), \quad d_j > 0$$

とする．

単独の場合（$m = 1$）は比較定理が成り立つので，(1.1) と (1.2) の解の多くの（少なくとも以下で述べるような）挙動においては差がない．以降，主に $m = 2$ の場合を考える．つまり，

$$\begin{cases} u_t = f(u, v), \\ v_t = g(u, v) \end{cases} \tag{1.3}$$

と

$$\begin{cases} u_t = d_1\Delta u + f(u, v) & (x \in \Omega, t > 0), \\ v_t = d_2\Delta v + g(u, v) & (x \in \Omega, t > 0), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & (x \in \partial\Omega, t > 0), \end{cases} \tag{1.4}$$

とする．以下，初期値を $(u(x,0), v(x,0)) = (u_0(x), v_0(x))$ で表す．

2. 平均化

まず，$f=0$ の場合を考えよう．このとき方程式は単独と見なせる．方程式 (1.3) は

$$u_t = v_t = 0$$

となり，解は t に依らない．つまり，

$$u(t) = u_0, \quad v(t) = v_0.$$

方程式 (1.4) は

$$\begin{cases} u_t = d_1 \Delta u & (x \in \Omega, t > 0), \\ v_t = d_2 \Delta v & (x \in \Omega, t > 0), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & (x \in \partial\Omega, t > 0) \end{cases}$$

となり，

$$(2.1) \quad u(x,t) = \sum_j (u(x,0), \varphi_j) e^{-d_1 \lambda_j t} \varphi_j(x), \quad v(x,t) = \sum_j (v(x,0), \varphi_j) e^{-d_2 \lambda_j t} \varphi_j(x).$$

ここでノイマン境界条件付きのラプラス作用素 $-\Delta$ の固有値を λ_j，対応する正規化された固有関数を $\varphi_j(x)$ とする．つまり

$$-\Delta \varphi_j = \lambda_j \varphi_j, \quad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots.$$

(2.1) からわかるように解は空間一様化していく．

次に拡散の影響が空間一様化という方向に働くのなら，(1.1) と (1.2) の挙動は近いはずである．実際，

$$d_* = \min_j d_j$$

が大きいときこの推論は正しい．Conway, Hoff, Smoller [2] によって次が示された．

$u(x,t)$ は方程式 (1.2) の解とする．コンパクト正不変集合 $\Sigma (\subset \mathbb{R}^m)$ が存在し，定数 M を Σ における f の微分の最大値として

$$d_* \lambda_1 > M$$

が成り立つと仮定する．このとき，定数 c_1, c_2 が存在して次が成り立つ．

$$\|\nabla u(\cdot,t)\|_{L^2(\Omega)} \leq c_1 e^{-(d\lambda_1 - M)t},$$
$$\|u(\cdot,t) - \frac{1}{|\Omega|}\int_\Omega u(y,t)dy\|_{L^2(\Omega)} \leq c_2 e^{-(d\lambda_1 - M)t}.$$

拡散係数 D が大きければ方程式 (1.1) と (1.2) の解の挙動が近いことを意味している．

3. 拡散不安定性

ここでは次のような方程式を考える．

$$(3.1)\qquad \begin{cases} u_t = u - u^3 - v, \\ v_t = 3u - 2v \end{cases}$$

と

$$(3.2)\qquad \begin{cases} u_t = d_1 u_{xx} + u - u^3 - v & (0 < x < 1, t > 0), \\ v_t = d_2 v_{xx} + 3u - 2v & (0 < x < 1, t > 0), \\ u_x(0,t) = u_x(1,t) = 0 & (t > 0), \\ v_x(0,t) = v_x(1,t) = 0 & (t > 0). \end{cases}$$

上の方程式は活性・抑制系（activator–inhibitor system）と呼ばれる．u が興奮性因子（activator），v が抑制因子（inhibitor）として働くからである．

方程式 (3.1) の解の挙動は図 3.1 のようになる．

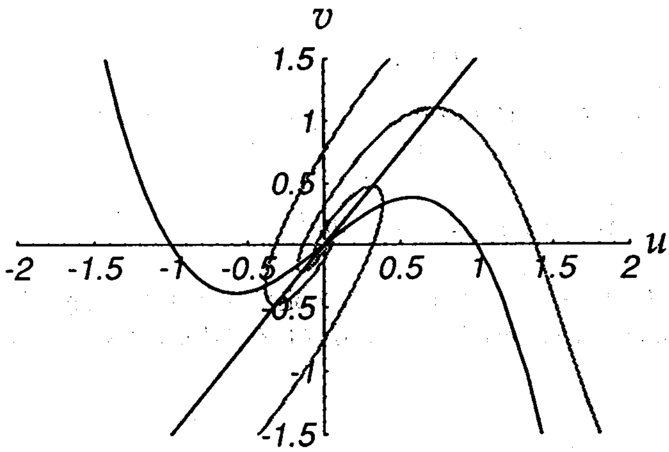

FIG. 3.1. 方程式 (3.1) の解の挙動

いずれの場合も定常解 $(u,v) = (0,0)$ をもつ．まず (3.1) での線形化安定性を調べよう．(3.1) を $(0,0)$ のまわりで線形化すると，

$$(3.3)\qquad \begin{cases} \phi_t = \phi - \psi, \\ \psi_t = 3\phi - 2\psi \end{cases}$$

となり，行列

$$\begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}$$

の固有値は $-1 \pm i\sqrt{3}$ となり，実部が共に負なので解 $(0,0)$ は漸近安定となることがわかる．

次に (3.2) の場合を調べよう．線形化固有値問題は

$$\begin{cases} \lambda\phi = d_1\phi_{xx} + \phi - \psi, \\ \lambda\psi = d_2\psi_{xx} + 3\phi - 2\psi \end{cases}$$

である．$(\phi(x), \psi(x)) = (\phi_0, \psi_0)\cos k\pi x$ とおくと

$$\begin{cases} \lambda\phi_0 = (1 - d_1k^2\pi^2)\phi - \psi, \\ \lambda\psi_0 = 3\phi - (2 + d_2k^2\pi^2)\psi \end{cases}$$

となるので，固有値 λ は次を満たす．

(3.4) $$\lambda^2 + (1 + (d_1 + d_2)k^2\pi^2)\lambda + (d_1k^2\pi^2 - 1)(d_2k^2\pi^2 + 2) + 3 = 0.$$

$$d = d_2/d_1, \nu = d_1k^2\pi^2$$

とおく．方程式 (3.4) の左辺最後の項 $(\nu - 1)(d\nu + 2) + 3$ が負となれば実部正の固有値が存在することになる．実際，$d = 10, \nu = 2/5$ とすると

$$(\nu - 1)(d\nu + 2) + 3 = 10\nu^2 - 8\nu + 1 = -\frac{3}{5} < 0$$

となる．拡散係数 $d_1 = d_2 = 0$ のときは安定だったので，拡散不安定性（Turing's instability, diffusion-driven instability）と呼ばれる（[10] 参照）．

4. 拡散誘導絶滅

次に生物学的な問題をみてみよう．生物種の共存・絶滅・棲み分け等の現象が競争を通じて現れることはよく知られている．種の絶滅や保存を左右する要因としては初期状態や増殖率・競争の強さなどが考えられる．ここでは2つの種の競争がロトカ・ボルテラ型の競争で支配される場合を考える．適当な変数変換すると，次のような方程式系で表される．

$$(4.1)\qquad \begin{cases} u_t = u(a-u-bv), \\ v_t = v(1-cu-v). \end{cases}$$

u, v は個体群のそれぞれの大きさ，a は u の増殖率，b, c は種間競争係数を表している．解の挙動は次の2つのタイプに分けられる．

(i) 安定な定数定常解が1つ存在して，大域的安定である．

(ii) 安定な定数定常解が2つある ($\frac{1}{c} < a < b$).

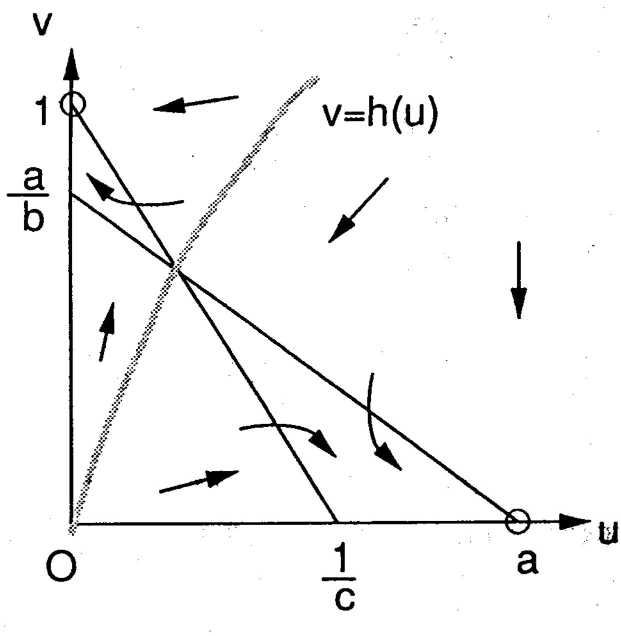

FIG. 4.1. 方程式 (4.1) の解の挙動

前者はすべての正の解（それぞれの成分が正）は安定な解に収束する．後者は2つの安定平衡解のどちらかに近づく．従って，初期値によって生き残り絶滅が変わる後者の方を考えよう（実際，前者の場合，方程式 (4.2) の解はすべて方程式 (4.1) の安定定常解に近づくことが示されている．[1, 3] 参照）．この場合，安定解の2つの吸引域が存在して，その境界をなす曲線がセパラトリックスと呼ばれる．つまり，方程式系 (4.1) のセパラトリックスは $v = h(u)$ というグラフで表現でき，$v(0) > h(u(0))$ なら解 $(u(t), v(t))$ は $(0,1)$ に収束し，$v(0) < h(u(0))$ なら解は $(a,0)$ に収束する（図 4.1 参照）．

では空間分布を考慮に入れたモデルではどうなるかみていこう．方程式系

$$
\begin{cases} u_t = d\Delta u + u(a-u-bv) & (x \in \Omega,\ t>0), \\ v_t = d\Delta v + v(1-cu-v) & (x\in\Omega,\ t>0) \end{cases} \tag{4.2}
$$

にノイマン境界条件

$$
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad (x \in \partial\Omega,\ t>0), \tag{4.3}
$$

初期条件を付加してものを考える．

常微分方程式系 (4.1) のセパラトリックスの定義から自然に種の優勢という概念が定義できる．ある点 x で種 v が種 u より優勢であるとはその点を初期値とする (4.1) の解は u は絶滅し v が生き残る場合と定義する．つまり点 x で

$$
v(x) > h(u(x))
$$

とする．各点で種 v が種 u より優勢であれば，種 v が生き残り，種 u が絶滅するのだろうか？　飯田・村松・二宮・柳田 [5] においては，優勢な種が絶滅するような場合があることが示されている．優勢な種の絶滅は拡散がなければ起きないことから，この現象は拡散誘導絶滅 (diffusion-induced extinction) と呼ばれる．

証明のアイデアを述べよう．3 つの段階に分かれる．

Step 1. セパラトリックスの特徴付け．
- $a > 1$ のとき，$h''(u) < 0$,
- $a = 1$ のとき，$h''(u) = 0$,
- $a < 1$ のとき，$h''(u) > 0$.

セパラトッリクスが正値不安定定数定常解 $((b-a)/(bc-1), (ac-1)/(bc-1))$ の安定多様体となることを用いて示す．以下，$a>1$ の場合に限定し W_A を次のように定義する．

$$
W_A = \{(u,v) \in \mathbb{R}^2 \mid 0 \le v \le h(u), u \ge 0\}
$$

Step 2. W_A の正不変性．$h''(u) < 0$ より，W_A は正不変となる．さらに次が成り立つ．

$$
(u_0(x), v_0(x)) \in W_A \quad (\forall x \in \Omega) \text{ ならば } (u(x,t), v(x,t)) \in \mathrm{int} W_A \quad (\forall x \in \Omega, t>0).
$$

Step 3. W_A に初期値をもつ解が $(a,0)$ に収束すること．正不変な領域を構成することと，比較定理を用いて示される（[9]）．

Step 2 から初期値をセパラトリックス上にとったとき，つまり，

$$
v_0(x) \equiv h(u_0(x)), \quad (u_0(x), v_0(x)) \not\equiv \text{定数}
$$

のとき，$t>0$ になると，W_A に入ることが従う．また，Step 3 から $(a,0)$ に収束することがわかる．至る所 $v_0(x) > h(u_0(x))$ であるが，セパラトリックスに十分近い初期値をとれば，初期値に対する連続性から，先の解に十分近いので W_A に入り，$(a,0)$ に収束することが従う．こうして，拡散誘導絶滅が示される．

5. 拡散誘導爆発

爆発は非線型問題特有のものであり，非線型項の影響によるものであると考えるのが自然であり，拡散の影響はあまりないように思われがちである．つまり，方程式 (1.1) の解の爆発と (1.2) の解の爆発は同値のように思われる．しかし，拡散に因って爆発が起こる例を挙げよう（[8] 参照）．

$$
\begin{cases} u_t = (u-v)^3 - v, \\ v_t = (u-v)^3 - v. \end{cases} \tag{5.1}
$$

この方程式も活性・抑制系である．

$$
(u-v)_t = 0
$$

より，

$$
u(t) = u_0 + \{(u_0 - v_0)^3 - v_0\}(1 - e^{-t}), \quad v(t) = v_0 e^{-t} + (u_0 - v_0)^3 (1 - e^{-t})
$$

が従う．つまり，解は時間大域的に存在するばかりか，有界にとどまり収束すらする．

次に拡散つきの方程式

$$
\begin{cases} u_t = (u-v)^3 - v & (0 < x < 1, t > 0), \\ v_t = v_{xx} + (u-v)^3 - v & (0 < x < 1, t > 0), \end{cases} \tag{5.2}
$$

にノイマン境界条件及び初期条件を付加したものを考える．初期値 $(u_0(x), v_0(x))$ は以下を満たすとする．

$$
\begin{aligned}
&u_{0x} = 0 \quad \text{at } x = 0, 1, \\
&\int_0^1 (u_0(x) - v_0(x))dx = 0, \\
&\|u_0(x)\|_{L^2(0,1)} - \|v_0(x)\|_{L^2(0,1)} > 0.
\end{aligned}
$$

更に，

$$
\|u_0(x) - v_0(x)\|_{H^{-1}(0,1)}
$$

が十分大きいとき，解は有限時間で爆発する（詳しくは [8] 参照）．

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1相ステファン問題の大域解の存在について

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1 Introduction

Let us consider the following one-phase Stefan problem DP (resp. NP) with homogenous Dirichlet (resp. Neumann) boundary condition in one-dimensional space: The problem is to find a curve $x = \ell(t) > 0$ on $[0,T]$, $(0 < T < \infty)$, and a function $u = u(t,x)$ on $Q_\ell(T) := \{(t,x); 0 < t < T, 0 < x < \ell(t)\}$ satisfying that $\ell \in C^1([0,T])$, $u \in C^{1,0}(\overline{Q_\ell(T)})$, u_{xx} and u_t are continuous in $Q_\ell(T)$ and

$$u_t = u_{xx} + u^{1+\alpha} \quad \text{in } Q_\ell(T), \tag{1}$$

$$u(0,x) = u_0(x) \quad \text{for } 0 \le x \le \ell_0, \tag{2}$$

$$u(t,0) = 0 \qquad \text{for } 0 < t < T, \tag{3}$$

$$(\text{resp. } \frac{\partial}{\partial x}u(t,0) = 0 \quad \text{for } 0 < t < T,) \tag{4}$$

$$u(t,\ell(t)) = 0 \quad \text{for } 0 < t < T, \tag{5}$$

$$\frac{d}{dt}\ell(t) = -u_x(t,\ell(t)) \quad \text{for } 0 < t < T, \tag{6}$$

$$\ell(0) = \ell_0 \tag{7}$$

where α and ℓ_0 are given positive constants and u_0 is a given initial function on $[0,\ell_0]$.

In [5] Fasano-Primicerio established the local existence in time and the uniqueness for solutions to the above DP and NP in the classical formulation (which means that u_t and u_{xx} are continuous functions). Besides, for solutions of DP and NP in the distribution sense (which means that u_t and u_{xx} belong to L^2-class) the existence, the comparison and the behavior were studied by Aiki-Kenmochi [1, 4, 9].

It is well known that there are blow-up solutions of the usual initial boundary value problem for semilinear equation (1) in bounded domain, accordingly, by using comparison principle it is clear that $DP(NP)$ has a blow-up solution. Here, we note the the following global existence result of a solution: Let $[0, T^*)$ be the maximal interval of existence of the solution to DP and NP, we see (cf. [4]) that the following cases (a) or (b) must occur:

(a) $T^* = +\infty$;

(b) $T^* < +\infty$ and $|u|_{L^\infty(Q_\ell(t))} \to +\infty$ as $t \uparrow T^*$.

However, from the above result we can get no information for the behavior of free boundary ℓ at blow-up time. In the present paper we shall show the behavior of blow-up solutions to DP and NP at finite blow-up time (Theorems 2 and 3).

Besides, we know the following result concerned with global existence for solutions of Cauchy problem for equation (1) (cf. Fujita [7] and Levine [10]): In case $\alpha > 2$ there are global solutions in time for sufficiently small initial data; in case $0 < \alpha \le 2$ the non-trivial solution always blows up at some finite time. Also, if a domain is bounded then for $\alpha > 0$ the problem has a global solutions. Hence, it follows from comparison principle that for $\alpha > 2$ $NP(DP)$ has a global solution in time for a small initial function u_0. One of purposes of this paper is to establish the global existence result for NP (Theorem 4).

Here, we give assumptions (H1) $\sim$ (H4) for initial data ℓ_0 and u_0.

(H1) $\ell_0 > 0$ and $u_0 \in C^2((0, \ell_0)) \cap C^1([0, \ell_0])$ and $u_0(x) > 0 \quad$ for $x \in (0, \ell_0)$,

(H2) $u_{0,xx}(x) + u_0^{1+\alpha}(x) \ge 0 \quad$ for $x \in (0, \ell_0)$,

(H3) $u_0(\ell_0) = 0$, $u_{0,x} < 0$ for $x \in (0, \ell_0)$ and $u_{0,x}(0) = 0$,

(H4) $u_0(0) = u_0(\ell_0) = 0$, $u_{0,x} > 0$ on $[0, x_0)$ and $u_{0,x} < 0$ on $(x_0, \ell_0]$ for some $x_0 \in (0, \ell_0)$.

We begin with the precise definition of a solution to DP and NP. Let $C^{1,0}(\overline{Q_\ell(T)})$ be the set of functions which are continuous on $\overline{Q_\ell(T)}$ with their $x-$derivatives.

Definition A couple $\{u, \ell\}$ of functions $u = u(t, x)$ and $x = \ell(t)$ is said to be a solution of DP (resp. NP) on a compact interval $[0, T]$, $0 < T < +\infty$, if the following conditions (S1) and (S2) are satisfied:

(S1) $\ell \in C^1([0, T])$, and $u \in C^{1,0}(\overline{Q_\ell(T)})$, u_{xx} and u_t are continuous in $Q_\ell(T)$;

(S2) (1) $\sim$ (3) and (5) $\sim$ (7) (resp. (1) $\sim$ (2) and (4) $\sim$ (7)) hold in the classical sense.

Also, we call a couple $\{u, \ell\}$ is a solution of DP(resp. NP) on an interval $[0, T')$, $0 < T' \leq \infty$, if it is a solution of DP(resp. NP) on $[0, T]$ in the above sense for any $0 < T < T'$.

First, we recall the theorem concerned with local existence of solutions to the above DP and NP.

Theorem 1 *(cf. [5, Theorem 1]) We assume that $u_0 \in C^1([0, \ell_0])$, $u_0 \geq 0$ on $[0, \ell_0]$, $u_0(\ell_0) = 0$ and $u_0(0) = 0$(resp. $(u_{0,x}(0) = 0)$. Then there exists a positive number T_0 depending only on $|u_0|_{C^1([0,\ell_0])}$, ℓ_0 and α such that problem DP(resp. NP) has a unique solution $\{u, \ell\}$ on $[0, T_0]$.*

For the problems DP and NP, we say that $[0, T)$, $0 < T \leq +\infty$, is the maximal interval of existence of the solution, if the problem has a solution on time-interval $[0, T']$, for every T' with $0 < T' < T$ and the solution can not be extended in time beyond T.

2 Blow-up points

Theorem 2 *(cf. [2]) Assume that (H1) $\sim$ (H3) hold. Let $\{u, \ell\}$ be a solution of NP. If $T^* < \infty$, then $\ell(t) \uparrow L < +\infty$ as $t \uparrow T^*$, $u(t, 0) \to +\infty$ as $t \uparrow T^*$, and for any $x \in (0, L)$ there exists a positive number $M(x)$ such that*

$$|u(t, x)| \leq M(x) \quad \text{for any } t \text{ with } (t, x) \in Q_\ell(T).$$

In [8] Fujita and Chen studied the following initial boundary value problem.

$$u_t = u_{xx} + u^{1+\alpha} \quad \text{in } (0, T) \times (0, 1),$$

$$u_x(t, 0) = 0 \quad \text{for } t \in (0, T],$$

$$u(t, 1) = 0 \quad \text{for } t \in (0, T],$$

$$u(0, x) = u_0(x) \quad \text{for } x \in [0, 1].$$

They showed that under the similar assumptions for u_0 to (H1) $\sim$ (H3) if the solution u blows up then blow-up point is one and only one point $x = 0$. In the proof of Theorem 2 we done with help of the idea in [8].

Theorem 3 *(cf. [3]) Assume that (H1),(H2) and (H4) hold. Let $[0, T^*)$ be the maximal interval of existence of the solution $\{u, \ell\}$ to DP. If T^* is finite, then either the following* *cases (A) or (B) always happens:*

(A) $\ell(t) \to \ell_\infty$ as $t \uparrow T^*$ where ℓ_∞ is some positive number, there exists one and only one point $x^* \in (0, \ell_\infty)$ such that $u(t,x) \to +\infty$ as $t \uparrow T^*$ and for $x \in (0, \ell_\infty)$ with $x \neq x^*$ there is a positive constant $M_1(x)$ such that $|u(t,x)| \leq M_1(x)$ for t with $(t,x) \in Q_\ell(T^*)$;

(B) $\ell(t) \to +\infty$ as $t \uparrow T^*$ and for any $x > 0$ there is a positive number $M_2(x)$ satisfying that $|u(t,\xi)| \leq M_2(x)$ for $(t,\xi) \in Q_\ell(T^*) \cap \{\xi < x\}$.

The proof of Theorem 3 is done in the following way.

First, we shall show continuation of solutions of our problem DP. Precisely speaking, for $0 < T_0 < \infty$ if a solution $\{u, \ell\}$ to SP on $[0, T_0)$ satisfies that

$$|u(t,x)| \leq K \quad \text{for } (t,x) \in \overline{Q_\ell(T_0)} \cap \{t < T_0\}$$

where K is a positive constant, then the solution $\{u, \ell\}$ is extended in time beyond T_0.

Next, we assume that (B) in the assertion of Theorem 3 does not hold, that is, (A1) or (A2) is valid:

(A1) There is a number $x_1 \in (0, \infty)$ satisfying that for some sequence $\{(t_n, \xi_n)\} \subset Q_\ell(T^*)$ $t_n \uparrow T^*$, $\xi_n \to x_1$ and $u(t_n, \xi_n) \to \infty$ as $n \to \infty$;

(A2) there exists a positive constant L_0 such that $\ell(t) \leq L_0$ for any $t \in [0, T^*)$.

Therefore, it is sufficient to show that (A1) is a sufficient condition for (A) in the statement of Theorem 3 and (A2) is, too.

Under the condition (A1) by the similar argument to those of Friedman-McLeod [6] we infer that the set of blow-up points consists of only one point $x_1 \in (0, \infty)$. Clearly, $u(t, x_1) \to +\infty$ as $t \uparrow T^*$ and for any $x_2 > x_1$ there is a positive constant $M_3(x_2)$ such that

$$|u(t,x)| \leq M_3(x_2) \quad \text{for any } (t,x) \in Q_\ell(T^*) \cap \{x > x_2\}.$$

By using the above estimate and a comparison theorem for solutions to one-phase Stefan problems we conclude that (A) holds. Similarly, we can prove that (A2) implies (A).

3 Global existence

Finally, we give a theorem concerned with a global existence for NP.

Theorem 4 *We assume that (H1) holds and $\alpha > 1$. There exist positive numbers $\delta > 0$ and $p_0 > 1$ such that if $\int_0^{\ell_0} u_0^{1+\alpha} dx \leq \delta$, $\int_0^{\ell_0} u_0^2 dx \leq \delta$, $\int_0^{\ell_0} u_0^{p_0} dx \leq 1$ and $\int_0^{\ell_0} u_{0,x}^2 dx \leq 1$, then the* *problem NP with initial condition u_0 and ℓ_0 has a solution $\{u,\ell\}$ on $[0,\infty)$ satisfying that*

$$\int_0^{\ell(t)} u(t,x)dx + \ell(t) \le C \quad \text{for } t>0,$$

$$\frac{d}{dt}|u_x(t)|^2_{L^2(0,\ell(t))} \le 0 \quad \text{for a.e. } t>0,$$

$$|u(t)|_{L^\infty(0,\ell(t))} \le C\exp(-\mu t) \quad \text{for } t>0$$

where C and μ are some positive constants.

Here, we note Gagliardo-Nirenberg inequalities.

Lemma 1 *(1) For $p \ge 2$ and $\alpha \ge 0$ we put $q = 2(p+\alpha)/p$ and $r \in (0,q)$. Then we have*

$$\int_0^d u^{p+\alpha}dx \le \left(\frac{q+2}{2}\right)^{\frac{2(q-r)}{r+2}} |(u^{\frac{p}{2}})_x|^{\frac{2(q-r)}{r+2}}_{L^2(0,d)} \left(\int_0^d u^{\frac{pr}{2}}dx\right)^{\frac{q+2}{r+2}}$$

for $u \in W^{1,2}(0,d)$ with $u(d)=0$.

(2) For $q \ge 1$ we have

$$|u|_{L^\infty(0,d)} \le \left(\frac{q+2}{2}\right)^{\frac{2}{q+2}} |u_x|^{\frac{2}{q+2}}_{L^2(0,d)} |u|^{\frac{q}{q+2}}_{L^q(0,d)} \quad \text{for } u \in W^{1,2}(0,d) \text{ with } u(d)=0.$$

In order to prove Theorem 4 we prepare the following energy estimates.

Lemma 2 *We suppose that $\alpha > 0$, $\ell_0 > 0$ and $u_0 \in W^{1,2}(0,\ell_0)$ such that $u_0 \ge 0$ on $[0,\ell_0]$ and $u_0(\ell_0)=0$. Let $\{u,\ell\}$ be a solution of NP on $[0,T]$, $0<T<\infty$. Then we have:*

(1) For each $p>1$

$$\frac{d}{dt}\int_0^{\ell(t)} u^p(t,x)dx$$
$$\le \left\{-\frac{4(p-1)}{p} + p(2+\frac{\alpha}{p})^2(\int_0^{\ell(t)} u^{1+\alpha}(t,x)dx)^{\frac{\alpha}{1+\alpha}}\ell(t)^{2-\frac{1}{1+\alpha}}\right\} |(u^{\frac{p}{2}}(t))_x|^2_{L^2(0,\ell(t))}$$

for any $t \in (0,T]$.

(2) $\frac{d}{dt}\{\int_0^{\ell(t)} u(t,x)dx + \ell(t)\} = \int_0^{\ell(t)} u(t,x)^{1+\alpha}dx$ for $t \in (0,T]$.

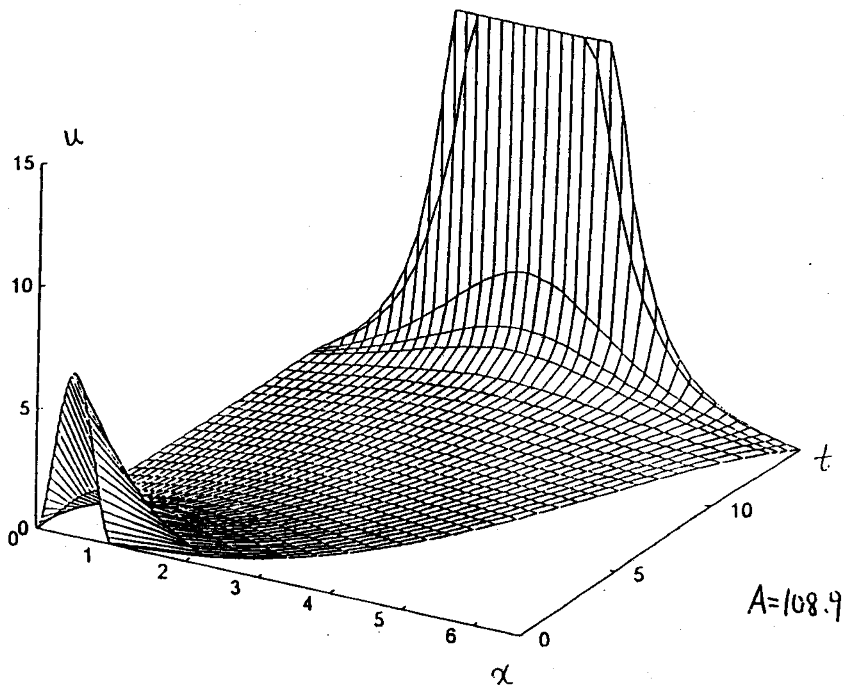
(3) For a.e. $t \in [0,T]$

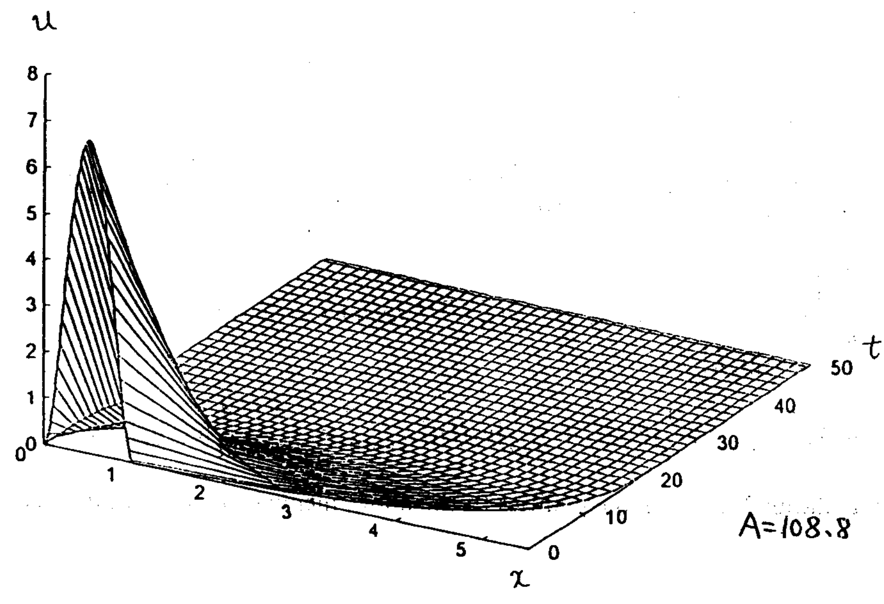
$$|u_t(t)|^2_{L^2(0,\ell(t))} + \frac{1}{2}|\ell'(t)|^3 + \frac{1}{2}\frac{d}{dt}|u_x(t)|^2_{L^2(0,\ell(t))} = \frac{1}{2+\alpha}\frac{d}{dt}|u(t)|^{2+\alpha}_{L^{2+\alpha}(0,\ell(t))}.$$

Applying Gagliardo-Nirenberg inequalities to the above inequalities we can prove Theorem 4.

4 Numerical experiments

We carried out numerical computations to DP with $\alpha = 1$, $\ell_0 = 1$ and $u_0(x) = u_0^A(x) \equiv Ax^2(x-1)^2$. We note that u_0 satisfies the condition (H2) for large A.





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Existence of a global attractor for evolution equation associated with the subdifferential operator

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1. Introduction and assumptions

Throughout this paper, let H be a real Hillbert space.

In this paper, we consider the existence of the global attractor $\mathcal{A}$ for the semigroup $\{S(t)\}_{t\geq 0}$ which is defined by $S(t)u_0 = u(t)$, where u is a global and unique solution of the following Cauchy problem (CP):=$\{$(1.1)-(1.2)$\}$:

$$u'(t) + \partial\varphi(u(t)) + g(u(t)) \ni f \quad \text{for a.e. } t \in R_+, \tag{1.1}$$

$$u(0) = u_0. \tag{1.2}$$

Here, φ is a l.s.c. convex function on H and $\partial\varphi$ is the subdifferential of φ; g is Lipschitz continuous function from H into itself; f and u_0 are given data.
Then, under assumptions (A1)-(A4), by using the technique of Temam, we can see that there exists a global attractor $\mathcal{A}$ for the semigroup $\{S(t)\}_{t\geq 0}$.

(A1) $\varphi : H \longrightarrow R \cup \{+\infty\}$ is a l.s.c. convex function whose effective domain is denoted by $D(\varphi) \neq \phi$ and $\partial\varphi$ is the subdifferential of φ and $0 \in \partial\varphi(0)$;

(A2) There exist positive constants α_1, α_2 and α_3 such that

$$|g(z_1) - g(z_2)|_H \leq \alpha_1 |z_1 - z_2|_H \tag{1.3}$$

and

$$(g(z_1), z_1)_H \geq \alpha_2 |z_1|_H^2 - \alpha_3 \tag{1.4}$$

for any z_1, $z_2 \in H$.
Moreover, there exists $\hat{g} : H \longrightarrow R$ such that

$$\hat{g} \text{ is Frechet differentiable}$$

and

$$D\hat{g}(z) = (g(z), \cdot)_H \quad \text{for any } z \in H, \tag{1.5}$$

where $D\hat{g}$ denotes the Frechet derivative of $\hat{g}$;

(A3) $f \in H$ and $u_0 \in \overline{D(\varphi)}$, where $\overline{D(\varphi)}$ denotes the closure of $D(\varphi)$ in H;

(A4) For each $r \geq 0$, the level set H_r in H, defined by

$$H_r := \{z \in H; |z|_H \leq r \text{ and } |\varphi(z)| \leq r\},$$

is relatively compact in H.

2. Main theorem

Before relating the main theorem, we give the definitions and lemmas.

Definition 2.1. $\{S(t)\}_{t \geq 0}$ is called C_0-semigroup on $\overline{D(\varphi)}$, if it satisfies the following properties (S1)-(S3);

(S1) $S(0) = I$, where H is the identity operator on H;

(S2) $S(t+s) = S(t)S(s)$ for any $s, t \in R_+$;

(S3) For each $t \in R_+$, $S(t)$ is continuous from $\overline{D(\varphi)}$ into itself.

Definition 2.2. $\mathcal{A}$ is called the global attractor for $\{S(t)\}_{t \geq 0}$ in H, if it satisfies the following properties (GA1)-(GA3);

(GA1) $\mathcal{A}$ is compact in H;

(GA2) $S(t)\mathcal{A} = \mathcal{A}$ for all $r \in R_+$;

(GA3) For any bounded set B in $\overline{D(\varphi)}$ and $\epsilon > 0$, there exists $t_0 := T_0(B, \epsilon) \in R_+$ such that

$$\mathrm{dist}_H(S(t)B, \mathcal{A}) \leq \epsilon \quad \text{for any } t \geq 0,$$

where $\mathrm{dist}_H(X, Y) := \sup\limits_{x \in X} \inf\limits_{y \in Y} |x - y|_H$ for any subset X, Y of H.

Definition 2.3. $\mathcal{B}$ is called an absorbing set in H, it satisfies the following property (AS1);

(AS1) For any bounded set B of $\overline{D(\varphi)}$, there exists $t_1 = t_1(B) \in R_+$ such that

$$S(t)B \subset \mathcal{B} \quad \text{for all } t \geq t_1.$$

Remark 2.1. In the results in [1], it is proved that there exists C_0-semigroup for (CP) on $\overline{D(\varphi)}$.

Here, we note the following lemma in [2] to show the existence of a global attractor $\mathcal{A}$ for $\{S(t)\}_{t\geq 0}$ in $\overline{D(\varphi)}$.

Lemma 2.1. [cf. 2] *We assume that X is a metric space and $\{T(t)\}_{t\geq 0}$ is C_0-semigroup on X. Moreover, we assume that (i) and (ii) hold.*

(i) *For any bounded set, there exists $t_2 = t_2(B) \in R_+$, which may depend upon B, such that*

$$\overline{\bigcup_{t\geq t_2} T(t)B}$$

is relatively compact in X;

(ii) *There exists an absorbing set $\mathcal{B}$ in X.*

Then, there exists a global attractor $\mathcal{A}$ for $\{T(t)\}_{t\geq 0}$ in X.

Theorem 2.1. *There exists an absorbing set $\mathcal{B}$ in $\overline{D(\varphi)}$.*
Proof. Let B be any bounded set in $\overline{D(\varphi)}$. Then, there exists a positive constant R_B such that

$$B \subset B_H(0, R_B),$$

where $B_H(0, R_B) := \{z \in \overline{D(\varphi)}; |z|_H < R_B\}$. For any $u_0 \in B$, (CP) has a unique and global solution.
Here, by multiplying (1.1) by $u(t)$ and from (A1), we can obtain

$$\frac{1}{2}\frac{d}{dt}|u(t)|_H^2 + (g(u(t)), u(t))_H \leq (f, u(t))_H \quad \text{for a.e. } t \in R_+.$$

Moreover, from (A2), we can obtain

$$\frac{d}{dt}|u(t)|_H^2 + \alpha_1|u(t)|_H^2 \le \frac{|f|_H^2}{\alpha_1} + \alpha_2 \quad \text{for a.e. } t \in R_+.$$

Hence, by applying Gronwall lemma, we can see that

$$\begin{aligned} |u(t)|_H^2 &\le \exp(-\alpha_1 t)|u_0|_H^2 + \frac{1}{\alpha_1}\left(\frac{|f|_H^2}{\alpha_1} + \alpha_2\right)(1 - \exp(-\alpha_1 t)) \\ &\le \exp(-\alpha_1 t)|u_0|_H^2 + \frac{1}{\alpha_1}\left(\frac{|f|_H^2}{\alpha_1} + \alpha_2\right) \\ &\le \exp(-\alpha_1 t)R_B^2 + \rho_0^2 \end{aligned}$$

,that is,

$$\sup_{u_0 \in B} |S(t)u_0|_H^2 \le \exp(-\alpha_1 t)R_B^2 + \rho_0^2 \quad \text{for any } t \in R_+.$$

We consider an open ball $B_H(0, \rho_1)$ with $\rho_1 > \rho_0$. Then, we can see that $B_H(0, \rho_1)$ is an absorbing set in $\overline{D(\varphi)}$.
Actually, we can obtain that

$$S(t)B \subset B_H(0, \rho_1) \quad \text{for any } t \ge t_1(B),$$

where $\frac{1}{\alpha_1}\log(\frac{R_B^2}{\rho_1^2 - \rho_0^2})$. $\diamond$

Theorem 2.2. *For any bounded set B in $\overline{D(\varphi)}$, there exists a positive constant ρ_2 such that*

$$\sup_{u_0 \in B} |\varphi(S(t)u_0)| \le \rho_2 \quad \textit{for any } t \ge t_0(B),$$

where $t_0(B)$ is the same in Theorem 2.1.
Proof. In this proof, we use the same notations in Theorem 2.1.
We note that

$$S(t_0(B))u_0 \in D(\varphi) \quad \text{for any } u_0 \in B \tag{2.1}$$

and

$$\sup_{u_0 \in B} \sup_{t \ge t_0(B)} S(t_0(B))u_0 < \rho_1. \tag{2.2}$$

By multiplying (1.1) by $u'(t)$ and from (A2), we can obtain

$$|u'(t)|_H^2 + \frac{d}{dt}\{\varphi(u(t)) + \hat{g}(u(t)) - (f, u(t))_H = 0 \quad \text{for a.e. } t \in R_+.$$

Hence, we can see that

$$\varphi(u(t)) + \hat{g}(u(t)) - (f, u(t))_H \leq \varphi(u(t_0(B))) + \hat{g}(u(t_0(B))) - (f, u(t_0(B)))_H$$
$$\text{for any } t \geq t_0(B).$$

¿From (2.1) and (2.2), we can easily see that there exists a positive constant ρ_2' such that

$$\varphi(u(t)) \leq \rho_2' \quad \text{for any } t \geq t_0(B) \text{ and } u_0 \in B. \tag{2.3}$$

Hence, by noting that there exists a positive constant α such that

$$\varphi(z) + \alpha|z|_H + \alpha \geq 0 \quad \text{for any } z \in H, \tag{2.4}$$

we can see that

$$|\varphi(u(t))| \leq \max\{\rho_2', \alpha\rho_1 + \alpha\} \quad \text{for any } t \geq t_0(B) \text{ and } u_0 \in B.$$

Hence, by putting $\rho_2 = \max\{\rho_2', \alpha\rho_1 + \alpha\}$, we can obtain this theorem. ◇

Theorem 2.3. *For any bounded set B,*

$$\bigcup_{t \geq t_0(B)} S(t)B$$

is relatively compact in H, where $t_0(B)$ is the same in Theorem 2.1.
Proof. From Theorems 2.1 and 2.2, we can see that

$$\bigcup_{t \geq t_0(B)} S(t)B \subset H_{\rho_3}$$

,where $\rho_3 = \max\{\rho_1, \rho_2\}$.
Hence, from (A4), we can see that this theorem holds. ◇

¿From Theorems 2.1 and 2.3, by applying Lemma 2.1, we can obtain the following main theorem.

Theorem 2.4. *There exists a global attractor $\mathcal{A}$ for $\{S(t)\}_{t \geq 0}$ in $\overline{D(\varphi)}$.*

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境界に摩擦の効果を考慮した波動方程式について

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1. The mixed initial-boundary value problem

We are concerned with the existence and decay properties to the mixed initial-boundary value problem for the following equation :

$$\text{(E)} \qquad \frac{\partial^2 u}{\partial t^2} - M(t)\frac{\partial^2 u}{\partial x^2} = f(x,t), \quad 0 < x < 1, \ t \geq 0,$$

$u = u(x,t)$, with initial condition

$$\text{(I)} \qquad u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \quad 0 < x < 1,$$

and boundary condition

$$\text{(B)} \qquad u(0,t) = 0, \quad \frac{\partial u}{\partial x}(1,t) = -\alpha\frac{\partial u}{\partial t}(1,t), \quad \alpha > 0, \ t > 0,$$

where we set

$$M(t) \equiv a + b\Big(\int_0^1 \Big|\frac{\partial u}{\partial x}(x,t)\Big|^2 dx\Big)^\gamma$$

with some constants $a > 0, b \geq 0, \gamma \geq 1$.

Let $H^m(\Omega), m = 0, 1, \cdots, \Omega = (0,1)$, (in particular $H^0(\Omega) = L^2(\Omega)$) denote the usual Sobolev space on Ω with norm $\|\cdot\|_{H^m}$. $L^p(\Omega), 1 \leq p \leq \infty$, denote the usual Banach space with norm $\|\cdot\|_p$ and we write often $\|\cdot\|$ for $\|\cdot\|_2$. $(\cdot,\cdot) = (\cdot,\cdot)_{L^2}$ denotes the inner product of $L^2(\Omega)$.

Our study (cf. [O2]) is motivated by Tucsnak's work [Tu], where global existence and exponetial decay estimate of the first energy of (E-I-B) are proved under that $f(x,t) \equiv 0$ and the initial data $(u_0, u_1) \in H^2(\Omega) \times H^1(\Omega)$ are small. One of the object of this report is to improve the results in [Tu] by simplifying the arguments. We derive the global existence and exponential decay estimate including the first and second energies of (E-I-B) for a set of initial data which is unbounded in $H^2(\Omega) \times H^1(\Omega)$ in a certain sense.

Many authors have treated the problem (E-I) with the homogeneous Dirichlet boundary condition :

$$(\mathrm{B_0}) \qquad u(x,t)|_{\partial\Omega} = 0 \quad \text{(i.e. } u(0,t) = u(1,t) = 0),$$

admitting the higher demensional equations. The existence of local solutions for this problem have been investigated well by Menzala [Me], Ebihara et at. [EM], Yamada [Ya], Mederios & Miranda [MM], Arosio & Garavaldi [AG], Crippa[Cr] and the references cited therein.

Concerning the global solutions to the problem (E-I-B_0), Pohožaev [Po], Arosio & Spagnolo [AS], and Nishihara [N1] proved the existence of such solutions in some classes of generalized analytic functions. But, there seems to be no result on the existence of global solutions in usual Sobolev spaces.

If we add an appropriate dissipative term such as $u_t, -\Delta u_t$, etc. to the equation (E) itself the situation is different, and various results on the global existence and decay in Sobolev spaces have been established including the case $a = 0$ (see Nishihara [N2], [N3], Matos & Pereira [MP], Nishihara & Ono [NO], [ON], Nakao [Na] and the references cited therein).

The equation we consider is added a dissipation at the boundary (cf. [Ch], [La], [KZ], [Li]) and the equation itself is not deformed by dissipation, which seems to be an interesting situation.

We make the following assumption of $f(x,t)$ appearing in the equation (E).

Hyp. 1. (i) $f(x,t)$ belongs to $C^1([0,\infty); L^2(\Omega))$ and satisfies

$$\|f(t)\|^2 \le d_0^2 e^{-\mu_0 t}, \qquad t \ge 0$$

with some constants $d_0 \ge 0, \mu_0 \ge 0$.

(ii) $f(x,t)$ belongs to $C^0([0,\infty); H_o^1(\Omega))$ and satisfies

$$\|f_x(t)\|^2 \le d_1^2 e^{-\mu_1 t}, \qquad t \ge 0$$

with some constants $d_1 \ge 0, \mu_1 \ge 0$.

We often use the following functionals to denote energies for the equation :

$$E_1(t) \equiv \|u_t(t)\|^2 + (a + \frac{b}{\gamma+1}\|u_x(t)\|^{2\gamma})\|u_x(t)\|^2$$

$$E_2(t) \equiv \|u_{xt}(t)\|^2 + (a + b\|u_x(t)\|^{2\gamma})\|u_{xx}(t)\|^2$$

Our result reads as follows :

Theorem 1. *Let (u_0, u_1) belong to $H^2(\Omega) \times H^1(\Omega)$ and satisfy*

$$(*) \qquad u_0(0) = 0, \quad \frac{du_0}{dx}(x) = -\alpha u_1(x)$$

and let $f(x,t)$ satisfy Hyp.1. For any ε such that

$$(a) \qquad 0 < \varepsilon \le \min\{\frac{1}{2}, \frac{a}{2}, \frac{2\alpha a}{1+\alpha^2 a}\},$$

suppose that (u_0, u_1) *and* f *satisfy*

(b) $$8\gamma a^{-(\gamma+1/2)} b B_1^{2\gamma-1} B_2 < \varepsilon.$$

Then the problem (E-I-B) admits a unique solution u *having the regularity*

$$u \in \bigcap_{k=0}^{2} C^k([0,\infty); H^{2-k}(\Omega)).$$

Moreover, this solution u *satisfies the following decay properties :*

$$\|u_t(t)\|^2 + \|u_x(t)\|^2 \leq \begin{cases} C_{[1]} e^{-\min\{\varepsilon/3, \mu_0\} t} & \text{if } \varepsilon/3 \neq \mu_0 \\ C_{[1]} t e^{-\varepsilon t/3} & \text{if } \varepsilon/3 = \mu_0 \end{cases}$$

$$\|u_{xt}(t)\|^2 + \|u_{xx}(t)\|^2 \leq \begin{cases} C_{[2]} e^{-\min\{\varepsilon/3, \mu_1\} t} & \text{if } \varepsilon/3 \neq \mu_1 \\ C_{[2]} t e^{-\varepsilon t/3} & \text{if } \varepsilon/3 = \mu_1 \end{cases}$$

and

$$\|u_{tt}(t)\|^2 \leq \begin{cases} C_{[2]} e^{-\min\{\varepsilon/3, \mu_1, \mu_0\} t} & \text{if } \varepsilon/3 \neq \mu_1 \\ C_{[2]} (t e^{-\varepsilon t/3} + e^{-\mu_0 t}) & \text{if } \varepsilon/3 = \mu_1 \end{cases}$$

where $C_{[1]}$ *(resp.* $C_{[2]}$*) is certain positive constants depending on* $E_1(0), d_0, \varepsilon$ *(resp.* $E_1(0), E_2(0), d_0, d_1, \varepsilon$*), and*

$$\begin{aligned} B_1^2 &\equiv 3E_1(0) + 2(1 + 2\varepsilon^{-1}) d_0^2 \\ B_2^2 &\equiv 3E_2(0) + 4(1 + 2\varepsilon^{-1}) d_1^2 \\ E_1(0) &\equiv \|u_1\|^2 + (a + \frac{b}{2}\|\frac{du_0}{dx}\|^2)\|\frac{du_0}{dx}\|^2 \\ E_2(0) &\equiv \|\frac{du_1}{dx}\|^2 + (a + b\|\frac{du_0}{dx}\|^2)\|\frac{d^2u_0}{dx^2}\|^2. \end{aligned}$$

Remark. Let S_ε be the set (u_0, u_1, f) satisfying (b) and define S by the union of S_ε for ε such that (a). Then the assumption of Theorem 1 is valid for $(u_0, u_1, f) \in S$. S is unbounded in $H^2(\Omega) \times H^1(\Omega)$.

The proof of Theorem 1 is given by the following three propositions. First, applying Banach fixed point theorem, we obtain the following local existence result for the problem (E-I-B) as in [Tu].

Proposition 1. *Let* $f \in C^1([0,\infty); L^2(\Omega)) \cap C^0([0,\infty); H_o^1(\Omega))$ *and* $(u_0, u_1) \in H^2(\Omega) \times H^1(\Omega)$ *satisfying* (*)*. Then there exists* $T > 0$ *and a unique function*

$$u \in \bigcap_{k=0}^{2} C^k([0,T); H^{2-k}(\Omega))$$

verifying (E-I-B). Moreover, one of the following affirmations holds true :

$$T = \infty \quad \text{or}$$
$$\lim_{t \nearrow T} \{\|u(t)\|_{H^2} + \|u_t(t)\|_{H^1}\} = \infty .$$

Next, we shall give a result with respect to the estimate of the first energy $E_1(t)$.

Proposition 2. *Let $(u_0, u_1) \in H^2(\Omega) \times H^1(\Omega)$ satisfy (*) and let $f(x,t)$ satisfy Hyp.1 (i). Then, for any ε with (a), the first energy $E_1(t)$ of (E-I-B) has the estimate*

$$E_1(t) \leq 3E_1(0)e^{-\varepsilon t/3} + 2(1 + 2\varepsilon^{-1})d_0^2 e^{-\min\{\varepsilon/3, \mu_0\}t}$$

with the following exceptional case : If $\varepsilon/3 = \mu_0$, we need the second term of the right hand side is replaced by

$$2(1 + 2\varepsilon^{-1})d_0^2 t e^{-\varepsilon t/3} .$$

Next, we shall give the following result with respect to the estimate of the second energy $E_2(t)$.

Proposition 3. *Under the assumptions of Theorem 1, the second energy $E_2(t)$ of (E-I-B) has the estimate*

$$E_2(t) \leq 3E_2(0)e^{-\varepsilon t/3} + 4(1 + 2\varepsilon^{-1})d_1^2 e^{-\min\{\varepsilon/3, \mu_1\}t}$$

with the following exceptional case : If $\varepsilon/3 = \mu_1$, we need the second term of the right hand side is replaced by

$$4(1 + 2\varepsilon^{-1})d_1^2 t e^{-\varepsilon t/3} .$$

2. The Periodic Problem

We are concerned with the existence and uniqunes of a bounded solution or a periodic solution for the following equation :

$$\text{(E')} \qquad \frac{\partial^2 u}{\partial t^2} - M(t)\frac{\partial^2 u}{\partial x^2} = f(x,t), \quad 0 < x < 1, \ t \in \mathbb{R},$$

$u = u(x,t)$, with boundary condition

$$\text{(B')} \qquad u(0,t) = 0, \quad \frac{\partial u}{\partial x}(1,t) = -\alpha\frac{\partial u}{\partial t}(1,t), \quad \alpha > 0, \ t \in \mathbb{R}.$$

Since we treat the case $f(x,t) \not\equiv 0$ in section 1, we can prove as a consequence of our analysis the existence of periodic solution when $f(x,t)$ is periodic in t. Our results correspond to those in Ono [O1], where some quasilinear wave equation with usual dissipative terms are treated.

We make the following assumption of $f(x,t)$ appearing in the equation (E').

Hyp. 2. $f(x,t)$ belongs to $C^1(\mathbb{R}; L^2(\Omega)) \cap C^0(\mathbb{R}; H_o^1(\Omega))$ and satisfies

$$\|f(t)\|^2 \le d_0^2, \quad \|f_x(t)\|^2 \le d_1^2, \quad t \in \mathbb{R}$$

with some constants $d_i \ge 0, i = 1, 2$.

Applying the results of previous theorem, we can get the following :

Theorem 2. *For any ε with (a),suppose that f satisfies Hyp.2 and*

$$4\gamma a^{-\gamma}(1 + 3a^{-1/2})b\tilde{B}_1^{2\gamma-1}\tilde{B}_2 < \varepsilon.$$

Then the problem (E'-B') admits a unique solution $u \in \bigcap_{k=0}^{2} C^k(\mathbb{R}; H^{2-k}(\Omega))$, satisfying

$$\sum_{k=0}^{2} \|\partial_t^k u(t)\|_{H^{2-k}} \le C_{[2]}^*,$$

where $C_{[2]}^*$ is a certain constant depending on d_0, d_1 and ε, and

$$\tilde{B}_1^2 \equiv 2(1 + 2\varepsilon^{-1})d_0^2$$
$$\tilde{B}_2^2 \equiv 4(1 + 2\varepsilon^{-1})d_1^2.$$

Remark. To apply the results of previous theorem, we consider the following mixed initial-boundary value problem :

$$u_{tt}^n - M(\|u_x^n(t)\|^2)u_{xx}^n = f^n(x,t), \quad 0 < x < 1, \quad t \in \mathbb{R}$$

with initial and boundary conditions :

$$u^n(x,-n) = u_t^n(x,-n) = 0, \quad 0 < x < 1,$$
$$u^n(0,t) = 0, \quad u_x^n(1,t) = -\alpha u_t^n(1,t), \quad \alpha > 0, \quad t \in \mathbb{R},$$

where we set

$$f^n(t) \equiv \phi(t + n - 1)f(t)$$

for $\phi(t) \in C^\infty(\mathbb{R})$ such that

$$\phi(t) = \begin{cases} 0 & \text{for} \quad t \le 0, \\ 1 & \text{for} \quad t \ge 1. \end{cases}$$

As a corollary, we can get immediately the following :

Corollary 3. (Periodic Solution) *In addition to the conditions in Theorem 2 we assume that $f(x,t)$ is ω-periodic in t. Then, the unique bounded solution u is ω-periodic in t.*

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固定境界と交わる自由境界の構成

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1 Introuction and Results

D を R^2 の滑らかな境界をもつ領域(非有界でよい)とし, Dにおいて自由境界問題

$$\Delta u = 0 \quad \text{in } D \cap \{u > 0\}, \tag{1.1}$$

$$u \geq 0 \quad \text{in } D, \tag{1.2}$$

$$|\nabla u| = 1 \quad \text{on } D \cap \partial\{u > 0\} \tag{1.3}$$

を考える. 但し, 境界条件は, Diriclet条件

$$u = u^0 \quad \text{on } \partial D \tag{1.4}$$

をおくことにする(u^0 は∂D上の与えられた非負関数). (1.1) - (1.4) をみたす関数uを stream function, $\gamma = D \cap \partial\{u > 0\}$ を自由境界という($\{u > 0\}$ は$\{ z \in D \mid u(z) > 0 \}$ を略記したもの). 曲線γが与えられているとすると, この問題は過剰決定系となる. したがって問題は(1.1) - (1.4) を満たす関数u が存在するような曲線γを定めることである.

自由境界の内部正則性についての研究は多い. V. M. Isakov (1974) は局所的に$C^{1+\alpha}$級の自由境界はそこで実解析的であることを示した. H. Alt と L. A. Caffarelli (1981) は変分法を用いて局所的に$C^{1+\alpha}$級の自由境界の存在を示した. したがって, 我々の興味は自由境界の大域的性質を調べることにある. 本稿では自由境界が固定境界と交わる場合に, 自由境界の交点の近傍での挙動を調べることにする.

先ず, 領域が半平面$D = \mathrm{R}^2_+ = \{ z = x + iy \mid x \in \mathrm{R},\ y > 0 \}$ で, 境界値が $u^0(x) = x(a + f(x))$ for $x > 0$, $= 0$ for $x \leq 0$ ($f(x)$ は $(0, +\infty)$上の与えられた関数, a $(0 < a < 1)$ は定数) の場合を考える. $f \equiv 0$のとき$u^*(z) = \max\{ax - \sqrt{1-a^2}\, y,\ 0\}$ が(1.1) - (1.4)のexact なstream function となる. したがって, f がある関数空間のノルムの意味で十分小さいときに, $f \equiv 0$の場合のperturbation として, 逐次近似法で解(u, γ)を構成することができる. この場合, $f(+0)$が存在すれば, 自由境界は固定境界と角度をもって交わり, fが$x = 0$の近傍で無限回振動していると, 自由境界も振動する(島倉紀夫氏との共同研究).

以下では, exact なstream functionが見つからない場合を考える. 問題を簡単にするために, 領域は単位円板

$$D = \{ z = re^{i\theta} \mid 0 \leq r < 1,\ 0 \leq \theta < 2\pi \}, \tag{1.5}$$

境界値は

$$u^0(z) = \max\{\varepsilon y,\ 0\} \quad (z = x + iy), \tag{1.6}$$

(ε は正の小さなパラメータ) と仮定する. このとき $u_0(z) = \max\{\log r + (\varepsilon \sin\theta)/r,\ 0\}$ とおくと, u_0 は次を満たす: $\Delta u_0 = 0$ in $D \cap \{u_0 > 0\}$, $u_0 \geq 0$ in D, $u_0 = u^0$ on ∂D, $|\nabla u_0| = 1 + O(\varepsilon^2)$ on $D \cap \partial\{u_0 > 0\}$. また, $\partial\{u_0 > 0\}$ は, $\gamma_0 = \{r = 1 - \varepsilon\sin\theta,\ 0 < \theta < \pi\}$ で近似される. εが十分小さいとき, この (u_0, γ_0) を修正することによって, (1.1) - (1.4) の解 (u, γ) を構成することができる.

定理 (1.5), (1.6) を仮定する. このとき, 次をみたす正数ε_0が存在する:

(i) $0 < \varepsilon \leq \varepsilon_0$なる任意の$\varepsilon$に対して, (1.1) - (1.4) の解 (u, γ) が存在する.

(ii) 自由境界γは $z = \pm 1$ を除いて実解析的であり, $z = \pm 1$ の近傍では, $0 < \alpha < 1$ なる任意のα に対して $C^{1+\alpha}$曲線である. また, γは凸曲線である.

(iii) $\Theta_{\pm 1}$をそれぞれ$z = \pm 1$ での自由境界と固定境界のなす角とすると, $\sin\Theta_1 = \sin\Theta_{-1} = \varepsilon$が成り立つ.

(iv) $\Omega_\gamma = D \cap \{u > 0\}$ とおくと, stream function u は次のように表される:

$$u(z) = \begin{cases} \log|z| + \dfrac{\varepsilon y}{|z|^2} + \varepsilon^2 v(z) & \text{for} \quad z \in \Omega_\gamma, \\ 0 & \text{for} \quad z \in D \setminus \Omega_\gamma. \end{cases}$$

ここでvは, $v|_{|z|=1} = 0$, $v \in C^{1+\alpha}(\overline{\Omega_\gamma})$ なるΩ_γ上の調和関数である.

(v) 次の公式が成り立つ:

$$\lim_{\substack{z \to \pm 1 \\ z \in \overline{\Omega_\gamma}}} u_x(z) = \pm\sqrt{1 - \varepsilon^2}, \qquad \lim_{\substack{z \to \pm 1 \\ z \in \overline{\Omega_\gamma}}} u_y(z) = \varepsilon.$$

(vi) $\overline{\Omega_\gamma}$で $|u_x| \leq \sqrt{1 - \varepsilon^2}/\varepsilon$, $\varepsilon \leq u_y \leq 1$ が成り立つ.

この結果は, もっと一般の境界値の場合に拡張される. 例えば境界値 (1.6) のかわりに

$$u^0(e^{i\theta}) = \max\{\varepsilon f(\theta), 0\}, \quad 0 \leq \theta < 2\pi$$

とおく. このとき fが次の3条件を満たせば上と同様の結果が成り立つ.

(H1) $f(\theta)$ は $f(\theta) = \sum_{p=0}^{\infty}(a_p \cos p\theta + b_p \sin p\theta)$ と Fourier 展開できる.

(H2) $\sum_{p=0}^{\infty} R^p(|a_p| + |b_p|) < \infty$ となる R ($R > 1$) が存在する.

(H3) ある θ_1, θ_2 ($0 \leq \theta_1 < \theta_2 < 2\pi$) が存在して, $f(\theta_1) = f(\theta_2) = 0$, $f(\theta) > 0$ for $\theta_1 < \theta < \theta_2$, $f'(\theta_1) > 0$, $f'(\theta_2) < 0$が成り立つ.

2 Mathmatical Formulation

以下では, 解の構成法についてのみ簡単に述べる. 次のように問題を formulate する. $S = \{ z = e^{i\theta} \mid 0 < \theta < \pi \}$ とし, 開区間 $(0,\pi)$ の関数 φ に対し,

$$\Gamma_\varphi = \{ z = re^{i\theta} \mid r = 1 - \varepsilon \sin\theta + \varepsilon^2 \varphi(\theta),\ 0 < \theta < \pi \}$$

とおく. また S と Γ_ψ で囲まれた内部領域を Ω_φ とする. φ が恒等的に 0 のとき, Γ_0, Ω_0 を単に Γ, Ω で表す. $\overline{\Omega_\varphi}$ 上の関数 $w(z) = w_\varphi(z)$ を, 次の境界値問題

$$\Delta w = 0 \quad \text{in} \quad \Omega_\varphi, \qquad w = \varepsilon \sin\theta \quad \text{on} \quad S, \qquad \frac{\partial w}{\partial \nu} = -1 \quad \text{on} \quad \Gamma_\varphi \tag{2.1}$$

の解とする (w_φ の存在は後に正当化される). 我々の目標は

$$w_\psi = 0 \quad \text{on} \quad \Gamma_\psi \tag{2.2}$$

をみたす $(0,\pi)$ 上の関数 ψ を求めることである. この ψ が求まると, 境界値 (1.6) に対する自由境界問題 (1.1) - (1.4) の stream function u は

$$u(z) = \begin{cases} w_\psi(z) & \text{for} \quad z \in \Omega_\psi, \\ 0 & \text{for} \quad z \in D \setminus \Omega_\psi \end{cases}$$

で与えられる. さて

$$w_0(z) = \log r + \frac{\varepsilon \sin\theta}{r}$$

とし, $w(z)$ を

$$w(z) = w_0(z) + \varepsilon^2 v(z) \tag{2.3}$$

の形で求めよう.

$$w_0|_{\Gamma_\varphi} = \varepsilon^2 \varphi + \frac{\varepsilon^2 \sin^2\theta}{2} - \varepsilon^3 \lambda[\varphi], \qquad \left.\frac{\partial w_0}{\partial \nu}\right|_{\Gamma_\varphi} = -1 + \varepsilon^2 \varphi + \varepsilon^2 (\sin^2\theta - \frac{\cos^2\theta}{2}) - \varepsilon^3 \mu[\varphi]$$

が成り立つ ($\lambda[\varphi]$, $\mu[\varphi]$ は φ と ε にのみ依存する量[1]) ので, (2.1) に代入して, $v = v_\varphi$ のみたすべき境界値問題

$$\Delta v = 0 \ \text{in} \ \Omega_\varphi, \quad v = 0 \ \text{on} \ S, \quad \frac{\partial v}{\partial \nu} = -\varphi - \sin^2\theta + \frac{\cos^2\theta}{2} + \varepsilon \mu[\varphi] \ \text{on} \ \Gamma_\varphi \tag{2.4}$$

[1] $\lambda[\varphi]$ と $\mu[\varphi]$ は次で与えられる:

$$\lambda[\varphi] = -\sigma^3 \int_0^1 \frac{(1-t)^2}{(1-t\varepsilon\sigma)^3} dt + 2\sigma^2 \sin\theta \int_0^1 \frac{1-t}{(1-t\varepsilon\sigma)^3} dt - \frac{\varepsilon\varphi}{2} \quad (\sigma = \sin\theta - \varepsilon\varphi),$$

$$\mu[\varphi] = -\frac{\sigma^3}{1-\varepsilon\sigma} + \frac{\varphi' \cos\theta}{(1-\varepsilon\sigma)^3} + \frac{\varepsilon(\varphi' - \sigma\cos\theta)^2}{2(1-\varepsilon\sigma)^4} - \frac{\cos\theta(\varphi' - \sigma\cos\theta)}{1-\varepsilon\sigma} + 6\sigma^2 \sin\theta \int_0^1 \frac{1-t}{(1-t\varepsilon\sigma)^4} dt$$

が得られる．したがって，v_φ を (2.4) の解とするとき，

$$-\psi - \frac{\sin^2\theta}{2} + \varepsilon\lambda[\psi] = v_\psi|_{\Gamma_\psi} \tag{2.5}$$

をみたす関数 ψ が求めるものである．

3 Ω における線型境界値問題と関数空間

前節で導入した写像 $\varphi \mapsto v_\varphi$ を正当化するために，次の境界値問題をφに依存しない領域で考えよう．g, $\mathrm{f} = {}^T[f_1, f_2]$ を $\Omega = \{\ z = re^{i\theta} \mid 1 - \varepsilon\sin\theta < r < 1,\ 0 < \theta < \pi\ \}$ 上の与えられた関数，h を $\Gamma = \partial\Omega \cap \{r = 1 - \varepsilon\sin\theta\}$ 上の与えられた関数とし，次の2つの境界値問題

$$\Delta u = g + \mathrm{div}\mathrm{f} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } S, \quad \frac{\partial u}{\partial \nu} = h \quad \text{on } \Gamma, \tag{3.1}$$

および

$$\Delta u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } S, \quad u - \frac{\partial u}{\partial \nu} = h \quad \text{on } \Gamma, \tag{3.2}$$

を考える．これらの境界値問題を解くために，開区間 $(0, \pi)$ 上の関数 ϕ に対する semi-norm:

$$[\phi]^*_{\alpha,(0,\pi)} = \sup\{\ \frac{|\phi(\theta) - \phi(\theta')|}{(|\log\theta - \log\theta'| + |\log(\pi - \theta) - \log(\pi - \theta')|)^\alpha} \mid 0 < \theta,\ \theta' < \pi, \quad \theta \neq \theta'\ \},$$

Ω 上の関数 u に対する semi-norm:

$$[u]^*_{\alpha,\Omega} = \sup\{\ \frac{|u(z) - u(z')|}{d(z, z')^\alpha} \mid z,\ z' \in \Omega, \quad z \neq z'\ \},$$

を導入する．但し，

$$d(z, z') = |z - z'| + |\log|z^2 - 1| - \log|z'^2 - 1|| + |\log\theta - \log\theta'| + |\log(\pi - \theta) - \log(\pi - \theta')|,$$

$z = |z|e^{i\theta}$, $z' = |z'|e^{i\theta'}$ $(0 \leq \theta,\ \theta' \leq \pi)$ である．これらを用いて，次のように関数空間を定義する．$0 < \alpha < 1$, m を非負整数とし，開区間 $(0, \pi)$ 上の C^m-級の関数 $h(\theta)$ で

$$\|h\|_{C^{m+\alpha}_*(0,\pi)} = \|h\|_{L^\infty(0,\pi)} + \|\sin^m\theta(d/d\theta)^m h\|_{L^\infty(0,\pi)} + [\sin^m\theta(d/d\theta)^m h]^*_{\alpha,(0,\pi)}$$

$$-3\sigma\cos\theta\int_0^1 \frac{dt}{(1 - t\varepsilon\sigma)^4} + \frac{3A\varepsilon\tau^4}{4(1 - \varepsilon\sigma)^4}\int_0^1 \frac{1 - t}{[1 + \{\varepsilon\tau/(1 - \varepsilon\sigma)\}^2 t]^{5/2}}dt - \frac{B\tau^2}{2(1 - \varepsilon\sigma)^2},$$

$$\tau = \cos\theta - \varepsilon\varphi', \quad A = -\frac{1}{1 - \varepsilon\sigma} + \frac{\varepsilon\sin\theta}{(1 - \varepsilon\sigma)^2} - \frac{\varepsilon^2\tau\cos\theta}{(1 - \varepsilon\sigma)^3}, \quad B = -\frac{\sigma}{1 - \varepsilon\sigma} + \frac{\sin\theta}{(1 - \varepsilon\sigma)^2} - \frac{\varepsilon\tau\cos\theta}{(1 - \varepsilon\sigma)^3}.$$

が有限なものの全体を $C_*^{m+\alpha}(0,\pi)$, Γ 上の実数値関数 $\varphi(z)$ で $\tilde{\varphi}(\theta)=\varphi((1-\varepsilon\sin\theta)e^{i\theta})\in C_*^{m+\alpha}(0,\pi)$ なるものの全体を $C_*^{m+\alpha}(\Gamma)$ で表す (norm は $\|\varphi\|_{C_*^{m+\alpha}(\Gamma)}=\|\tilde{\varphi}\|_{C_*^{m+\alpha}(0,\pi)}$).
norm

$$\|\varphi\|_{E_*^{1+\alpha}(\Gamma)}=\|\varphi/\sin\theta\|_{C_*^{1+\alpha}(\Gamma)}$$

をもつ Banach 空間を $E_*^{1+\alpha}(\Gamma)$ で表す. Ω 上の関数 $u(z)$ で

$$\|u\|_{C_*^\alpha(\Omega)}=\|u\|_{L^\infty(\Omega)}+[u]_{\alpha,\Omega}^*$$

が有限なものの全体を $C_*^\alpha(\Omega)$ で表す. norm

$$\|u\|_{E_*^{1+\alpha}(\Omega)}=\|u/y\|_{L^\infty(\Omega)}+\|\nabla u\|_{C_*^\alpha(\Omega)}$$

をもつ Banach 空間を $E_*^{1+\alpha}(\Omega)$ で表す.

以下では, 正数 ε_0 を十分小さくとって固定し, つねに $0<\varepsilon<\varepsilon_0$ と仮定する. 次の 2 つの補題が基本的である.

補題 1 $0<\alpha<1$ とする. 任意の g ($yg\in L^\infty(\Omega)$), $\mathrm{f}\in C_*^\alpha(\Omega)$, $h\in C_*^\alpha(\Gamma)$ に対して, 境界値問題 (3.1) の一意的な解 $u\in E_*^{1+\alpha}(\Omega)$ が存在し,

$$\|u\|_{E_*^{1+\alpha}(\Omega)}\le C(\|yg\|_{L^\infty(\Omega)}+\|\mathrm{f}\|_{C_*^\alpha(\Omega)}+\|h\|_{C_*^\alpha(\Gamma)})$$

が成り立つ. 但し C は (α,ε_0) にのみ依存する定数である.

補題 2 $0<\alpha<1$ とする. 任意の $h\in C_*^\alpha(\Gamma)$ に対して, 境界値問題 (3.2) の一意的な解 $u\in E_*^{1+\alpha}(\Omega)$ が存在し,

$$\|u\|_{E_*^{1+\alpha}(\Omega)}\le C\|h\|_{C_*^\alpha(\Gamma)}$$

が成り立つ. 但し C は (α,ε_0) にのみ依存する定数である.

4 Construction of the solution

まず 境界値問題 (2.4) を解く. 問題を固定された領域 Ω で考えよう. $\varphi\in E_*^{1+\alpha}(\Gamma)$ とし, $\varphi(\theta)=\phi(\theta)\sin\theta$ ($\phi\in C_*^{1+\alpha}(\Gamma)$) とおく. Ω から Ω_φ への写像

$$\Phi:(\rho,\omega)\mapsto(r,\theta),\quad r=(1-\varepsilon\phi(\omega))\rho+\varepsilon\phi(\omega),\quad \theta=\omega \tag{4.1}$$

を用いて $V(\rho,\omega)=v(\Phi(\rho,\omega))$ とおく. すると, 境界値問題 (2.4) は次の V を未知関数とする Ω における問題

$$\Delta V+\varepsilon L_\varphi V=0 \ \text{ in } \ \Omega,\qquad V=0 \quad \text{on } \ S, \tag{4.2}$$

$$\frac{\partial V}{\partial\nu}+\varepsilon B_\varphi V=-\varphi-\sin^2\omega+\frac{\cos^2\omega}{2}+\varepsilon\mu[\varphi]\quad \text{on } \ \Gamma \tag{4.3}$$

と同値である．ここで L_φ は次の形の divergence form の2階の偏微分作用素，B_φ は1階の境界作用素である:

$$L_\varphi w = \mathrm{div}(A_\varphi \nabla w) + \frac{\mathrm{b}_\varphi}{y} \cdot \nabla w, \qquad B_\varphi w = \mathrm{c}_\varphi \cdot \nabla w.$$

但し，A_φ は 成分が $C^\alpha_*(\Omega)$ である 2×2 - 行列，b_φ は 成分が $C^\alpha_*(\Omega)$ である ベクトル，c_φ は 成分が $C^\alpha_*(\Gamma)$ である ベクトルである．さらに，写像 $\varphi \mapsto A_\varphi$, $\varphi \mapsto \mathrm{b}_\varphi$, $\varphi \mapsto \mathrm{c}_\varphi$ は解析的であり，φ について1次以上である．

与えられた関数 g, $\mathrm{f} = {}^T[f_1, f_2]$, h に対して，で境界値問題 (3.1) の解を

$$u = Gg + H\mathrm{f} + Kh \tag{4.4}$$

で表す．補題1によって，G, H, K はそれぞれ $\{g \mid yg \in L^\infty(\Omega)\}$, $C^\alpha_*(\Omega)^2$, $C^\alpha_*(\Gamma)$ から $X_* = \{ v \in E^{1+\alpha}_*(\Omega) \mid v = 0 \text{ on } S \}$ への全単射である．

補題 3 $X_* = \{ v \in E^{1+\alpha}_*(\Omega) \mid v = 0 \text{ on } S \}$ とおく．$\Lambda_\varphi : X_* \to X_*$ を

$$\Lambda_\varphi u = G[(\mathrm{b}_\varphi/y)\cdot\nabla u] + H[A_\varphi \nabla u] + K[(\mathrm{c}_\varphi \cdot \nabla u)|_\Gamma]$$

で定義する．ε が十分小さいとき，$\varphi \in E^{1+\alpha}_*(\Gamma)$ ならば，$I + \varepsilon\Lambda_\varphi$ は X_* で可逆である．

これを用いると

$$V_\varphi = -K[\varphi] - K\left[\sin^2\theta - \frac{\cos^2\theta}{2}\right] + \varepsilon R[\varphi, \varepsilon] \tag{4.5}$$

が (4.2) - (4.3) の解であることが示される．但し

$$R[\varphi,\varepsilon] = -\Lambda_\varphi(I + \varepsilon\Lambda_\varphi)^{-1} K\left[-\varphi - \sin^2\theta + \frac{\cos^2\theta}{2} + \varepsilon\mu[\varphi]\right] + K\left[\mu[\varphi]\right]$$

である．$v_\varphi|_{\Gamma_\varphi} = V_\varphi|_\Gamma$ なので，方程式 (2.5) は

$$\psi - Q\psi = f_0 + \varepsilon q[\psi] \tag{4.6}$$

と同値である．但し

$$Q\varphi = (K\varphi)|_\Gamma, \tag{4.7}$$

$$f_0 = -\frac{\sin^2\theta}{2} + K\left[\sin^2\theta - \frac{\cos^2\theta}{2}\right]\Bigg|_\Gamma, \qquad q[\varphi] = \lambda[\varphi] - R[\varphi,\varepsilon]|_\Gamma$$

である．ここで，Q は φ について線型，f_0 は与えられた関数，$q[\varphi]$ は φ, ε でのみきまる関数，また f_0, $q[\varphi] \in E^{1+\alpha}_*(\Gamma)$ である．

補題 4 Q を(4.7)で定義される作用素とする. Q は $E_*^{1+\alpha}(\Gamma)$ から自分自身への連続線型作用素であり, $I-Q$ は $E_*^{1+\alpha}(\Gamma)$ で可逆である(補題2による). また

$$\|(I-Q)^{-1}h\|_{E_*^{1+\alpha}(\Gamma)} \leq C\|h\|_{E_*^{1+\alpha}(\Gamma)}$$

が成り立つ. C は ε に依らない定数である.

方程式(4.6)の解 ψ は, 半線型作用素

$$\mathcal{F}\varphi = (I-Q)^{-1}(f_0+\varepsilon q[\varphi]). \tag{4.8}$$

の不動点として与えられる.

補題 5 $\mathcal{F}$ を(4.8)で与えられる作用素とし, $M = 2\|(I-Q)^{-1}f_0\|_{E_*^{1+\alpha}(\Gamma)}$ とおく. ε が十分小さいとき, 次が成り立つ.

(i) $\mathcal{F}$ は $N = \{\ \varphi \in E_*^{1+\alpha}(\Gamma) \mid \|\varphi\|_{E_*^{1+\alpha}(\Gamma)} \leq M\ \}$ からそれ自身への連続作用素である.

(ii) $\varphi,\ \psi \in N$ ならば,

$$\|\mathcal{F}\varphi - \mathcal{F}\psi\|_{E_*^{1+\alpha}(\Gamma)} \leq c\|\varphi-\psi\|_{E_*^{1+\alpha}(\Gamma)} \tag{4.9}$$

が成り立つ. 但し, c は $0<c<1$ をみたす定数である.

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Application of Almost Analytic Continuation to the Summabilty of Riesz Mean of Eigenfunction Expansions.

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Abstract. In this paper, we shall show the summabilty of Riesz mean of eigenfunction expansions, that is, for $\phi \in L^p(R^n)$ $(1 \le p < \infty), \delta > n|1/2 - 1/p|$,

$$\lim_{t \to +0} f_\delta(tH)\phi = \phi$$

in L^p sense. Here, $f_\delta(\sigma) = \max\{(1-\sigma)^\delta, 0\}$, and the Schrödinger operator $H = -\Delta + V$. The main idea of the proof is to use " almost analytic continuation " of the function $f_\delta(\sigma)$ with one point singularity at 1. By using such method, we have only to concentrate our attention to to the $L^p - L^p$ boundedness properties of the resolvent $(H - z)^{-1}$ for $z \notin \sigma(H)$ in order to obtain that of $f_\delta(tH)$.

1 Introduction and Theorems

There are so many papers and books on the convergence or divergence of the Fourier series expansion in the almost everywhere sense or in the L^p sense (for instance , cf [7]). This problem which originated with Fourier in 1811 has had a rich and eventful history. It has given rise to an entire branch of modern mathematics (abstract harmonic analysis).

As generalization of the Fourier series expansion , what happen to the case of the eigenfunction expansions of ϕ where $\{\psi_j\}_{j=0}^{\infty}$ are the eigenfunctions of some self-adjoint operator H on L^2? It is well-known as the Hilbert - Schmidt theorem that the eigenfunction expansion of $\phi \in L^2(R^n)$ converges to ϕ in L^2 sense (cf. [4]). However, how about for $\phi \in L^p(R^n)$ $(p \neq 2)$? As in the case of Fourier series expansion, such a problem is negatively proved (cf. [7] for Fourier series expansion case , [3] for other types of expansions).

On the other hand, we know that, in the Fourier series case, the pointwise convergence is shown for some weighted Fourier series expansions (Cesàro mean , etc. cf. [7]). So it is of some interest to investigate the convergency for the weighted eigenfunction expansions :

$$S_t(\phi) := \sum_{j=0}^{\infty} f(t\mu_j) < \psi_j, \phi > \psi_j \tag{1}$$

where μ_j is an eigenvalue of H associated with ψ_j , $< \psi_j, \phi >= \int_{R^n} \overline{\psi_j}\phi_j dx$ and the weight function f is , roughly speaking, a continuous function on $\mathbf{R}$ such that $f(0) = 1$, $\lim_{\sigma \to \infty} f(\sigma) = 0$.

Note that it seems that $S_t(\phi)$, formally , tends to ϕ as $t \to +0$ since $f(0) = 1$. In particular, when the weight function

$$f(\sigma) = f_\delta := (1-\sigma)_+^\delta, \tag{2}$$

we call (1) "*the Riesz mean*" of eigenfunction expansion. In this paper, we investigate the summability of the Riesz mean of eigenfunction expansions.

By using the spectral decomposition theorem in spectral theory (cf. [4]), we can represent (1) as follows,

$$S_t(\phi) = f_\delta(tH)\phi, \tag{3}$$

for the self-adjoint operator H. Hence we can generalize the problem much more, that is, what happens in the case of the self adjoint operator H with continuous spectra as well?

Before stating our main theorem, we list up some assumptions.

Assuptions.

Let $H = -\Delta + V$, where Δ is the n dimensional Laplacian $(= \Sigma_{j=1}^n \partial^2/\partial x_j^2)$ and V is a mesurable function on R^n such that

$V = V_+ - V_-$,

(V.1) $V_+ \in L^2_{loc}(R^n)$,

(V.2) $V_- \in S_n \cap K_n$,

where $V_\pm(x) = \max\{\pm V(x), 0\}$, and S_n, K_n are Stummel class and Kato class of ptential functions, respectively, defined by

$V_- \in S_n \overset{\text{def}}{\Longleftrightarrow}$

$$\sup_{x\in R^n} \int_{|x-y|\le 1} |V_-(y)|^2 dy < \infty \qquad (if\, n \le 3),$$
$$\lim_{\alpha\to+0} \sup_{x\in R^n} \int_{|x-y|\le\alpha} \log(|x-y|^{-1})|V_-(y)|^2 dy = 0 \qquad (if\, n = 4),$$
$$\lim_{\alpha\to+0} \sup_{x\in R^n} \int_{|x-y|\le\alpha} \frac{|V_-(y)|^2}{|x-y|^{n-4}} dy = 0 \qquad (if\, n \ge 5),$$

$V_- \in K_n \overset{\text{def}}{\Longleftrightarrow}$

$$\sup_{x\in R^n} \int_{|x-y|\le 1} |V_-(y)| dy < \infty \qquad (if\, n = 1),$$
$$\lim_{\alpha\to+0} \sup_{x\in R^n} \int_{|x-y|\le\alpha} \log(|x-y|^{-1})|V_-(y)| dy = 0 \qquad (if\, n = 2),$$
$$\lim_{\alpha\to+0} \sup_{x\in R^n} \int_{|x-y|\le\alpha} \frac{|V_-(y)|}{|x-y|^{n-2}} dy = 0 \qquad (if\, n \ge 3).$$

Under these assumptions, V is $H_0 = -\Delta$ bounded. Therefore, according to the Kato-Rellich's theorem, the Hamiltonian H is an essentially self-adjoint operator on $L^2(R^n)$ with its domain $\mathcal{D}(H) = C_0^\infty$. In what follows, we denote the self-adjoint extension of H by the same symbol H.(cf. [5])

Note that H is lower semi-bounded, i.e.

$$< u, Hu > \ge -c\|u\|_{L^2}^2, \tag{4}$$

for $u \in \mathcal{D}(H)$.

To tell the truth, the condition $V_- \in K_n$ can be excluded if we want to obtain only the essentially self-adjointness of H. But when we obtain the $L^p - L^p$ boundedness properties of the resolvent $(H - z)^{-1}$ in § 3, this condition is indispensable since we have to use Feynmann-Kac formula. As to this point, the brief explanation will be done in § 3.

Now let us state the main theorem.

Theorem 1 *Let $\delta > n|1/2 - 1/p|$ $(1 \leq p \leq \infty)$, then the operator $f_\delta(H)$ can be extended as an operator on $L^p(R^n)$, furthermore*

$$\|f_\delta(tH)\|_{B(L^p)} \leq c_\delta, \tag{5}$$

for $t \in (0, 1]$, where the positive constant C_δ is independent of $t \in (0, 1]$, but dependent on δ .

Note that we denote the extended operator by the same symbol as $f_\delta(tH)$, and in what follows, we use this notation.

we obtain our main results as a corollary of the theorem.

Corollary 2 *For $1 \leq p < \infty$, and for $\delta > n\,|1/2 - 1/p|$,*

$$\lim_{t \to +0} f(tH)\phi = \phi, \tag{6}$$

in $L^p(R^n)$, for $\phi \in L^p(R^n)$.

The outline of proof for main theorem will be done in § 4, and for the corollary, in § 5.

Habicsh (cf.[1]) obtain the same results under the assumption $V \geq 0, V \in L^q(R^n)$ $(q > \max\{n/2, 1\})$. So our results are stronger at the point where we deal with much wider class of potentials.

2 Representation of $f_\delta(tH)$ due to Almost Analytic Continuation

In this section, we shall exploit the "*almost analytic continuation*" of f_δ, that is, the extention of the function defined on R to the whole complex plain C. By virtue of this tool, we can represent $f_\delta(tH)$ by the resolvent $(H - z)^{-1}$. So, once we know the $L^p - L^p$ boundedness of $(H - z)^{-1}$, we can obtain the $L^p - L^p$ boundedness property of $f_\delta(tH)$.

For the function with some smoothness and decay properties, the almost analytic continuation is well-known (cf [2]). However, in our case, f_δ has a singularity at $\sigma = 1$, we have to constract the continuation carefully.

Since the Hamiltonian H is lower semi-bounded (therefore, its spectral measure has support on $\lambda \geq -c$, and $\sigma(H) \subset \{\lambda | \lambda \geq -c\}$), it suffices to consider $\tilde{f}_\delta(tH)$ where $\tilde{f}_\delta$ is a smooth cut-off of f_δ ($\tilde{f}_\delta(\sigma) = 0$ if $\sigma < -2c$).

Proposition 3 (almost analytic continuation of $\bar{f}_\delta$) *There exists a function $\tilde{F}_\delta$ defined on* C *such that*

1. $\tilde{F}_\delta(z) \in C(\mathbf{C}) \cap C^\infty(\mathbf{C} \setminus \{1\})$,
2. $\tilde{F}_\delta(z) = \bar{f}_\delta(z)$ *if* $\Im z = 0$,
3. $supp \tilde{F}_\delta$ *is compact in* C,
4. $supp \frac{\partial \tilde{F}_\delta}{\partial \bar{z}}$
$\subset \{z \in \mathbf{C} \setminus \{1\} | \frac{2\pi}{3} \leq |Arg\,(z-1)| \leq \frac{5\pi}{6}$ *or* $\frac{7\pi}{6} \leq |Arg\,(z-1)| \leq \frac{4\pi}{3}\}$,
furtheremore
$|\frac{\partial \tilde{F}_\delta}{\partial \bar{z}}(z)| \leq C_\beta |z-1|^{\delta-\beta-1} |\Im z|^\beta$ *for* $\beta \geq 0$,
5. $(z-1)^{-1} \frac{\partial \tilde{F}_\delta}{\partial \bar{z}} \in L^1(\mathbf{C})$.

where $\frac{\partial \phi}{\partial \bar{z}}(z) = \frac{1}{2}(\frac{\partial}{\partial x} - \frac{\partial}{i\partial y})\phi(x+iy)$ *for* $z = x+iy$, *and* $L^1(\mathbf{C}) \simeq L^1(R^2)$.

Outline of the proof.

Note that $\bar{f}_\delta$ can be extended to the whole complex plane by taking a proper branch. We introduce two functions η, χ such that

(1) $\eta \in C^\infty(\mathbf{C}), \eta(z) = 0$ if $\Re z < -2c$, and $\eta(z) = 1$ if $\Re z > -c$,

(2) $\chi \in C^\infty(\mathbf{C} \setminus \{0\})$, χ is homogeneous of degree zero, i.e.$\chi(\alpha z) = \chi(z)$ for $\alpha > 0$ and furtheremore $\chi(z) = 0$ if $|Arg\,z| < 2\pi/3$ or $\pi/3 < |Arg\,z| \leq 2\pi$, $\chi(z) = 1$ if $5\pi/6 < |Arg\,z| < 7\pi/6$.

Then $\tilde{F}_\delta(z) = \eta(z) f_\delta(z) \chi(z-1)$ is one asserted in the proposition.□

Proposition 4 *Let $\tilde{F}_\delta$ to be as in proposition 2.1. Then for $\sigma \in R$,*

$$\bar{f}_\delta(\sigma) = \frac{1}{2\pi i} \int_{\mathbf{C}} \frac{\partial \tilde{F}_\delta}{\partial \bar{z}}(z) \frac{1}{z-\sigma} dz \wedge d\bar{z}, \tag{7}$$

where $dz \wedge d\bar{z} = -2i dx \wedge dy$ when $\mathbf{C} \simeq R^2$.

Outline of the proof.

Let Ω be a domain in C containing 1 with a smooth boundary $\partial\Omega$ and let $B_\epsilon(1)$ be a ball with its center at 1 and radius ϵ. Then carrying out the proof as in the case of the Cauchy's integral formula, we obtain

$$\tilde{F}_\delta(\zeta) = \frac{1}{2\pi i}\left(\int_{\partial\Omega} \frac{\tilde{F}_\delta(z)}{z-\zeta} dz - \int_{\partial B_\epsilon(1)} \frac{\tilde{F}_\delta(z)}{z-\zeta} dz\right) + \frac{1}{2\pi i} \int_{\Omega \setminus B_\epsilon(1)} \frac{\partial \tilde{F}_\delta}{\partial \bar{z}}(z) \frac{1}{z-\zeta} dz \wedge d\bar{z}, \tag{8}$$

for $\zeta \in \Omega, \zeta \neq 1$. Since $\tilde{F}_\delta(z) = 0$ for z with $|z|$ so large and $\frac{\partial \tilde{F}_\delta}{\partial \bar{z}}(z)\frac{1}{z-\zeta}$ is absolutely integrable in C, we have the result by taking Ω to whole C and $\epsilon \to +0$. For the case of $\zeta = 1$, we can show that the right hand side of (2.2) is equal to 0 by the analogy of the proof of Cauchy's integral formula. Note that $\tilde{F}_\delta(1) = 0$, $\tilde{F}_\delta$ is continuous and that $\frac{\partial \tilde{F}_\delta}{\partial \bar{z}}(z)\frac{1}{z-1}$ is absolutely integrable, we obtain the results in this case as well. □

Remark. When we apply the analogous proof of the Cauchy's integral formula, the function must possess some smoothness. So we had to exclude the ϵ- neighborhood of 1 in the above.

3 $L^p - L^p$ boundedness of the resolvent $(H - z)^{-1}$

In this section, we list up some results, wihtout proofs, on the $L^p - L^p$ boundedness of the resolvent $(H - z)^{-1}$. Theorem 5 below is the consequence of Jensen and Nakamura (cf. [2]). Note that $(H-z)^{-1} = \int_0^\infty \exp(-tH+tz)dt$ for z with $\Re z$ very negative, then $L^p - L^p$ boundedness of $(H - z)^{-1}$ follows from that of the semigroup e^{-tH}.

In order to obtain the $L^p - L^p$ boundedness of e^{-tH}, the Feynmann-Kac formula is applied.(It is a useful and powerful tool.) The semigroup is represented with the measure on "*the space of paths*" as follows :

$$\left[e^{-tH}\phi\right](x) = \int_\Omega exp\{-\int_0^t V(\omega(s))\,ds\}\phi(\omega(t))\,d\mu_x(\omega), \tag{9}$$

where ω is the set of all paths (precisely speaking, $\omega = \Pi_{0\le t<\infty}\dot{R}^n$ and $\dot{R}^n$ is one pint compactification of R^n) and μ_x is the measure on ω. If V is bounded from below, there is no problem on the boundedness of the term $exp\{-\int_0^t V(\omega(s))\,ds\}$, of course. However, how about the case where V is unbounded from below? If V_- (negative part of V) belongs to K_n (Kato class, see §1), the bundedness of the exponential term is guaranteed by Khasmin'skii's lemma (cf [6]). Hence $L^p - L^p$ boundedness of e^{-tH} easily follows from the Hölder's inequality and interpolation theory etc. Note that $\int_\Omega \phi(\omega(t))\,d\mu_x(\omega) = (4\pi t)^{-\frac{n}{2}}\int e^{-\frac{|x-y|^2}{4t}}\phi(y)\,dy$ (case of $V = 0$) (for detail, see [...]).

In Theorem 5 below, it is very important to note that the $L^p - L^p$ boundedness is obtained for all z with $z \in \mathbf{C}\setminus R$.

Theorem 5 (Jensen-Nakamura [2]) *Let $1 \le p \le \infty$ and $\beta = n|1/2 - 1/p|$. Then there exists $C > 0$ such that*

$$\|(H - z)^{-1}\|_{B(L^p)} \le C\frac{<z>^\beta}{|\Im z|^{\beta+1}}, \tag{10}$$

for $z \in \mathbf{C}\setminus R$, where $<z> = (1 + |z|)^{\frac{1}{2}}$.

The following proposition is very useful when we obtain a certain inequality (see Proposition 7 in the next section), which will be used to prove $\lim_{t\to+0} f_s(tH)\phi = \phi$ in $L^1(R^n)$, for $\phi \in L^1(R^n)$. The proof is almost similer to that of Theorem 5 in the case of $p = 1$, we omit it.

Proposition 6 *For any positive ℓ and any $\phi \in L^1_{comp}(R^n)$ with $supp\phi \subset B_{R_0}$, there exists a positive constant $C_{\ell,R_0} > 0$ such that*

$$\|\chi_{|x|\ge R}(H - z)^{-1}\phi\|_{L^1} \le C_{\ell,R_0}\,R^{-\ell}\frac{<z>^{\frac{n}{2}-\ell}}{|\Im z|^{\frac{n}{2}+\ell+1}}\|\phi\|_{L^1(R^n)}, \tag{11}$$

for $R > 2R_0$ and for $z \in \mathbf{C}\setminus\mathbf{R}$, where $\chi_{|x|\ge R}$ is the characteristic function with support out of the ball B_R.

4 Proof of Theorem 1

In virture of proposition 2.2, Theorem 3.1 and the spectal decomposition theorem, we obtain the following : for $\delta > n|1/2 - 1/p|$,

$$f_\delta(tH) = \frac{1}{2\pi i}\int_{\mathbf{C}} \frac{\partial \tilde{F}_\delta}{\partial \bar{z}}(z)\,(z - tH)^{-1}\,dz\,d\bar{z}, \tag{12}$$

changing the scale of variables,

$$= \frac{1}{2\pi i}t\int_{\mathbf{C}} \frac{\partial \tilde{F}_\delta}{\partial \bar{z}}(tz)\,(z - H)^{-1}\,dz\,d\bar{z}. \tag{13}$$

Note that $f_\delta(tH) = \tilde{f}_\delta(tH)$ for $t \in (0,1]$ (see §2) and that the integration converges in the $B(L^p)$-operator norm sense. Thus, by the easy calculation, we can show that

$$\|f_\delta(tH)\|_{B(L^p)} \le C_\delta, \tag{14}$$

where C_δ is independent of $t \in (0,1]$. Hence we completes the proof of the main theorem. □

From proposition 6 in the former section, the next follows.

Proposition 7 *For any $\delta > \frac{n}{2}$, there exists $\ell > 0$ such that*

$$\|\chi_{|x|\ge R} f_\delta(tH)\phi\|_{L^1} \le C_{\ell,R_0,\delta} R^{-\ell}\|\phi\|_{L^1}, \tag{15}$$

for large R ($\gg R_0$), where $\phi \in L^1_{comp}(R^n)$ with $supp\,\phi \subset B_{R_0}$ and the positive constant $C_{\ell,R_0,\delta}$ is independent of $t \in (0,1]$.

proof.

Choose sufficiently small $\ell > 0$ so that $\frac{n}{2} < \frac{n}{2} + \ell < \delta$. Then by (13),

$$\chi_{|x|\ge R} f_\delta(tH)\phi = \frac{1}{2\pi i}t\int_{\mathbf{C}} \frac{\partial \tilde{F}_\delta}{\partial \bar{z}}(tz)\,\chi_{|x|\ge R}(z - H)^{-1}\phi\,dz\,d\bar{z}. \tag{16}$$

Calculating as in the proof of the main theorem, we obtain the proposition.□

5 proof of the corollary

It is obvious in the case $p = 2$ by the spectral decomposition and dorminated convergence theorem, so we separate the proof into the three cases $2 < p < \infty$, $1 < p < 2$ and $p = 1$.

(1) $2 < p < \infty$. Take $\phi_\epsilon \in C_0^\infty(R^n)$ such that $\|\phi - \phi_\epsilon\|_{L^p} < \epsilon$. Then for $\phi \in L^p(R^n)$,

$$\|f_\delta(tH)\phi - \phi\|_{L^p} \le \|f_\delta(tH)(\phi - \phi_\epsilon)\|_{L^p} + \|f_\delta(tH)\phi_\epsilon - \phi_\epsilon\|_{L^p} + \|\phi_\epsilon - \phi\|_{L^p}, \tag{17}$$

by virtue of the main theorem, there exists a constant $C > 0$ independent of $t \in (0,1]$,

$$\le C\|\phi - \phi_\epsilon\|_{L^p} + \|f_\delta(tH)\phi_\epsilon - \phi_\epsilon\|_{L^p}. \tag{18}$$

Note that there exists $p' > p$ and $\theta \in (0,1)$ such that

$$\frac{\theta}{p'} + \frac{1-\theta}{2} = \frac{1}{p} \text{ and } n(\frac{1}{2} - \frac{1}{p}) < n(\frac{1}{2} - \frac{1}{p'}) < \delta. \tag{19}$$

Thus

$$\begin{aligned} \|f_\delta(tH)\phi_\epsilon - \phi_\epsilon\|_{L^p} &\le \|f_\delta(tH)\phi_\epsilon - \phi_\epsilon\|_{L^{p'}}^{\theta} \|f_\delta(tH)\phi_\epsilon - \phi_\epsilon\|_{L^2}^{1-\theta}, & (20) \\ &\le C\|\phi_\epsilon\|_{L^{p'}}^{\theta} \|f_\delta(tH)\phi_\epsilon - \phi_\epsilon\|_{L^2}^{1-\theta}. & (21) \end{aligned}$$

We completes the proof in this case.

(2) $1 < p < 2$.

The way of the proof is similar to that of (1), so we omit.

(3) $p = 1$.

In (18), noting that for large $R > 0$,

$$\|f_\delta(tH)\phi_\epsilon - \phi_\epsilon\|_{L^1} \le \|\chi_{|x|>R} f(tH)\phi_\epsilon\|_{L^1} + \|\chi_{|x|>R}(f(tH)\phi_\epsilon - \phi_\epsilon)\|_{L^1},$$

by virtue of Proposition 7,

$$\le C R^{-\ell}\|\phi_\epsilon\|_{L^1} + R^{n/2}\|f(tH)\phi_\epsilon - \phi_\epsilon\|_{L^2}.$$

Hence we completes the proof of corollary.□

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Phase field 方程式の周期解の存在について

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次の問題(PP)を考える。

Find u, w s.t.

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \frac{\partial w}{\partial t} - \Delta u = f(t,x) \quad \text{in } Q := R_+ \times \Omega,\\
&\nu \frac{\partial w}{\partial t} - \kappa \Delta w + \beta(w) + g(w) \ni u \quad \text{in } Q,\\
&\frac{\partial u}{\partial n} + n_0 u = h_N(t,x) \quad \text{on } \Sigma := R_+ \times \Gamma,\\
&\frac{\partial w}{\partial n} = 0 \quad \text{on } \Sigma,\\
&u(t+T_0,\cdot) = u(t,\cdot),\ w(t+T_0,\cdot) = w(t,\cdot),\ t \geq 0
\end{aligned}
$$

ここで、Ωは$R^N(N \geq 1)$の滑らかな境界Γをもつ有界領域；ν,κは正定数；βは$R \times R$の極大単調グラフ；gはR上の連続関数；n_0は正定数；T_0は正定数；f, h_Nは与えられた関数とする。この問題においては、β, gはそれぞれ次のグラフをもつようなものを考えている。

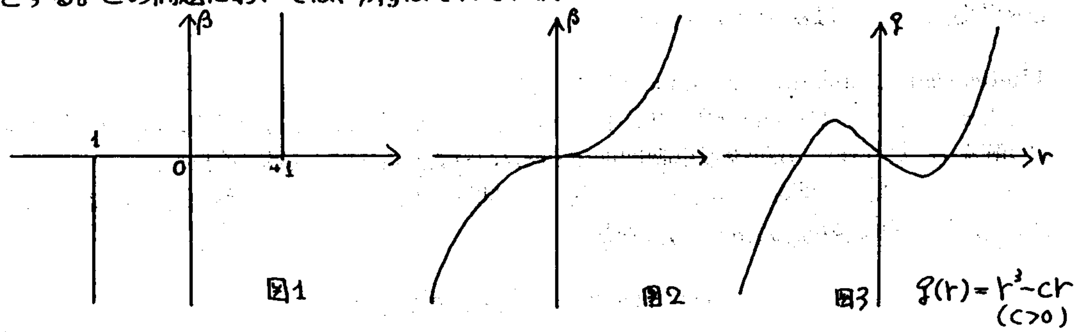

図1　図2　図3

一般に、βは図1の様に多価なので問題(PP)の第2式では"∋"が用いられている。

問題(PP)は相転移現象をモデル化したもので、Ωに水(H_2O)が満たされており、uを温度、wを相の状態を表すオーダーパラメーターと考えるとイメージしやすい。βが図1の

グラフのような場合wは次のようになる。

$$\begin{cases} w=1 & \text{液体(水)} \\ -1<w<1 & \text{mushy} \\ w=-1 & \text{個体(氷)} \end{cases}$$

この問題の導出についてはKenmochi[7]がわかりやすい。相転移問題は多くの研究者によって研究がなされている。(Kenmochi[7]のReferenceを参照)

まず、問題(PP)の第2式の$g(w)$の項を与えられた関数$\ell(t,x)$で置き換えた問題(PP_ℓ)を考えよう: Find u, w s.t.

$$\begin{aligned}
&\frac{\partial u}{\partial t}+\frac{\partial w}{\partial t}-\Delta u=f(t,x) \quad \text{in } Q:=R_+\times\Omega,\\
&\nu\frac{\partial w}{\partial t}-\kappa\Delta w+\beta(w)+\ell(t,x)\ni u \quad \text{in } Q,\\
&\frac{\partial u}{\partial n}+n_0u=h_N(t,x) \quad \text{on } \Sigma:=R_+\times\Gamma,\\
&\frac{\partial w}{\partial n}=0 \quad \text{on } \Sigma,\\
&u(t+T_0,\cdot)=u(t,\cdot),\ w(t+T_0,\cdot)=w(t,\cdot), t\geq 0
\end{aligned}$$

この問題については次の結果がある。

Theorem 1.(Damlamian-Kenmochi-Sato[3])

$\ell(t,x)\in L^2(R_+;\Omega)$ with $\ell(t+T_0,\cdot)=\ell(t,\cdot), t\geq 0$とすると、問題$(PP_\ell)$は少なくとも1つ解をもつ。特に、$\beta$が狭義単調増加であるならば解$u,w$は一意に定まる。

さらに、一般に、解の1つをu,wとすると、問題(PP_ℓ)のすべての解の集合$\wp$はβとwに依存した2つの定数$\sigma_1,\sigma_2(\sigma_1\leq\sigma_2)$を用いて、

$$\wp=\{\{u,w+\sigma\};\sigma_1\leq\sigma\leq\sigma_2\}$$

と表される。

この結果から、問題(PP_ℓ)において、温度uは一意に決まることがわかる。wについては、例えば、βが図1のような場合、一つの解が$-1\leq w(t,x)\leq 1,(t,x)\in Q$ならば、平行移動して$-1\leq w(t,x)+\sigma\leq 1,(t,x)\in Q$となっていれば、$w+\sigma$も解になることを意味している。つまり、$\beta$のグラフに"平らな"部分があれば$w$は一般に一意には決まらないということである。

さて、問題(PP)についてはどうだろうか。Theorem 1とSchauderの不動点定理を用いれば問題(PP)の解の存在が示される。

Theorem 2. 問題(PP)は少なくとも1つ解をもつ。

我々の次の目標は問題(PP)の解の構造を調べることであるが、これはそう簡単ではない。そこで、内部熱源fと境界のサーモスタットの能力を表すh_Nを調整して温度uを十分大きくした場合を考えてみることにした。数学的には、uの値が十分大きいときに、問題:Find w s.t.

$$\nu\frac{\partial w}{\partial t} - \kappa\Delta w + \beta(w) + w^3 - cw \ni u \quad \text{in } Q,$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{on } \Sigma,$$

$$w(t + T_0, \cdot) = w(t, \cdot), \ t \geq 0$$

に、一意性があるかどうかということになる。(ここで、cは正定数) 直感的に温度が十分大きければ、Ω は水(液体)(すなわち、$w \equiv 1$)で満たされるはずである。確かに、$W \equiv 1$は解の1つになることが示されるが、一意性は減少項$-cw$があるため示すことが困難である。現象を考えれば直感的に明らかなことであるから数学的に証明することもできる筈である。モデリングに問題がある可能性もある。今後、もうしばらくこの問題を考えていくつもりである。

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相転移モデルの解の漸近挙動

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We consider the following coupled system of nonlinear PDEs, referred as $(P_{\nu\kappa})$:

$$
\begin{gathered}
(\rho(u)+\lambda(w))_t-\Delta u=f(t,x) \quad \text{in } Q:=(0,T)\times\Omega,\\
\nu w_t-\kappa\Delta w+\beta(w)+g(w)\ni\lambda'(w)u \quad \text{in } Q,\\
\frac{\partial u}{\partial n}+n_0u=h(t,x) \quad \text{on } \Sigma:=(0,T)\times\Gamma,\\
\frac{\partial w}{\partial n}=0 \quad \text{on } \Sigma,\\
u(0,\cdot)=u_0,\ w(0,\cdot)=w_0 \quad \text{in } \Omega.
\end{gathered}
$$

Here Ω is a bounded domain in $\mathbf{R}^N$ $(1\le N\le 3)$ with smooth boundary $\Gamma:=\partial\Omega$ and $0<T<+\infty$; κ is a non-negative (real) parameter; n_0 and ν are positive constants; ρ and β are maximal monotone graphs in $\mathbf{R}\times\mathbf{R}$; g and λ are smooth functions; λ' is the derivative of λ; f, h, u_0 and w_0 are prescribed data.

This system is a model for solid-liquid phase transition which is the so-called phase-field model with constraint (or obstacles). In such a context, $\theta:=\rho(u)$ represents the temperature and w a non-conserved order parameter, i.e. variable characterizing phases which has constraint "$w\in D(\beta)$", and κ and ν are respectively the interfacial energy and relaxation time for the order parameter w; see Caginalp [1] and Penrose-Fife [5] for the derivation of system $(P_{\nu\kappa})$.

In this paper, assuming that λ is convex on $\overline{D(\beta)}$ and $\lambda''(w)u\le 0$ for all $w\in\overline{D(\beta)}$ and $u\in D(\rho)$, we shall show that for $\kappa>0$, $(P_{\nu\kappa})$ has one and only one solution under a more general condition on ρ than that in [3, 4], and $(P_{\nu 0})$ has one and only one solution, and its solution is the limit of the solution of $(P_{\nu\kappa})$ as $\kappa\searrow 0$.

Notation. In general, for a (real) Banach or Hilbert space X we denote by $|\cdot|_X$ the norm in X and by X^* the dual space.

Throughout this paper, we denote by H the usual space $L^2(\Omega)$ and by V the Sobolev space $H^1(\Omega)$ with norm $|z|_V:=\{|\nabla z|^2_{L^2(\Omega)}+n_0|z|^2_{L^2(\Gamma)}\}^{1/2}$.

Also, we denote by $(\cdot,\cdot)$ and $(\cdot,\cdot)_\Gamma$ the inner product in H and $L^2(\Gamma)$, respectively, and by $\langle\cdot,\cdot\rangle$ the duality pairing between V^* and V.

Let $a(v,z):=\int_\Omega \nabla v\cdot\nabla z\,dx$ for all $v,z\in V$ and F_0 be the operator from V into V^* defined by $\langle F_0v,z\rangle=a(v,z)$ for all $v,z\in V$; in particular, if $F_0v\in H$, then $v\in H^2(\Omega)$ and $F_0v=-\Delta v$ in Ω with $\dfrac{\partial v}{\partial n}=0$ on Γ.

By $C_w([0,T];X)$ for a Banach space X we mean the space of all weakly continuous X-valued functions from $[0,T]$ into X. By " $v_n \to v$ in $C_w([0,T];X)$ (as $n \to +\infty$) " we mean that for each $z^* \in X^*$ the function $\langle z^*, v_n(t) - v(t)\rangle_X$ converges to 0 uniformly on $[0,T]$, where $\langle \cdot,\cdot\rangle_X$ stands for the duality pairing between X^* and X.

We often use symbol " $'$ " to indicate the time-derivative $\dfrac{d}{dt}$.

Problem $(P_{\nu\kappa})$ is discussed under the following assumptions (A1)-(A5):

(A1) ρ is a maximal monotone graph in $\mathbf{R}\times\mathbf{R}$ whose domain $D(\rho)$ and range $R(\rho)$ are open in $\mathbf{R}$ and it is locally bi-Lipschitz continuous as a function from $D(\rho)$ onto $R(\rho)$.

(A2) β is a maximal monotone graph in $\mathbf{R}\times\mathbf{R}$ such that $\overline{D(\beta)} = [\sigma_*, \sigma^*]$ for constants σ_*, σ^* with $-\infty < \sigma_* < \sigma^* < +\infty$.

(A3) λ and its derivative λ' are Lipschitz continuous functions on $[\sigma_*, \sigma^*]$, and λ is convex on $[\sigma_*, \sigma^*]$ and $\lambda''(w)u \le 0$ for a.e. $w \in [\sigma_*, \sigma^*]$ and all $u \in D(\rho)$.

(A4) g is a Lipschitz continuous function on $[\sigma_*, \sigma^*]$.

(A5) n_0 and ν are positive constants.

Next we introduce the variational formulation for $(P_{\nu\kappa})$, given $f \in L^2(0,T;H)$, $h \in L^2(0,T;L^2(\Gamma))$ and $u_0, w_0 \in V$.

Definition 1.1. A couple of functions $u := u_{\nu\kappa} : [0,T] \to V$ and $w := w_{\nu\kappa} : [0,T] \to V$ is called a (weak) solution of $(P_{\nu\kappa})$ with $\kappa > 0$, if the following conditions $(\mathrm{w}1)_{\nu\kappa}$-$(\mathrm{w}3)_{\nu\kappa}$ are fulfilled:

$(\mathrm{w}1)_{\nu\kappa}$ $u \in L^2(0,T;V)$, $\rho(u) \in L^\infty(0,T;H) \cap W^{1,2}(0,T;V^*)$, $w \in L^\infty(0,T;V) \cap W^{1,2}(0,T;H)$ and $u(0) = u_0, w(0) = w_0$.

$(\mathrm{w}2)_{\nu\kappa}$ For all $z \in V$ and a.e. $t \in [0,T]$,

$$\langle \rho(u)'(t) + \lambda(w)'(t), z\rangle + a(u(t), z) + (n_0 u(t) - h(t), z)_\Gamma = (f(t), z). \qquad (1)$$

$(\mathrm{w}3)_{\nu\kappa}$ There is $\xi \in L^2(0,T;H)$ such that $\xi \in \beta(w)$ a.e. on Q and

$$\nu(w'(t), z) + \kappa a(w(t), z) + (\xi(t) + g(w(t)) - \lambda'(w(t))u(t), z) = 0 \qquad (2)$$

for all $z \in V$ and a.e. $t \in [0,T]$.

Definition 1.2. A couple of functions $u := u_{\nu 0} : [0,T] \to V$ and $w := w_{\nu 0} : [0,T] \to H$ is called a (weak) solution of $(P_{\nu 0})$, if the following conditions $(\mathrm{w}1)_{\nu 0}$-$(\mathrm{w}3)_{\nu 0}$ are fulfilled:

$(\mathrm{w}1)_{\nu 0}$ $u \in L^2(0,T;V)$, $\rho(u) \in L^\infty(0,T;H) \cap W^{1,2}(0,T;V^*)$, $w \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V)$ and $u(0) = u_0, w(0) = w_0$.

$(\mathrm{w2})_{\nu 0}$ (1) holds for all $z \in V$ and a.e. $t \in [0,T]$.

$(\mathrm{w3})_{\nu 0}$ There is $\xi \in L^2(0,T;H)$ such that $\xi \in \beta(w)$ a.e. on Q and

$$\nu(w'(t),z) + (\xi(t) + g(w(t)) - \lambda'(w(t))u(t), z) = 0$$

$$\text{for all } z \in H \text{ and a.e. } t \in [0,T].$$

As is easily understood from $(\mathrm{w3})_{\nu\kappa}$ and $(\mathrm{w3})_{\nu 0}$ in the above definitions, the function w of the solution $\{u,w\}$ of $(P_{\nu\kappa})$ has constraint $\sigma_* \le w \le \sigma^*$ on Q. This means that our weak formulation of $(P_{\nu\kappa})$ is independent of the behavior of functions λ and g on the outsides of $[\sigma_*, \sigma^*]$. Therefore, without loss of generality we may assume that

g, λ and λ' are Lipschitz continuous on $\mathbf{R}$ and they have compact support in $\mathbf{R}$. (3)

In the rest of this paper, (3) is assumed.

Now we recall the following results.

Theorem 1. *(cf.[6]) Assume that (A1)-(A5) hold with the following conditions (H1)-(H4) for the data:*
(H1) $f \in W^{1,2}(0,T;H)$.
(H2) $h \in W^{1,2}(0,T;L^2(\Gamma)) \cap L^\infty(0,T;L^\infty(\Gamma))$ *such that*

$$n_0 \sup D(\rho) \ge h(t,x) \ge n_0 \inf D(\rho) \quad \text{for a.e. } (t,x) \in \Sigma$$

and there are positive constants A_1 *and* A_1' *such that*

$$\rho(r)(n_0 r - h(t,x)) \ge -A_1|r| - A_1' \quad \text{for all } r \in D(\rho) \text{ and a.e. } (t,x) \in \Sigma.$$

(H3) $u_0 \in V$ *with* $\rho(u_0) \in H$.
(H4) $w_0 \in H^2(\Omega)$ *with* $\dfrac{\partial w_0}{\partial n} = 0$ *a.e. on* Γ *and there is* $\xi_0 \in H$ *such that* $\xi_0 \in \beta(w_0)$ *a.e. on* Ω.
Then $(P_{\nu\kappa})$ *with* $\kappa > 0$ *admits one and only one solution* $\{u_{\nu\kappa}, w_{\nu\kappa}\}$ *such that*

$$\begin{cases} u_{\nu\kappa} \in L^\infty(0,T;V), \quad w_{\nu\kappa} \in L^\infty(0,T;H^2(\Omega)), \\ w'_{\nu\kappa} \in L^\infty(0,T;H) \cap L^2(0,T;V), \quad \xi_{\nu\kappa} \in L^\infty(0,T;H), \end{cases}$$

where $\xi_{\nu\kappa}$ *is the function* ξ *as in condition* $(\mathrm{w3})_{\nu\kappa}$ *of Definition 1.1. Moreover we have* *the following uniform estimates (i) and (ii):*

(i) There are positive constants δ_0 and M_0, independent of parameter κ, such that

$$\begin{aligned}
&\int_\Omega \{\widehat{\rho^{-1}}(\rho(u_{\nu\kappa}(t))) - r_0(\rho(u_{\nu\kappa}(t)) + \lambda(w_{\nu\kappa}(t)))\}dx \\
&\quad + \frac{\kappa}{2}|\nabla w_{\nu\kappa}(t)|_H^2 + \int_\Omega \{\widehat{\beta}(w_{\nu\kappa}(t)) + \widehat{g}(w_{\nu\kappa}(t))\}dx \\
&\quad + \delta_0 \left\{ \int_0^t |u_{\nu\kappa}(s)|_V^2 ds + \nu \int_0^t |w'_{\nu\kappa}(s)|_H^2 ds + \nu |\rho(u_{\nu\kappa}(t))|_H^2 \right\} \qquad (4) \\
&\leq \int_\Omega \{\widehat{\rho^{-1}}(\rho(u_0)) - r_0(\rho(u_0) + \lambda(w_0))\}dx + \int_\Omega \{\widehat{\beta}(w_0) + \widehat{g}(w_0)\}dx \\
&\quad + \frac{\kappa}{2}|\nabla w_0|_H^2 + M_0\{|\rho(u_0)|_H^2 + |f|^2_{L^2(0,T;H)} + |h|^2_{L^2(0,T;L^2(\Gamma))} + 1\}
\end{aligned}$$

for all $t \in [0,T]$, where $r_0 \in D(\rho)$, $\widehat{\rho^{-1}}$ is the primitive of ρ^{-1} with $\widehat{\rho^{-1}}(\rho(r_0)) = 0$, $\widehat{\beta}$ is a non-negative proper l.s.c. convex function on R *with $\partial\widehat{\beta} = \beta$, and $\widehat{g}$ is a non-negative primitive of g.*

(ii) For each $\nu > 0$ there is a positive constant $M_1(\nu)$, independent of $\kappa > 0$, such that

$$\begin{aligned}
&\frac{1}{4}|u_{\nu\kappa}(t)|_V^2 + \frac{\nu}{2}|w'_{\nu\kappa}(t)|_H^2 + \kappa \int_0^t |\nabla w'_{\nu\kappa}(s)|_H^2 ds \\
&\quad \leq M_1(\nu)\{|u_0|_V^2 + |\rho(u_0)|_H^2 + |w_0|^2_{H^2(\Omega)} + |\xi_0|_H^2 \qquad (5) \\
&\qquad + |f|^2_{W^{1,2}(0,T;H)} + |h|^2_{W^{1,2}(0,T;L^2(\Gamma))} + 1\}
\end{aligned}$$

for all $t \in [0,T]$.

Lemma 1. *(cf.[6]) Under the same assumptions as in Theorem 1, let $\{u_i, w_i\}$ be a solution of $(P_{\nu_i\kappa_i})$ with $\nu_i > 0$ and $\kappa_i \geq 0$ for $i = 1,2$. Then, with $e_i := \rho(u_i) + \lambda(w_i)$ for $i = 1,2$, the following inequality holds:*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|e_1(t) - e_2(t)|^2_{V^*} + (\nu_1 w_1'(t) - \nu_2 w_2'(t), w_1(t) - w_2(t)) \\
&\quad + \int_\Omega \nabla(\kappa_1 w_1(t) - \kappa_2 w_2(t)) \cdot \nabla(w_1(t) - w_2(t))dx \qquad (6) \\
&\quad + (g(w_1(t)) - g(w_2(t)), w_1(t) - w_2(t)) \leq 0
\end{aligned}$$

for a.e. $t \in [0,T]$.

Our result is stated as follows.

Theorem 2. *Assume that (A1)-(A5), (H1)-(H4) hold, and let* $0 < \nu \le 1$. *Then, as* $\kappa \searrow 0$, *the solution* $\{u_{\nu\kappa}, w_{\nu\kappa}\}$ *of* $(P_{\nu\kappa})$ *converges to a couple of functions* $\{u_{\nu 0}, w_{\nu 0}\}$ *in the sense that*

$$u_{\nu\kappa} \to u_{\nu 0} \quad \text{in } C_w([0,T];V), \qquad \rho(u_{\nu\kappa}) \to \rho(u_{\nu 0}) \quad \text{in } C_w([0,T];H), \tag{7}$$

and

$$w_{\nu\kappa} \to w_{\nu 0} \quad \text{weakly in } W^{1,2}(0,T;H), \text{ in } C_w([0,T];V) \cap L^2(0,T;V). \tag{8}$$

Moreover, the limit $\{u_{\nu 0}, w_{\nu 0}\}$ *is a unique solution of* $(P_{\nu 0})$.

Proof. Fix $\nu > 0$ and denote by $\{u_\kappa, w_\kappa\}$ the solution of $(P_{\nu\kappa})$ for each $\kappa \in (0,1]$.
Noting that

$$\widehat{\rho^{-1}}(\rho(u_\kappa)) - \rho(u_\kappa) r_0 \ge \widehat{\rho^{-1}}(\rho(r_0)) - \rho(r_0) r_0 \quad \text{a.e. on } Q,$$

we observe from inequalities (4) and (5) that

$$\{u_\kappa\} \text{ is bounded in } L^\infty(0,T;V), \tag{9}$$

$$\{\rho(u_\kappa)\} \text{ is bounded in } L^\infty(0,T;H) \cap W^{1,2}(0,T;V^*), \tag{10}$$

$$\{w_\kappa\} \text{ is bounded in } W^{1,2}(0,T;H), \tag{11}$$

$$\{\sqrt{\kappa} w_\kappa\} \text{ is bounded in } L^\infty(0,T;V).$$

Also, multiply (2) for $\{u_\kappa, w_\kappa\}$ by $F_0 w_\kappa(t) (= -\Delta w_\kappa(t))$ to obtain

$$\frac{\nu}{2}\frac{d}{dt}|\nabla w_\kappa(t)|_H^2 + \kappa|\Delta w_\kappa(t)|_H^2 \le (L(g) + M(\lambda'))|\nabla w_\kappa(t)|_H^2 + M(\lambda')|\nabla u_\kappa(t)|_H^2 \tag{12}$$

for a.e. $t \in [0,T]$, where $L(g)$ is the Lipschitz constant of g and $M(\lambda')$ is the maximum of λ' on $[\sigma_*, \sigma^*]$; to get (12) we used

$$(\xi_\kappa(t), -\Delta w_\kappa(t)) \ge 0$$

and

$$\begin{aligned}
&(\lambda'(w_\kappa(t))u_\kappa(t), -\Delta w_\kappa(t)) \\
&= \int_\Omega \lambda''(w_\kappa(t))u_\kappa(t)|\nabla w_\kappa(t)|^2 dx + \int_\Omega \lambda'(w_\kappa(t))\nabla u_\kappa(t)\cdot\nabla w_\kappa(t) dx \\
&\le \int_\Omega \lambda'(w_\kappa(t))\nabla u_\kappa(t)\cdot\nabla w_\kappa(t) dx \\
&\le M(\lambda')(|\nabla w_\kappa(t)|_H^2 + |\nabla u_\kappa(t)|_H^2),
\end{aligned}$$

since $\lambda''(w_\kappa)u_\kappa \le 0$ by (A3). Now, applying the Gronwall's lemma to (12) and noting (9), we see that

$$\begin{cases} \{w_\kappa\} \text{ is bounded in } L^\infty(0,T;V), \\ \{\sqrt{\kappa}\Delta w_\kappa\} \text{ is bounded in } L^2(0,T;H). \end{cases} \tag{13}$$

Next, let $\{\kappa_n\}$ be any decreasing sequence in $(0,1]$ which converges to 0 (as $n \to +\infty$). Then, by (6) of Lemma 1,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\{|e_n(t) - e_m(t)|_{V^*}^2 + \nu|w_n(t) - w_m(t)|_H^2\} \\ &\quad + \int_\Omega \nabla(\kappa_n w_n(t) - \kappa_m w_m(t)) \cdot \nabla(w_n(t) - w_m(t))dx \\ &\quad \le L(g)|w_n(t) - w_m(t)|_H^2 \end{aligned} \tag{14}$$

for a.e. $t \in [0,T]$, where $\{u_n, w_n\} := \{u_{\kappa_n}, w_{\kappa_n}\}$ and $e_n := \rho(u_n) + \lambda(w_n)$.

It is easy to get from (14) that

$$\begin{aligned} &\frac{d}{dt}\{\exp(-\frac{2L(g)}{\nu}t)(|e_n(t) - e_m(t)|_{V^*}^2 + \nu|w_n(t) - w_m(t)|_H^2)\} \\ &\quad +2\exp(-\frac{2L(g)}{\nu}t)\int_\Omega \nabla(\kappa_n w_n(t) - \kappa_m w_m(t)) \cdot \nabla(w_n(t) - w_m(t))dx \le 0, \end{aligned}$$

hence

$$\begin{aligned} &\exp(-\frac{2L(g)}{\nu}s)(|e_n(s) - e_m(s)|_{V^*}^2 + \nu|w_n(s) - w_m(s)|_H^2) \\ &\quad +2\int_0^s\int_\Omega \exp(-\frac{2L(g)}{\nu}t)\nabla(\kappa_n w_n - \kappa_m w_m) \cdot \nabla(w_n - w_m)dxdt \le 0 \end{aligned}$$

for all $s \in [0,T]$. Especially, with $\tilde{w}_n := \exp(-\frac{L(g)}{\nu}t)w_n$ we have

$$\int_0^T\int_\Omega \nabla(\kappa_n\tilde{w}_n - \kappa_m\tilde{w}_m) \cdot \nabla(\tilde{w}_n - \tilde{w}_m)dxds \le 0.$$

Therefore, by virtue of [2, Lemma 2.4] it results from the above inequality with (13) that

$$\{\nabla\tilde{w}_n\} \text{ is Cauchy in } L^2(0,T;H)^N.$$

Hence

$$\{w_n\} \text{ is Cauchy in } L^2(0,T;V) \cap C([0,T];H) \tag{15}$$

and

$$\{\rho(u_n)\} \text{ is Cauchy in } C([0,T];V^*). \tag{16}$$

Furthermore, combining (15) and (16) with (13) and (10), respectively, we have

$$w_n \to w \quad \text{in } C_w([0,T];V) \tag{17}$$

and

$$\rho(u_n) \to \chi \quad \text{in } C_w([0,T];H) \tag{18}$$

for some functions w and χ. Now, choose a subsequence from $\{n\}$, denoted by $\{n\}$ again, for which $\{u_n\}$ converges weakly* in $L^\infty(0,T;V)$ to a function u. Then $\chi = \rho(u)$, i.e. $\rho(u_n) \to \rho(u)$ in $C_w([0,T];H)$. In fact, since $\rho(u_n) \to \chi$ in $C([0,T];V^*)$, it follows that

$$\lim_{n\to+\infty} \int_0^T (\rho(u_n), u_n - u)dt = \lim_{n\to+\infty} \int_0^T \langle \rho(u_n), u_n - u\rangle dt = 0,$$

which shows by the standard maximal monotone argument in $L^2(0,T;H)$ that $\chi = \rho(u)$. Moreover, we have $u_n,\ u \in C_w([0,T];V)$ and $u_n \to u$ in $C_w([0,T];V)$.

By the way, we have

$$\begin{cases} \xi_n := -\nu w_n' + \kappa_n \Delta w_n - g(w_n) + \lambda'(w_n)u_n \\ \quad \to -\nu w' - g(w) + \lambda'(w)u =: \xi \quad \text{weakly in } L^2(0,T;H), \\ \text{and } \xi \in \beta(w) \text{ a.e. on } Q. \end{cases} \tag{19}$$

Finally, from (9)-(11), (13), (17)-(19) and Lemma 1 it immediately follows that the limit $\{u,w\}$ is a unique solution of $(P_{\nu 0})$. Convergences (7) and (8) hold as $\kappa \searrow 0$ without taking any subsequence $\kappa_n \searrow 0$. □

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時間に依存する劣微分作用素によって支配される非線形発展方程式の anti-periodic 解の存在と一意性，外部領域における放物型変分不等式への応用について

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1 Introduction 実 Hilbert 空間 H に於いて，時間に依存する非線形作用素 $\partial\varphi^t$ に支配される次の非線形抽象発展方程式

$$\text{(E)} \qquad \frac{du}{dt}(t) + \partial\varphi^t(u(t)) \ni f(t), \quad t \in [0,T].$$

の anti-periodic 問題 $u(0) = -u(T)$ の解の存在について得られた結果について報告する．ここで $\partial\varphi^t(\cdot)$ は時間依存性を有する下半連続凸関数 $\varphi^t : H \to (-\infty, +\infty]$ の劣微分作用素である．劣微分 $\partial\varphi$ とは適正 $(\varphi \not\equiv +\infty)$ 下半連続凸関数 $\varphi : H \to (-\infty, +\infty]$ に対して

Definition $f \in \partial\varphi(u) \Leftrightarrow \varphi(v) - \varphi(u) \geq (f, v-u) \quad {}^\forall v \in H$

で定義される 非線形作用素で一般的には多価であるが，Fréchet 微分の一般化になっている．Anti-periodic 問題とは，Anti-periodic 関数の定義を

Definition $u(t)$ is T-anti-periodic $\Leftrightarrow \; u(t+T) = -u(t) \quad {}^\forall t \in \mathbb{R}$

とするとき，方程式 (E) の anti-periodic な解を求める問題をいう．

Remark $u(t)$ is T-anti-periodic
$\Rightarrow$ $u(t)$ is $2T$-periodic, and $\displaystyle\int_t^{t+2T} u(\tau)d\tau = 0, \ {}^\forall t \in \mathbb{R}$
Example $u(t) = \sin(\frac{\pi}{T}t)$ is T-anti-periodic

Anti-periodic 問題の特徴としては，coerciveness と evenness の定義を

Definition $\varphi^t(\cdot)$ is coercive
$\displaystyle\Leftrightarrow \liminf_{|u|\to+\infty} \frac{\varphi^t(u)}{|u|} = l(t), \text{ with } \int_0^T l(t)dt = +\infty.$
Definition $\varphi^t(\cdot)$ is even $\Leftrightarrow \varphi^t(-u) = \varphi^t(u) \quad {}^\forall u \in H \quad {}^\forall t \in [0,T].$

とする時，periodic 問題では，解の存在を示すのに，$\varphi^t(\cdot)$ の coerciveness を仮定することが本質的に必要なのに対し，anti-periodic 問題では，必ずしも $\varphi^t(\cdot)$ が coercive でなくても，$\varphi^t(\cdot)$ が even であることを仮定するだけで，解の存在を示せる事にある．

2 Cauchy problem, periodic problem, anti-periodic problem で今まで得られている結果の概説

Cauchy Problem に関しては，次の φ^t の t に関する滑らかさの条件

$(\mathrm{A}.\varphi^t)$: $^\exists m_1, m_2 > 0$ such that $^\forall t_0 \in [0,T]$, $^\forall x_0 \in D(\partial\varphi^{t_0})$,

$^\exists x(t) : [0,T] \to H$ such that

$$(1)\quad |x(t) - x_0| \leq m_1|t - t_0|(\varphi^{t_0}(x_0) + 1)^{1/2} \qquad ^\forall t \in [0,T],$$

$$(2)\quad \varphi^t(x(t)) - \varphi^{t_0}(x_0) \leq m_2|t - t_0|(\varphi^{t_0}(x_0) + 1) \quad ^\forall t \in [0,T].$$

のもとで, Kenmochi [2], [3], Yamada [9], Ôtani [8] などによって解の存在が示されている.

Periodic Problem に関しては, $(\mathrm{A}.\varphi^t)$ + 「$\varphi^0 = \varphi^T$」 + 「φ^t の coerciveness 」のもとで, Nagai [5], Kenmochi [2], Yamada [10] などによって解の存在が示されている.

Anti-periodic Problem に関しては, $(\mathrm{A}.\varphi^t)$ + 「$\varphi^0 = \varphi^T$」 + 「φ^t の evenness 」のもとで, 最初,Okochi [6] によって $\varphi^t \equiv \varphi$, の場合について解の存在が示され,φ^t が t に依存する場合については, Okochi [7] により, ある条件の下で解の存在が示されている. その条件とは, まず, $F(t) : H \to H$ odd, monotone として, 方程式 (E) が

$$(\mathrm{E})_1 \quad \frac{du}{dt}(t) + \partial\varphi(u(t)) + F(t)u(t) \ni 0, \quad t \in [0,T].$$

という形に書けることを仮定し,$\partial\varphi$ と $F(t)$ に関しては次の条件が仮定されるというものである.

(α) $^\exists B : H \to H$ bounded linear operator, $^\exists b \in L^1(0,T)$, $^\exists c > 0$,
satisfying

- $(F(t)u, (I - B)v) \leq b(t)|(I - B)v|$
  $u \in D(\partial\varphi) \cap D(F(t))$, $t \in \mathbf{R}$, $u \in \bigcup_t D(\partial\varphi) \cap D(F(t))$
- $(F(t)u, B^*Bu) \geq -b(t)|u|$, $u \in D(\partial\varphi) \cap D(F(t)), t \in \mathbf{R}$
- $(F(t)u, u) \geq -b(t)|u|$, $u \in D(\partial\varphi) \cap D(F(t))$, $t \in \mathbf{R}$,
- $\varphi(c^{-1}Bv) \leq \varphi(v)$, $v \in \bigcup_t D(\partial\varphi) \cap D(F(t))$
- $\varphi(c^{-1}B^*Bv) \leq \varphi(v)$, $v \in \bigcup_t D(\partial\varphi) \cap D(F(t))$
- $\liminf_{r\to+\infty} \left\{ \frac{\varphi(Bv)}{|Bv|}; v \in \bigcup_t D(\partial\varphi) \cap D(F(t)), |Bv| \geq r \right\} = +\infty$

Theorem (Okochi)
$(\mathrm{A}.\varphi^t)$ + 「$\varphi^0 = \varphi^T$」 + 「φ^t:evenness 」 + (α)
$\Rightarrow$ $^\exists$ anti-periodic solution of $(\mathrm{E})_1$

さらに,[7] では,moving boundary での外部領域での非線形熱方程式の Dirichlet 問題に応用している.

3 主結果 我々の結果は,[7] で仮定された条件が無くても, 本質的に $(\mathrm{A}.\varphi^t)$ + 「$\varphi^0 = \varphi^T$」 + 「φ^t:evenness 」だけで, (E) の anti-periodic 解の存在を示すことが出来たというものである.

Theorem 1 $(\mathrm{A}.\varphi^t)$ のもとで, $\varphi^0 = \varphi^T$, φ^t : even ならば,(E) の anti-periodic 解が存在する. このときさらに $u \in W^{1,2}(0,T;H)$ かつ $\varphi^t(u(t))$ は $[0,T]$ で絶対連続である.

Anti-Periodic 解の一意性は, 周期解の一意性を保証する条件よりも弱い条件のもとで保証される. すなわち任意の 2 つの周期解の差が常に定ベクトルであれば,Anti-Periodic 解は一意的である. こ

の為の十分条件は例えば, 次の定理で示される (c.f. Kenmochi-Ôtani [4]).

Theorem 2 次の (0),(i),(ii) のいずれかの条件を満たすとき, (E) の解は一意的である.

(0) $\varphi^t(u) = \varphi(u)$ $\forall t \in [0,T]$ $\forall u \in H$.

(i) ある $t_0 \in [0,T]$ に対して φ^{t_0} が $D(\varphi^{t_0})$ 上で狭義凸である.

(ii) (E) のすべての解 u に対して $-\frac{du}{dt}(t) = (\partial\varphi^t(u(t)) - f(t))^0$ a.e. $t \in [0,T]$ が成立する.

4 Theorem 1 の証明の概略. まず最初に,

Lemma 0 $(A.\varphi^t)$, φ^t is even ならば, 一般性を失わずに, $\varphi^t(u) \geq 0$, $\varphi^t(0) \leq C$, $\forall u \in H$, $\forall t \in [0,T]$ とすることが出来る.

が成り立つので,φ^t はそのようなものとする. そして, 次の近似方程式を導入する.

$$(\mathrm{E})_\varepsilon \begin{cases} \dfrac{du_\varepsilon}{dt}(t) + \partial\varphi^t(u_\varepsilon(t)) + \varepsilon u_\varepsilon(t) \ni f(t), \\ u_\varepsilon(0) = -u_\varepsilon(T) \end{cases}$$

$\partial\varphi^t(u) + \varepsilon u$ に対応する functional ϕ^t_ε $(\partial\phi^t_\varepsilon(u) = \partial\varphi^t(u) + \varepsilon u)$ は coercive であるので,$(\mathrm{E})_\varepsilon$ の Cauchy 問題の解 $v_\varepsilon(t)$ は t に関して有界になる. したがって,periodic 問題と同様にして, $(\mathrm{E})_\varepsilon$ の anti-Periodic 問題の解の存在がいえる. その一意性は ϕ^t_ε が strictly convex である事より保証される.

アプリオリ評価 (I) : まず, $(\mathrm{E})_\varepsilon$ に $\frac{du_\varepsilon}{dt}$ を掛ける. そして, 次の

Lemma 1 $(A.\varphi^t) \Rightarrow$

$$|\frac{d}{dt}\varphi^t(u(t)) - (\partial\varphi^t(u(t)), \frac{du}{dt}(t))|$$
$$\leq m_1|\partial\varphi^t(u(t))|(\varphi^t(u(t)) + 1)^{1/2} + m_2(\varphi^t(u(t)) + 1), \quad \forall t \in [0,T]$$

と,$\varphi^t(u_\varepsilon(t))$, $|u_\varepsilon(t)|^2$ が T-periodic である事に注意して,$[0,T]$ で積分すると, 次の式を得る.

$$(\mathrm{A}) \quad \frac{1}{2}\int_0^T |\frac{du_\varepsilon}{dt}(t)|^2 dt \leq C\Big(\int_0^T |f(t)|^2 dt + \int_0^T (\varphi^t(u_\varepsilon(t)) + 1)dt\Big) + \varepsilon^2 \int_0^T |u_\varepsilon(t)|^2 dt$$

アプリオリ評価 (II) : $(\mathrm{E})_\varepsilon$ に u_ε を掛ける. そして, $\forall\delta > 0$ を固定する. $\varphi^t(0) \leq C$, $\forall t \in [0,T]$ に注意して,$[0,T]$ で積分すると, 次の式を得る.

$$(\mathrm{B}) \quad \int_0^T \varphi^t(u_\varepsilon(t))dt + 2\varepsilon\int_0^T |u_\varepsilon(t)|^2 dt \leq \delta\int_0^T |u_\varepsilon(t)|^2 dt + C_\delta\int_0^T (|f(t)|^2 + 1)dt$$

ここで,Haraux [1] によって示された次の不等式

Lemma 2 $u(0) = -u(T)$, $u \in W^{1,2}(0,T;H)$
$\Rightarrow |u|_{L^\infty(0,T;H)} \leq T^{1/2}|\frac{du}{dt}|_{L^2(0,T;H)}$

を用いて,(A) と (B) を辺々をたせば,$0 < \varepsilon \leq 1$ に対し,

$$(\frac{1}{2} - \delta T^2)\int_0^T |\frac{du_\varepsilon}{dt}(t)|^2 dt \leq C'_\delta \int_0^T (|f(t)|^2 + 1)dt$$

を得る．よって，$\left\{\frac{du_\varepsilon}{dt}\right\}$ は $L^2(0,T;H)$ で有界となり，もう一度 Lemma 2 を用いれば，$\{u_\varepsilon\}$ は $L^\infty(0,T;H)$ で有界となる．したがって，適当な部分列を引き抜けば，以下を得る．

$$u_n \rightharpoonup u \ \text{ weakly in } L^2(0,T;H)$$

$$\frac{du_n}{dt} \rightharpoonup \frac{du}{dt} \ \text{ weakly in } L^2(0,T;H)$$

ここで,Ascoli-Arzela の定理を用いて証明される次の Lemma

Lemma 3 $\quad u_n \rightharpoonup u$ weakly in $L^p(0,T;H)$, $p>1$
$\frac{du_n}{dt} \rightharpoonup \frac{du}{dt}$ weakly in $L^q(0,T;H)$, $q>1$
$\Rightarrow u_n \to u \quad$ in $C([0,T];H_w)$
($C([0,T];H_w)$ is equipped with uniform convergence topology)

を用いれば，

$$u_n \to u \quad \text{in } C([0,T];H_w)$$

を得る．$u_n(t)$ は T-anti-periodic であるので，$(u_n(0)+u_n(T),v)_H = 0 \quad {}^\forall v \in H$ となる．よって，$(u(0)+u(T),v)_H = 0 \quad {}^\forall v \in H$ となるので，$u(0) = -u(T)$ を得る．$u(t)$ が (E) の解である事を見るには，まず $\partial\varphi^t(u_\varepsilon)$ の定義に $(\mathrm{E})_\varepsilon$ を代入すれば，${}^\forall w \in L^2(0,T;H)$ に対し，

$$\int_0^T \varphi^t(w(t))dt - \int_0^T \varphi^t(u_n(t))dt \geq \int_0^T (f(t) - \frac{du_n}{dt}(t) - \varepsilon u_n(t), w(t) - u_n(t))dt$$

となる．そこで，

$$\varepsilon u_\varepsilon \to 0 \ \text{ strongly in } L^2(0,T;H)$$

$$\int_0^T (\frac{du_\varepsilon}{dt}(t), u_\varepsilon(t))dt \ = \tfrac{1}{2}|u_\varepsilon(T)|^2 - \tfrac{1}{2}|u_\varepsilon(0)|^2 \ = 0$$

$$\int_0^T (\frac{du}{dt}(t), u(t))dt \ = \tfrac{1}{2}|u(T)|^2 - \tfrac{1}{2}|u(0)|^2 \ = 0$$

$$\Psi(u) = \int_0^T \varphi^t(u(t))dt \text{ is weakly semicontinuous on } L^2(0,T;H)$$

に注意して，$\varepsilon \searrow +0$ とすれば，

$$\int_0^T \varphi^t(w(t))dt - \int_0^T \varphi^t(u(t))dt \geq \int_0^T (f(t) - \frac{du}{dt}(t), w(t) - u(t))dt$$

となる．したがって $\partial\varphi^t(u)$ の定義より,u は (E) の解となる．

[Q.E.D.]

5 外部領域 Ω における時間依存性をもつ obstacle 問題への応用． obstacle 問題というのは,obstacle 条件という $u(t)$ の動く範囲に制限をつけて，方程式の解を考えることを言う．ここでは，Neumann 条件での熱方程式の obstacle 問題を考える．問題を考える空間を $H = L^2(\Omega)$ とし，与えられた関数 $g(x,t)$ に対して，$K(t) = \{u \in L^2(\Omega)\ ;\ |u(x,t)| \leq g(x,t) \text{ a.e. } \Omega\}$ とする．つまり $K(t)$ は時間に依存する obstacle 条件 $|u(x,t)| \leq g(x,t)$ a.e. $x \in \Omega$ をみたす H の集合である．そこで $-\Delta u$ に対応する functional の effective domain を $K(t)$ に制限したものを φ^t とする．つ

まり，

$$\varphi^t(u) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 dx, & u \in W^{1,2}(\Omega)\bigcap K(t), \\ +\infty, & u \notin W^{1,2}(\Omega)\bigcap K(t). \end{cases}$$

とすると，この φ^t の劣微分 $\partial\varphi^t$ に支配される (E) は次の変分不等式と同値になる．

$$\int_\Omega (-\frac{\partial u}{\partial t}(t) + f(t))(u(t) - v(t))dx + \int_\Omega \left(|\nabla v(t)|^2 - \nabla v(t)\nabla u(t)\right) dx \geq 0$$
$$^\forall v \in W^{1,2}(\Omega)\bigcap K(t) \quad \text{a.e. } t \in [0, T].$$

この変分不等式を見ただけでは,(E) がどういった現象を表しているのかよくわからないが，$u(t) \in$ interior$(K(t))$ が成り立つような t では，次の Neumann 条件での熱方程式を満たしている．

$$\begin{cases} \frac{\partial u}{\partial t}(t) - \Delta u(t) &= f(t), \quad \text{on } \Omega, \\ \frac{\partial u}{\partial n} &= 0, \quad \text{on } \partial\Omega. \end{cases}$$

また,(A.φ^t) のもとで，$u(t) \in W^{1,2}(\Omega)\bigcap K(t), ^\forall t > 0$ であるので，$u(t)$ は obstacle 条件をみたすような所を動き回り，さらに $u(t)$ が $K(t)$ の内点になっているような t では，熱方程式に従っていることが分る．この問題の anti-periodic 解の存在を示すには,$K(t)$ は $L^2(\Omega)$ 上の対称集合より φ^t は even になるので，あとは (A.φ^t) と $\varphi^0 = \varphi^T$ が満たされればよい．そのためには，例えば $g(x,t) = G(x)k(t)$ として次の (g.1)-(g.3) を仮定すれば，これは (A.φ^t) と $\varphi^0 = \varphi^T$ の十分条件となっている (c.f. [4]).

(g.1) $k(0) = k(T)$.
(g.2) $G(x) \geq 0$, a.e. $x \in \Omega$, and $G \in W^{1,2}(\Omega)$
(g.3) $k(t) \geq \rho > 0$, $^\forall t \in [0, T]$, and $k(t)$ is Lipschitz continuous on $[0.T]$

したがって,Theorem 1 より (E) の anti-periodic 解の存在 がいえて，さらにこの φ^t は strictly convex になっているので，Theorem 2 より (E) の anti-periodic 解の一意性 がいえる．

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代用電荷法による多次元ポアソン方程式の数値計算とその応用

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本研究は、小林尚弘(日本電気(株))、天野要(愛媛大・工)、四ツ谷晶二(龍谷大・理工)の共同研究である。

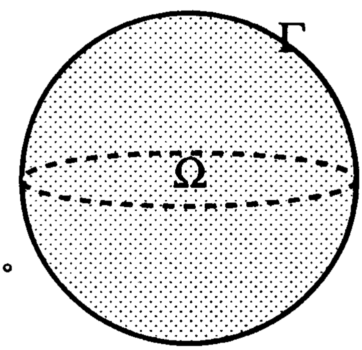
代用電荷法は、簡単な原理・短い計算時間でラプラス方程式の高精度の近似解を得る方法として知られており数学的な解析も進んできた([1],[2])。

本講演では、ポアソン方程式の近似解を代用電荷法で求める方法を述べる。代用電荷法の特徴である領域の変形に対して融通性があることを利用して、半線形楕円型方程式の解の形状の領域に対する依存性の研究に用いることが最終目標である。

安田([3])は熱方程式を時間方向に差分化し各時間毎に楕円型方程式を解くのに代用電荷法を用いた。本質的なアイデアは、数学での常套手段である。特殊解をグリーン関数を用いて積分表示し、斉次方程式に帰着することである。グリーン関数が特異性を持つのでそのことにより計算精度が落ちている様に思われる。2次元の場合に小林 - 巽 - 天野 - 四ッ谷([4])は、特異点の回りで極座標を導入し、適当な変数変換の後、積分計算を実行することにより精度を上げることができることを示した。ここでは、3次元の場合を述べる。

1　代用電荷法によるポアソン方程式の解法

$$\begin{cases} \Delta u = f & in\ \ \Omega \\ u = g & on\ \ \Gamma \end{cases} \qquad (*)$$



(∗)式のような有界領域でのポアソン方程式の境界値問題を以下の3つのステップにより近似解を求める。ただし、Ω は球の内部、Γ は球の表面とする。

<u>step1</u>　$\Delta v = f \quad in\ \ \Omega$

の特解をニュートンポテンシャルを用いて求める。

<u>step2</u>　$\begin{cases} \Delta w = 0 & in\ \ \Omega \\ w = g - v & on\ \ \Gamma \end{cases}$

のようなラプラス方程式を代用電荷法を用いて求める。

step3　$u := v + w$

step1 で求めた v と step2 で求めた w を重ね合わせる。

それでは、step1 から step3 までそれぞれについて詳しく説明していく。

1.1　step1 の詳しい説明

ここでは、ニュートンポテンシャルを用いた $\Delta v(x,y,z) = f(x,y,z)$ $in\,\Omega$ の特解をどのように数値積分するのかについて説明する。

$v(x,y,z)$ は、ニュートンポテンシャルによって

$$v(x,y,z) = -\frac{1}{4\pi}\int\int\int_{\Omega}\frac{f(\xi_1,\xi_2,\xi_3)}{\sqrt{(x-\xi_1)^2+(y-\xi_2)^2+(z-\xi_3)^2}}d\xi_1 d\xi_2 d\xi_3 \quad (1)$$

のようにあらわされる。ただし、$(x,y,z),(\xi_1,\xi_2,\xi_3)$ は、それぞれ図 1のような球面内の点とする。

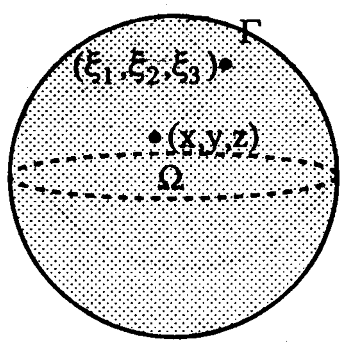


図 1:

ここで、(ξ_1,ξ_2,ξ_3) を特異点 (x,y,z) の周りで極座標 (r,θ,φ) に変換する。そのことにより (ξ_1,ξ_2,ξ_3) は、

$$\begin{cases} \xi_1 = r\cos\theta\sin\varphi + x \\ \xi_2 = r\sin\theta\sin\varphi + y \\ \xi_3 = r\cos\varphi + z \end{cases}$$

のようになる。また、極座標に変換したことにより、Ω は $\tilde{\Omega}$ になる。ここで、$\tilde{\Omega}$ は、

$$\tilde{\Omega} = \{(r,\theta,\varphi) : 0 \le r < \rho(\theta,\varphi),\ 0 \le \theta < 2\pi,\ 0 \le \varphi \le \pi\}$$

のようにあらわされる。また、$\tilde{\Omega}$ を図示すると図 2のようになる。

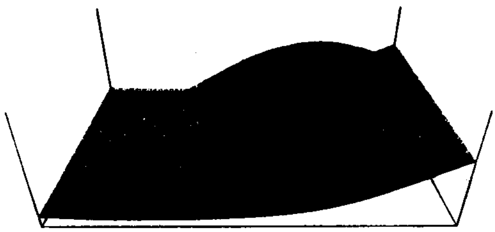


図 2:

ただし、r は (x,y,z) と (ξ_1,ξ_2,ξ_3) との距離、θ,φ はそれぞれ図 3のような偏角、$\rho(\theta,\varphi)$ は (x,y,z) から (θ,φ) 方向の Γ への距離とする。

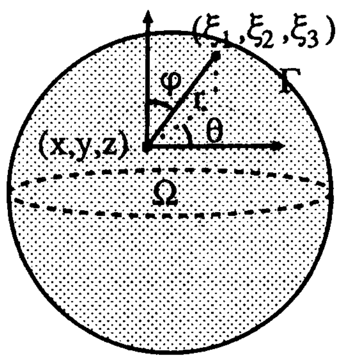


図 3:

また、極座標に変換したことにより (1) 式は、

$$v(x,y,z) = -\frac{1}{4\pi}\int\int\int_{\tilde{\Omega}} f\frac{1}{r}r^2\sin\varphi drd\theta d\varphi \tag{2}$$

のようになる。このように特異点を中心とする極座標変換を用いることにより、数値積分をするときに問題となる基本解の特異性を解消することができる。

次に、図 2のような積分区間 $\tilde{\Omega}$ を直方体領域の積分区間に変換する。まず、r の積分区間を $(0,1)$ にするために、r を $r = \rho(\theta,\varphi)t$ のように変換する。そうすると (2) 式は、

$$v(x,y,z) = -\frac{1}{4\pi}\int_0^{\pi}\sin\varphi\int_0^{2\pi}\rho^2\int_0^1 ftdtd\theta d\varphi \tag{3}$$

のようになる。ただし、$\rho = \rho(\theta,\varphi)$、$f = f(\rho t\cos\theta\sin\varphi + x, \rho t\sin\theta\sin\varphi + y, \rho t\cos\varphi + z)$ とする。

1.2 step2 の詳しい説明

ここでは、(4) 式のようなラプラス方程式の近似解を代用電荷法を用いて求める方法について説明する。

$$\begin{cases} \Delta w(x,y,z) = 0 & in \ \ \Omega \\ w(x,y,z) = g(x,y,z) - v(x,y,z) & on \ \ \Gamma \end{cases} \tag{4}$$

まず始めに、図 4のように拘束点 (x_i, y_i, z_i) を球の表面 Γ に置く。そして、電荷点 (a_j, b_j, c_j) を球面を何倍かに膨らました場所 Ω^c に置く。ただし、$i = 0,1,2,\cdots,n-1$、$j = 0,1,2,\cdots,n-1$ とする。

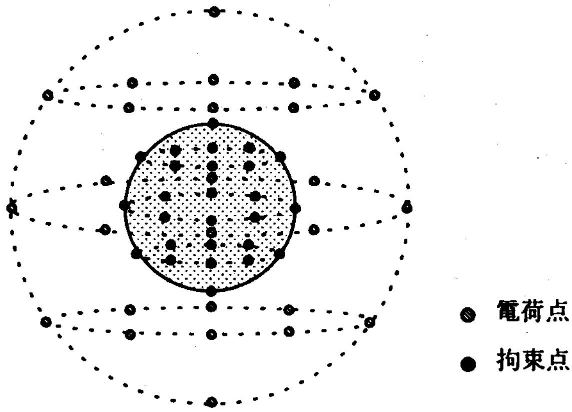


図 4:

そうすると、近似解 $w(x,y,z)$ は、

$$w(x,y,z) = \sum_{j=0}^{n-1} C_j \frac{1}{\sqrt{(x-a_j)^2 + (y-b_j)^2 + (z-c_j)^2}}$$

のよう与えられる。ただし、C_j は未定係数とする。次に、その未定係数 C_j を求める方法として、$C_0, C_1, \cdots, C_{n-1}$ についての (5) 式のような連立 1 次方程式の解を求める。ただし、$i = 0,1,2,\cdots,n-1$ とする。

$$\sum_{j=0}^{n-1} C_j \frac{1}{\sqrt{(x_i-a_j)^2 + (y_i+b_j)^2 + (z_i+c_j)^2}} = g(x_i,y_i,z_i) - v(x_i,y_i,z_i) \tag{5}$$

α	U_S	E_S	U_{GL}	E_{GL}
1	1.000000	3.7×10^{-8}	1.000000	2.8×10^{-8}
2	2.000182	1.8×10^{-4}	2.000176	1.8×10^{-4}
3	3.011791	1.2×10^{-2}	3.011553	1.2×10^{-2}
4	4.043175	4.3×10^{-2}	4.038417	3.8×10^{-2}
5	5.084566	8.5×10^{-2}	5.042214	4.2×10^{-2}
6	6.108528	1.1×10^{-1}	6.000479	4.8×10^{-4}
7	6.827204	1.7×10^{-1}	6.909578	9.0×10^{-2}

表1：積分の計算をシンプソン法で求めた場合

どちらの場合も、α の値が大きくなるにつれて、精度が低下していくが、ガウス・ルジャンドルの積分公式を用いた方が、この精度の低下が小さいことがわかる。よって、これからの数値積分には、ガウス・ルジャンドルの公式を用いる。

1.5 半線形楕円型方程式への応用

ここでは、(8) 式のような半線形楕円型方程式について考える。

$$\begin{cases} \Delta u + u^3 = 0 & in \quad \Omega \\ u = 0 & on \quad \Gamma \end{cases} \tag{8}$$

(8) 式を解く方法として、*Nehari* の方法を用いる ([5],[6])。*Nehari* の方法とは、初期関数 u_0 を $u_0 \geq 0, \neq 0$ として、$u_1, u_2, \cdots$ を以下のように求めていく方法である。

$$u_k = \left(\frac{\int_\Omega u_{k-1}^p U_k dx}{\int_\Omega U_k^{p+1} dx} \right)^{\frac{1}{p-1}} U_k$$

ただし、

$$\begin{cases} \Delta U_k = -u_{k-1}^p & in \quad \Omega \\ U_k = 0 & on \quad \Gamma \end{cases} \qquad (k = 1, 2, 3, \cdots)$$

このように $\{u_k\}$ を定義すると、少なくとも部分列は真の解に収束することが Mizutani-Suzuki[6] によって示されている。ここでは、各 U_k の計算に、代用電荷法を用いる。

この *Nehari* の方法を使って (8) 式の近似解を求めるには、初期関数 u_0 が必要である。よって、ここでは、初期関数を $u_0 = 1$ として数値実験を行なう。

1.3 step3 の詳しい説明

ここでは、step1 で求めた $v(x,y,z)$ と step2 で求めた $w(x,y,z)$ を重ね合わせる。

$$u(x,y,z) = v(x,y,z) + w(x,y,z) \tag{6}$$

ここで問題となるのは、(6) 式が (*) 式を満たしているかどうかである。よって、(6) 式が (*) 式を満たしているかどうかを証明する。(*) 式の第 1 式は、(6) 式より

$$\begin{aligned} \Delta u(x,y,z) &= \Delta(v(x,y,z) + w(x,y,z)) \\ &= \Delta v(x,y,z) + \Delta w(x,y,z) \\ &= f(x,y,z) \qquad\qquad in \ \ \Omega \end{aligned}$$

となる。(*) 式の第 2 式は、(6) 式より

$$\begin{aligned} u(x,y,z) &= v(x,y,z) + (g(x,y,z) - v(x,y,z)) \\ &= g(x,y,z) \qquad\qquad on \ \ \Gamma \end{aligned}$$

となり、(6) 式が (*) 式を満たしていることがわかる。

1.4 ポアソン方程式の数値実験例

ここでは、(7) 式のようなポアソン方程式の数値実験について説明する。

$$\begin{cases} \Delta u(x,y,z) = -\alpha^5 \left\{1 + \frac{\alpha^4}{3}(x^2+y^2+z^2)\right\}^{-5/2} & in \ \ \Omega \\ u(x,y,z) = \alpha \left\{1 + \frac{\alpha^4}{3}\right\}^{-1/2} & on \ \ \Gamma \end{cases} \tag{7}$$

(7) 式を解く方法として、1.1 節から 1.3 節で説明してきた方法を用いる。また、(7) 式は、$\Delta u + u^5 = 0$ の解をあらわしたポアソン方程式である。

(7) 式の数値実験結果は以下のようになる。ここでは、2 種類の方法で数値積分を行なった。(3) 式の 3 重積分の計算をすべてシンプソン法を用いて解いた場合 U_S と t についての積分だけをルジャンドル・ガウスの積分公式、それ以外の積分をシンプソン法を用いて解いた場合 U_{GL} である。ただし、次の表 1 は α を 1 から 7 まで変化させたときの原点 (0,0,0) での近似値とその誤差 E_S, E_{GL} である。

(8) 式の数値実験結果は以下のようになる。ただし、別の方法で求めた近似解の値は、6.897 ぐらいである。

回数	U_{GL}	回数	U_{GL}	回数	U_{GL}
0	1.000000	6	6.562074	12	6.870039
1	4.658887	7	6.687010	13	6.876421
2	5.092090	8	6.764882	14	6.880265
3	5.643809	9	6.812715	15	6.882577
4	6.069547	10	6.841818	16	6.883928
5	6.366208	11	6.859422		

表２：原点 $(0,0,0)$ での値

このように、回数を重ねていくにつれて、(8) 式の近似解に近付いていくことがわかる。ここでは、１６回で計算を止めたが、この後を計算していくとさらに良い近似解が得られると思われる。

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2次元外部領域における elliptic system の
low frequency asymptotic expansion について

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§1. Introduction and main results

Let Ω be an unbounded domain in the 2-dimensional Euclidean space $\mathbf{R}^2$ having a boundary $\partial\Omega$ which is infinitely smooth and compact. Let $A(x,\partial)$ ($\partial = (\partial_1,\partial_2)$, $\partial_j = \partial/\partial x_j$, $x = (x_1,x_2) \in \mathbf{R}^2$) be an $m \times m$ matrix of differential operators of the form:

$$(1.1) \qquad A(x,\partial) = A(\partial) + P(x,\partial)$$

$$A(\partial)\mathbf{u}(x) = \sum_{i,j=1}^{2} A^{ij}\partial_i\partial_j\mathbf{u}(x), \quad P(x,\partial)\mathbf{u}(x) = \sum_{i,j=1}^{2} \partial_i(P^{ij}(x)\partial_j\mathbf{u}(x)),$$

where $\mathbf{u}$ is an m-dimensional column vector, A^{ij} and $P^{ij}(x)$ are $m \times m$ matrices. The elements of A^{ij} are all real constants and the elements of $P^{ij}(x)$ are real valued functions in $C_0^\infty(\mathbf{R}^2)$. Put $A^{ij}(x) = A^{ij} + P^{ij}(x)$ and $A(\xi) = \sum_{i,j=1}^{2} A^{ij}\xi_i\xi_j$ for $\xi = (\xi_1,\xi_2) \in \mathbf{R}^2$. In the present paper, we consider the boundary value problem with spectral parameter k in the domain Ω:

$$(1.2) \qquad (k^2 I + A(x,\partial))\mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \quad B(x,\partial)\mathbf{u} = 0 \quad \text{on } \partial\Omega,$$

where I is the $m \times m$ unit matrix and $B(x,\partial)$ is either the identity or the following operator:

$$B(x,\partial)\mathbf{u} = B_N(x,\partial)\mathbf{u} = \sum_{i,j=1}^{2} \nu_i(x)A^{ij}(x)\partial_j\mathbf{u}.$$

Here, $\nu(x) = (\nu_1(x),\nu_2(x))$ represents the unit normal of $\partial\Omega$ at $x \in \partial\Omega$.

Now, we introduce the assumptions (A.1) – (A.4) below.

$$(\text{A.1}) \qquad {}^tA^{ij}(x) = A^{ji}(x).$$

$$(\text{A.2}) \qquad \exists C > 0 \text{ s.t. } \sum_{i,j=1}^{2}(A^{ij}(\cdot)\partial_j\mathbf{u},\partial_i\mathbf{u})_\Omega \geq C\|\nabla\mathbf{u}\|_\Omega^2,$$

for $\mathbf{u} \in L^2_{loc}(\Omega)$, $\nabla\mathbf{u} \in L^2(\Omega)$ and $\mathbf{u} = 0$ on $\partial\Omega$, if $B(x,\partial)\mathbf{u} = \mathbf{u}$, for $\mathbf{u} \in L^2_{loc}(\Omega)$ and $\nabla\mathbf{u} \in L^2(\Omega)$, if $B(x,\partial)\mathbf{u} = B_N(x,\partial)\mathbf{u}$. Here, $\|\cdot\|_\Omega$ and $(\ ,\ )_\Omega$ denote usual L^2 norm and inner product on Ω.

In view of (A.1) and (A.2), $A(x,\partial)$ is a strongly elliptic system. If we note that $P^{ij}(x)$ vanish for large $|x|$, we have $A(\xi)$ is a symmetric matrix and that

$$\exists\delta > 0 \quad \text{s.t. } A(\xi) \geq \delta|\xi|^2 I \quad \text{for any } \xi \in \mathbf{R}^2.$$

Let $\lambda_1(\xi), \ldots, \lambda_N(\xi)$ denote distinct characteristic roots of $A(\xi)$, then

(A.3) $\quad \lambda_j(\xi),\ j = 1, \ldots, N$ have constant multiplicity for all $\xi \in \mathbf{R}^2 \setminus \{0\}$.

Put $\Sigma_j = \{\xi \in \mathbf{R}^2 \mid \lambda_j(\xi) = 1\},\ j = 1, \ldots, N$. The final assumption is that

(A.4) $\quad$ the Gaussian curvatures of the curves $\Sigma_j,\ j = 1, \ldots, N$ do not vanish.

From the assumptions, $\lambda_j(\xi)$ $(j = 1, \ldots, N)$ have the following properties:

(1) $\lambda_j(\xi) \in C^\infty(\mathbf{R}^2 \setminus \{0\}),\ j = 1, \ldots, N$ are real and N is independent of $\xi \in \mathbf{R}^2 \setminus \{0\}$.
(2) $\lambda_j(t\xi) = t^2 \lambda_j(\xi)$ for all $t \in \mathbf{R}$, $\xi \in \mathbf{R}^2$ $(j = 1, \ldots, N)$,
(3) $\exists \delta > 0$ s.t. $\lambda_j(\xi) \geq \delta |\xi|^2$ for all $\xi \in \mathbf{R}^2$ $(j = 1, \ldots, N)$,
(4) $\nabla \lambda_j(\omega) \cdot \omega = 2$ for all $\omega \in \Sigma_j$ $(j = 1, \ldots, N)$,
(5) Σ_j $(j = 1, \ldots, N)$ are C^∞, bounded and non-intersecting closed curves, enclosing the origin (cf.[22]).

Now, we shall give the notation. Let G be a domain in $\mathbf{R}^2$. Put

$$H^p_{loc}(G) = \{\mathrm{u} \mid \varphi \mathrm{u} \in H^p(G) \text{ for any } \varphi \in C_0^\infty(\mathbf{R}^2)\},$$
$$L^2_R(G) = \{\mathrm{u} \in L^2(G) \mid \mathrm{u}(x) = 0 \text{ for } |x| > R\},$$
$$\dot{H}^p(\Omega) = \{\mathrm{u} \in H^p_{loc}(\Omega) \mid \partial_x^\alpha \mathrm{u} \in L^2(\Omega), 1 \leq |\alpha| \leq p\},$$
$$S_R = \{x \in \mathbf{R}^2 \mid |x| = R\},\quad B_R = \{x \in \mathbf{R}^2 \mid |x| < R\},\quad \Omega_R = B_R \cap \Omega.$$

Take a constant $a > 0$ so that $\partial\Omega \subset B_{a-1}$ and $A(x, \partial) = A(\partial)$ when $|x| > a - 1$. Let $\mathbf{C}$ be the set of all complex numbers. Put

$$D = \{k \in \mathbf{C} \setminus \{0\} \mid -\pi/2 < \arg k < 3\pi/2\},\quad D_\pm = \{k \in D \mid \Im k \gtrless 0\}.$$

When we assume that (A.1) – (A.4) are satisfied, Vainberg [15, 16, 18] proved that there exists an operator $R_k \in \mathbf{B}(L^2_a(\Omega), H^2_{loc}(\Omega))$ which depend meromorphically on the parameter $k \in D$ and the asymptotic expansion of the operator R_k as $|k| \to 0$ has the form

$$R_k = k^{-\alpha} \sum_{m=0}^{\infty} \sum_{n=0}^{m\ell} \left[\frac{k}{P(\log k)}\right]^m \log^n k P_{m,n}, \tag{1.3}$$

where α is an integer, ℓ is a non-negative integer, P is a polynomial with constant coefficients, and $P_{m,n}$: $L^2_a(\Omega) \to H^2_{loc}(\Omega)$ are bounded operators independent of k. We can get expansion (1.3) from the corresponding expansion for the case of general elliptic problems. But the integers α and ℓ and the polynomial P was not known, even for equations of second order. Recently, Kleinmann and Vainberg [5] obtained the complete asymptotic expansion in the case that $A(x, \partial)$ coincides with the Laplace operator in some neighbourhood of infinity. We apply the idea of [5] to the system case.

In the case of displacement problem, we denote by $\mathbf{u}_0$ (m-dimensional column) and U_1 ($m \times m$ matrix) the solutions of the problems:

$$
\begin{aligned}
&(1.4) \qquad && A(x,\partial)\mathbf{u}_0 = \mathbf{f} \quad \text{in } \Omega, \\
& && \mathbf{u}_0 = 0 \quad \text{on } \partial\Omega, \quad |\mathbf{u}_0| = O(1) \quad \text{as } |x| \to \infty, \\
&(1.5) \qquad && A(x,\partial)U_1 = 0 \quad \text{in } \Omega, \\
& && U_1 = 0 \quad \text{on } \partial\Omega, \quad |U_1 - E_0| = O(1) \quad \text{as } |x| \to \infty,
\end{aligned}
$$

where E_0 is a fundamental solution: $A(\partial)E_0 = \delta I$. Here, δ is Dirac's distribution in $\mathbf{R}^2$. We can show that the uniqueness and the existenee of solutions of (1.4). Then, the unique solvability of (1.5) follows from that of (1.4). In fact, if we write the solution of (1.5) in the form $U_1 = \psi E_0 + W_1$, where $\psi = \psi(x)$ is a cut-off function such that $\psi = 1$ for $|x| > a$ and $= 0$ near $\partial\Omega$, then (1.5) is reduced to (1.4).

Since we can show that the solution of (1.4) converges to some constant, we can define the constant matrix and vector as follows:

$$L = \lim_{|x|\to\infty} (U_1 - E_0) = \lim_{|x|\to\infty} W_1, \quad \mathbf{b} = \lim_{|x|\to\infty} \mathbf{u}_0.$$

With this notation, we shall state our main results.

Theorem 1.1. *For the solution $\mathbf{u} = R_k\mathbf{f}$ of (1.2) with $\mathbf{f} \in L^2_a(\Omega)$ in the case that $B(x,\partial)$ is the identity, the following asymptotic expansion is valid when $k \in D$, $|k| \to 0$:*

$$R_k\mathbf{f} = \sum_{m=0}^{N} \sum_{n=-m}^{\infty} k^m \log^{-n} k G_{m,n}\mathbf{f} + \hat{\mathbf{u}}_N, \tag{1.6}$$

where $G_{m,n} \in \mathbf{B}(L^2_a(\Omega), H^2(\Omega_a))$ are independent of k and

$$\|\hat{\mathbf{u}}_N\|_{H^2(\Omega_a)} \le C(a)|k \log k|^{N+1} \|\mathbf{f}\|_{L^2_a(\Omega)}.$$

The leading terms of the asymptotic expansion have the form

$$\mathbf{u} = \mathbf{u}_0(x) + U_1 \left(\left(\log k - \frac{i\pi}{2} \right) M + B - L \right)^{-1} \mathbf{b} + O(k \log k), \tag{1.7}$$

where

$$M = c_\pi \int_{S_1} A(\omega)^{-1} dS_1, \quad c_\pi = \left(\frac{1}{2\pi}\right)^2,$$

$$B = c_\pi \sum_{j=1}^{N} \int_{\Sigma_j} \log|\omega|^2 \frac{P_j(\omega)}{|\nabla \lambda_j(\omega)|} d\Sigma_j,$$ *$d\Sigma_j$ is the surface element of Σ_j,*

the matrices $P_j(\xi)$ are projections on the eigenspaces corresponding to the $\lambda_j(\xi)$.

In the case of the traction problem, let us denote by $\mathbf{v}_1$ the solution of the problem:

$$(1.8) \quad A(x,\partial)\mathbf{v}_1 = \mathbf{f} \quad \text{in } \Omega,$$
$$B_N(x,\partial)\mathbf{v}_1 = 0 \quad \text{on } \partial\Omega, \quad \mathbf{v}_1 - \left(E_0 - \frac{i\pi}{2}M + B\right)\mathbf{d} \to 0 \quad \text{as } |x| \to \infty,$$

where $\mathbf{d} = \mathbf{d}(\mathbf{f})$ is constant m–dimensional column and $\mathbf{f} \in L^2_a(\Omega)$. We can show that uniqueness and the existence of solution of (1.8) and

$$\mathbf{d} = \frac{1}{\pi}M^{-1}\left(\sum_{i=1}^{2} A^{ii}\right)^{-1}\int_\Omega \mathbf{f}dx. \tag{1.9}$$

Note that the unit matrix satisfies the equations: $A(x,\partial)I = 0$ in Ω, $B_N(x,\partial)I = 0$ on $\partial\Omega$.

Theorem 1.2. *For the solution $\mathbf{u} = R_k\mathbf{f}$ with $\mathbf{f} \in L^2_a(\Omega)$ in the case that $B(x,\partial) = B_N(x,\partial)$, the following asymptotic expansion is valid where $k \in D$, $|k| \to 0$:*

$$R_k\mathbf{f} = \sum_{m=0}^{N}\sum_{n=0}^{2m+1} k^m \log^n k H_{n,m}\mathbf{f} + \hat{\mathbf{u}}_N, \tag{1.10}$$

where $H_{n,m} \in \mathbf{B}(L^2_a(\Omega), H^2(\Omega))$ are independent of k and

$$\|\hat{\mathbf{u}}_N\|_{H^2(\Omega_a)} \le C(a)|k|^{N+1}|\log k|^{2N+3}\|\mathbf{f}\|_{L^2_a(\Omega)}.$$

The leading terms of the asymptotic expansion have the form:

$$\mathbf{u} = \log kM\mathbf{d} + \mathbf{v}_1 + O(k\log^3 k). \tag{1.11}$$

Let Λ denote the set of all poles of R_k in D.

Theorem 1.3. *Assume that* (A.1)–(A.4) *are valid. Then,*

$$\Lambda \cap D_+ = \emptyset. \tag{1}$$

When $A(x,\partial) = A(\partial)$, we have

$$\Lambda \cap (\mathbf{R} \setminus \{0\}) = \emptyset. \tag{2}$$

§2. Sketch of the proof

As a fundamental solution: $A(\partial)E_0 = \delta I$, we adopt $E_0(x) = -\mathfrak{F}^{-1}[\text{p.v.}A(\xi)^{-1}]$. Then we have a representation of $E_0(x)$ such that

$$E_0(x) = \log|x|M + Q, \tag{2.1}$$

where M is a constant matrix. Furthermore, projecting on the eigenspaces corresponding to the $\lambda_j(\xi)$, we have that for fundamental solution: $(k^2I + A(\partial))E_k = \delta I$,

$$E_k(x) = E_0(x) + \left(\log k - \frac{i\pi}{2}\right) M + B + F_k^1(x) + F_k^2(x), \tag{2.2}$$

where

$$\sup_{x\in\Omega}|F_k^1(x)| = O(k\log k) \text{ and } \|F_k^2 * \mathbf{f}\|_{H^2(\mathbf{R}^2)} = O(k^2) \tag{2.3}$$

for $\mathbf{f} \in L_a^2(\Omega)$ as $k \to 0$.

Let us denote by η a particular function which is infinitely smooth, $\eta(x) = 0$ for $|x| < a-1$ and $= 1$ for $|x| > a - 1/2$. For any smooth $\mathbf{u}$ let us denote by $\mathbf{g}$ the following function:

$$\mathbf{g}(\mathbf{u}) = A(\partial)(\eta\mathbf{u}) - \eta A(\partial)\mathbf{u}, \quad x \in \mathbf{R}^2.$$

Since $A(x, \partial) = A(\partial)$ for $|x| > a - 1$, for $\mathbf{u} = R_k\mathbf{f}$, $\eta\mathbf{u}$ satisfies the following in $\mathbf{R}^2$

$$\left(k^2 I + A(\partial)\right)(\eta\mathbf{u}) = \mathbf{g}(\mathbf{u}) + \eta\mathbf{f},$$

where the right-hand side has compact support. Representation

$$\eta\mathbf{u} = E_k * (\mathbf{g}(\mathbf{u}) + \eta\mathbf{f}) \tag{2.4}$$

follows immediately.

From (1.3),

$$\mathbf{u} = k^s \frac{Q(\log k)}{S(\log k)} + O(k^{s+1}\log^\gamma k), \quad \text{in } \Omega_a,\ k \to 0, \tag{2.5}$$

where s in an integer, S is a polynomial with constant coefficients and Q is a polynomial, not identically zero, whose coefficients are functions of x.

Substituting (2.2) and (2.5) into both sides of the equality (2.4) and equating the leading terms which contain the multiplier k^s and using the uniqueness of the problems (1.4) and (1.5) or (1.8), we get the leading terms. If we have the leading terms, we can obtain the asymptotic expansions by the usual perturbation method.

§3. An application

We shall discuss the rate of the local energy decay of solutions to the dynamical system:

$$\begin{cases} \partial_t^2\mathbf{u}(t,x) - A(\partial)\mathbf{u}(t,x) = 0 & \text{in } \mathbf{R}\times\Omega, \\ \mathbf{u}(t,x) = 0 & \text{on } \mathbf{R}\times\partial\Omega, \\ \mathbf{u}(0,x) = \mathbf{f}^1(x), \ \ \partial_t\mathbf{u}(0,x) = \mathbf{f}^2(x) & \text{in } \Omega, \end{cases} \tag{3.1}$$

where t denotes the time variable and $\partial_t = \partial/\partial t$. Put

$$\mathfrak{f} = (\mathbf{f}^1, \mathbf{f}^2), \quad \mathfrak{L}\mathfrak{f} = (\mathbf{f}^2, A(\partial)\mathbf{f}^1),$$

$$(\mathfrak{f}, \mathfrak{g})_{\mathcal{H}} = \sum_{i,j=1}^{2} \int_\Omega A^{ij}\partial_j \mathbf{f}^1 \cdot \overline{\partial_i \mathbf{g}^1} dx + (\mathbf{f}^2, \mathbf{g}^2)_\Omega, \quad \|\mathfrak{f}\|^2_{\mathcal{H}} = (\mathfrak{f}, \mathfrak{f})_{\mathcal{H}},$$

$$\mathcal{H} = \{\mathfrak{f} \mid \mathbf{f}^1 \in \dot{H}^1(\Omega), \mathbf{f}^2 \in L^2(\Omega), \mathbf{f}^1 = 0 \text{ on } \partial\Omega\},$$

$$D(\mathfrak{L}) = \{\mathfrak{f} \mid \mathbf{f}^1 \in \dot{H}^2(\Omega), \mathbf{f}^2 \in H^1(\Omega), \mathbf{f}^j = 0 \ (j = 1, 2) \text{ on } \partial\Omega\}.$$

Let us adopt $(\mathfrak{f}, \mathfrak{g})_{\mathcal{H}}$ as the inner product of $\mathcal{H}$, then $\mathcal{H}$ is a Hilbert space. Since $\mathfrak{L}$ is skew self-adjoint in $\mathcal{H}$, from Stone's theorem it follows that $\mathfrak{L}$ generates one parameter unitary group $\{U(t) \mid t \in \mathbf{R}\}$ (cf.[13]). Put $U(t)\mathfrak{f} = (\mathbf{u}(t,x), \partial_t \mathbf{u}(t,x))$. Then, $\mathbf{u}(t,x)$ is a unique solution to (3.1) with initial data $\mathbf{f}^1 \in \dot{H}^1(\Omega)$ and $\mathbf{f}^2 \in L^2(\Omega)$.

Definition. *We shall say that Ω is non-trapping if there exists a $T > 0$ depending only on a and Ω such that all components of $U(t)\mathfrak{f}$, $\mathfrak{f} = (0, \mathbf{f}^2)$, belong to $C^\infty([T,\infty) \times \overline{\Omega_a})$ for any $\mathbf{f}^2 \in L^2_a(\Omega)$.*

Vainberg [17, 18] proves the following theorem.

Theorem 3.1. *Assume that Ω is non-trapping and that (A.1)–(A.4) are valid. Then there exist positive constants α, β, C and T such that for integers s and j, $0 \le s \le 1$, $0 \le j \le 2$,*

$$\sum_{|\gamma| \le 2+s-j} \|\partial^\gamma R_k \mathbf{f}\|_{\Omega_a} \le C|k|^{1-j} e^{T|\Im k|} \sum_{|\gamma| \le s} \|\partial^\gamma \mathbf{f}\|_\Omega$$

for any $k \in \{k \in D \mid |\Im k| < \alpha \log|\Re k| - \beta\}$ and $\mathbf{f} \in L^2_a(\Omega)$.

Put $\mathcal{H}_\infty = \{\mathfrak{f} \mid \mathbf{f}^j \in C_0^\infty(\overline{\Omega}) \ j = 1,2\}$. As is well known, we have

$$U(t)\mathfrak{f} = \frac{1}{2\pi} \int_{-\infty+i\sigma}^{+\infty+i\sigma} e^{-ikt}(ikI + \mathfrak{L})^{-1}\mathfrak{f} dk \tag{3.2}$$

for any $\mathfrak{f} \in \mathcal{H}_\infty$ and $\sigma > 0$. Then, we have the local energy decay of solutions.

Theorem 3.2. *Assume that Ω is non-trapping and that (A.1)–(A.4) are valid. Let $\mathfrak{f} \in D(\mathfrak{L})$ and $\mathrm{supp}\mathbf{f}^j \subset \Omega_a$, $j = 1, 2$. Put $U(t)\mathfrak{f} = (\mathbf{u}(t,x), \partial_t \mathbf{u}(t,x))$. Then, we have the local energy decay estimate:*

$$\sum_{|\alpha| \le 1} \|\partial^\alpha \mathbf{u}(t,\cdot)\|_{\Omega_a} + \|\partial_t \mathbf{u}(t,\cdot)\|_{\Omega_a} \le C_1 |t|^{-1} \log^{-2} t \left[\sum_{|\alpha|=1} \|\partial^\alpha \mathbf{f}^1\|_\Omega + \|\mathbf{f}^2\|_\Omega \right] \tag{3.3}$$

with some positive constant C_1 depending only on a and Ω for $t \to \infty$.

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磁場をもつシュレーディンガー方程式の基本解の構成について

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電磁場中の荷電粒子に対するシュレーディンガー方程式の初期値問題を考える:

$$i\hbar\frac{\partial}{\partial t}u(t,x) = [\frac{1}{2}\sum_{k=1}^{d}(-i\hbar\frac{\partial}{\partial x_k} - A_k(t,x))^2 + V(x)]u(t,x), \tag{1}$$

$$u(0,x) = u_0(x), \quad t \in \mathbf{R},\, x = (x_1,\cdots,x_d) \in \mathbf{R}^d. \tag{2}$$

ここで $0 < \hbar \leq 1$ はパラメター（プランク定数を 2π で割ったもの）で, $A(t,x) = (A_1(t,x),\cdots, A_d(t,x))$ と $V(x)$ は, ベクトルポテンシャルとスカラーポテンシャルで次を仮定する:

Assumption (A). $A_k(t,x)$, $k=1,\ldots,d$, は $(t,x) \in \mathbf{R}\times\mathbf{R}^d$ の実数値関数で, 任意の α に対して, $\partial_x^\alpha A_k(t,x)$ は $(t,x)\in\mathbf{R}\times\mathbf{R}^d$ について C^1 である。$\varepsilon > 0$ が存在して,

$$|\partial_x^\alpha A_k(t,x)| + |\partial_x^\alpha \partial_t A_k(t,x)| \leq C_\alpha, \quad |\alpha| \geq 1, \tag{3}$$

$$|\partial_x^\alpha B(t,x)| \leq C_\alpha(1+|x|)^{-1-\epsilon}, \quad |\alpha| \geq 1. \tag{4}$$

ここで $B(t,x)$ は 反対称行列で, (k,l)-成分は $B_{kl}(t,x) = (\partial A_l/\partial x_k - \partial A_k/\partial x_l)(t,x)$ であり, $|B|$ は B の $\mathbf{R}^d$ 上の作用素ノルムである. $V(x)$ は, 実数値 C^∞ 関数で, 次を満たす.

$$|\partial_x^\alpha V(x)| \leq C_\alpha, \quad |\alpha| \geq 2. \tag{5}$$

この仮定は, 谷島 [7] によって与えられ, 彼自身 (1)(2) の基本解を構成した.

$q(\tau)$ を, 時刻 s で y を出て, 時刻 t で x に達する, 電磁場中の粒子の古典軌道とする: $q(\tau)$ は Lagrange 方程式の境界値問題,

$$\dot{q}(\tau) = v(\tau), \quad \dot{v}(\tau) = B(\tau,q(\tau))v(\tau) + F(\tau,q(\tau)), \quad s \leq \tau \leq t \tag{6}$$

$$q(s) = y, \qquad q(t) = x,$$

の解である. ここで $F(t,x) = -(\partial_t A)(t,x) - (\partial_x V)(x)$. Assumption (A) が満たされる時, $|t-s|$ が十分小さいならば, 任意の $x, y \in \mathrm{R}^d$ に対して $q(s) = y$, $q(t) = x$ となる (6) の解が一意的に存在する [7]. そこで, この古典軌道 $q(\tau)$ に沿う作用関数 $S(t,s,x,y)$ をラグランジアン $L(\tau,q,v)$

$$L(\tau,q,v) = \frac{v^2}{2} + A(\tau,q)v - V(q) \tag{7}$$

によって

$$S(t,s,x,y) = \int_s^t L(\tau, q(\tau), v(\tau))d\tau \tag{8}$$

とする.

時間区間 $[s,t]$ の任意の分割をΔ : $s = \tau_0 < \tau_1 < \cdots < \tau_L = t$ とする. $\Delta\tau_j = \tau_j - \tau_{j-1}$, $j = 1, \cdots, L$, そして $|\Delta| = \max_{1\leq j \leq L} \Delta\tau_j$ とおく. ファインマン経路積分の, 有限次元近似は

$$K(\Delta; \hbar, t, s, x, y) = \prod_{j=1}^{L}\left(\frac{1}{2\pi i\hbar\Delta\tau_j}\right)^{d/2} \int_{\mathrm{R}^{d(L-1)}} e^{i\hbar^{-1}\sum_{j=1}^{L} S(\tau_j, \tau_{j-1}, x_j, x_{j-1})} \prod_{j=1}^{L-1} dx_j \tag{9}$$

で与えられる. ここで $x_j \in \mathrm{R}^d$ であり, そして $x = x_L$, $y = x_0$ とおいた.

$A \equiv 0$ で, V が (5) をみたし時間に依存する場合, 藤原 [3] は, ある積分方程式を解くことで基本解を構成し, $K(\Delta; \hbar, t, s, x, y)$ が $|\Delta| \to 0$ で基本解に強い位相で (amplitude function が $\mathcal{B}(\mathrm{R}^d \times \mathrm{R}^d)$ で) 収束することを示した. さらに谷島 [7] は, [3] と同様の方法を, Assumption (A) を満たす磁場をもつシュレーディンガー方程式に適用し, 基本解を構成し $K(\Delta; \hbar, t, s, x, y)$ の収束を示した. 一方, 藤原 [5] は, $A \equiv 0$ で V が (5) をみたすとき, 基本解の構成に積分方程式を用いずに, stationary phase method [4] をもとにした方法で積分 (9) を直接扱い, 強い位相での収束とその極限が基本解である事を示した. 本稿では, stationary phase method [6] をもとにして, [5] の方法をまねて, Assumption (A) を満たす磁場をもつシュレーディンガー方程式の基本解を構成する.

積分 (9) の phase function $\sum_{j=1}^{L} S(\tau_j, \tau_{j-1}, x_j, x_{j-1})$ は, τ_{j-1} で x_{j-1} と, τ_j で x_j を結ぶ区分的な古典軌道に沿う作用関数で, 次の性質を満たす ([6]):

(S) $S(\tau_j, \tau_{j-1}, x_j, x_{j-1})$ は次の形である:

$$S(\tau_j, \tau_{j-1}, x_j, x_{j-1}) = \frac{|x_j - x_{j-1}|^2}{2(\tau_j - \tau_{j-1})} + \omega(\tau_j, \tau_{j-1}, x_j, x_{j-1}), \quad j = 1, \cdots, L. \tag{10}$$

τ_j と τ_{j-1}を省略して

$$S_j(x_j, x_{j-1}) = S(\tau_j, \tau_{j-1}, x_j, x_{j-1}) \tag{11}$$

$$\omega_j(x_j, x_{j-1}) = \omega(\tau_j, \tau_{j-1}, x_j, x_{j-1}) \tag{12}$$

とおくと, $\omega_j(x_j, x_{j-1})$ は次の (i)(ii) を満たす:

(i) 任意の $m \geq 2$ に対し, j に依らない定数 $\kappa_m > 0$ が存在して

$$\max_{2\leq|\alpha+\beta|\leq m} \sup_{x,y\in\mathbf{R}^d} |\partial_x^\alpha \partial_y^\beta \omega_j(x,y)| \leq \kappa_m. \tag{13}$$

(ii) $(\bar{x}_L, \cdots, \bar{x}_0)$ を, 次の方程式系の任意の解とする:

$$\partial_{x_j} S_{j+1}(\bar{x}_{j+1}, \bar{x}_j) + \partial_{x_j} S_j(\bar{x}_j, \bar{x}_{j-1}) = 0, \quad j = 1, \cdots, L-1. \tag{14}$$

そのとき, 任意の $m \geq 1$ に対し, 定数 $B_m > 0$ が存在して

$$\sum_{j=1}^{L-1} \sum_{\substack{1\leq|\alpha|\leq m \\ |\beta|=1}} |[(\partial_{x_{j-1}} + \partial_{x_j} + \partial_{x_{j+1}})^\alpha \partial_{x_j}^\beta (\omega_j + \omega_{j+1})](\bar{x}_{j-1}, \bar{x}_j, \bar{x}_{j+1})| \leq B_m \tag{15}$$

となる. B_m は次元 d に依存するが, $(\bar{x}_L, \cdots, \bar{x}_0)$ と分割 Δには依存しない. ここで multi-index $\alpha = (\alpha_1, \cdots, \alpha_d)$ に対し, $(\partial_{x_{j-1}} + \partial_{x_j} + \partial_{x_{j+1}})^\alpha = \prod_{k=1}^d (\partial_{(x_{j-1})_k} + \partial_{(x_j)_k} + \partial_{(x_{j+1})_k})^{\alpha_k}$ である.

phase function $\sum_{j=1}^L S(\tau_j, \tau_{j-1}, x_j, x_{j-1})$ の $d(L-1)\times d(L-1)$-Hessian 行列 $H_\Delta + W_\Delta(X)$ は, 次のように書ける.

$$H_\Delta = \begin{pmatrix} (\frac{1}{\Delta\tau_1} + \frac{1}{\Delta\tau_2})I & -\frac{1}{\Delta\tau_2}I & 0 & \cdots & \cdots \\ -\frac{1}{\Delta\tau_2}I & (\frac{1}{\Delta\tau_2} + \frac{1}{\Delta\tau_3})I & -\frac{1}{\Delta\tau_3}I & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & -\frac{1}{\Delta\tau_{L-1}}I \\ 0 & \cdots & \cdots & \cdots & (\frac{1}{\Delta\tau_{L-1}} + \frac{1}{\Delta\tau_L})I \end{pmatrix}, \tag{16}$$

ここで I は, $d\times d$ の単位行列である. そして $X = (x_0, x_1, \cdots, x_L)$ に対して,

$$W_\Delta(X) = \begin{pmatrix} \partial_1^2(\omega_1+\omega_2) & \partial_1\partial_2\omega_2 & 0 & \cdots & \\ \partial_1\partial_2\omega_2 & \partial_2^2(\omega_2+\omega_3) & \partial_2\partial_3\omega_3 & 0 & \\ \vdots & \vdots & \vdots & \ddots & \partial_{L-1}\partial_{L-2}\omega_{L-1} \\ 0 & \cdots & \cdots & \cdots & \partial_{L-1}^2(\omega_{L-1}+\omega_L) \end{pmatrix} \tag{17}$$

となる、ここで $\partial_{j-1}\partial_j\omega_j$ は, x_{j-1} と x_j の微分に関する $d\times d$ 行列である. (9) の phase function が性質 (S) を満たすので, 次の stationary phase method が成り立つ.

Proposition 1. ([6]). $\delta>0$ が存在して, $|t-s|\leq\delta$ ならば $K(\Delta;\hbar,t,s,x,y)$ は, 次の形になる:

$$
\begin{aligned}
K(\Delta;\hbar,t,s,x,y) =&\left(\frac{1}{2\pi i\hbar(t-s)}\right)^{d/2}e^{i\hbar^{-1}S(t,s,x,y)}\\
&\times D(\Delta;t,s,x,y)^{-1/2}(1+r(\Delta;\hbar,t,s,x,y)). \qquad (18)
\end{aligned}
$$

但し,

$$
D(\Delta;t,s,x,y)=\det(I+H_\Delta^{-1}W_\Delta(X))|_{X=(y,x_1^*,\cdots,x_{L-1}^*,x)}. \qquad (19)
$$

ここで $(y,x_1^*,\cdots,x_{L-1}^*,x)$ は, $\sum_{j=1}^L S(\tau_j,\tau_{j-1},x_j,x_{j-1})$ の *critical point* である. そして, 任意の α, β に対して, $C_{\alpha\beta}>0$ が存在して

$$
|\partial_x^\alpha\partial_y^\beta r(\Delta;\hbar,t,s,x,y)|\leq C_{\alpha\beta}\hbar|t-s|^2. \qquad (20)
$$

$C_{\alpha\beta}$ は, 分割 Δ に依存しない.

注. $A(t,x)\equiv 0$ のとき, (20) において $|t-s|$ の巾は, 2 でなく 3 とできる (藤原 [4]).

次の 2 つの定理が, ここでの目的である.

Theorem 2. $\delta>0$ が存在して $|t-s|<\delta$ のとき, 関数 $d(t,s,x,y)$, $r(\hbar,t,s,x,y)\in\mathcal{B}(\mathbf{R}^d\times\mathbf{R}^d)$ が存在して,

$$
D(\Delta;t,s,x,y)\to 1+(t-s)d(t,s,x,y),\quad in\,\mathcal{B}(\mathbf{R}^d\times\mathbf{R}^d),\quad as\,|\Delta|\to 0, \qquad (21)
$$

$$
r(\Delta;\hbar,t,s,x,y)\to r(\hbar,t,s,x,y),\quad in\,\mathcal{B}(\mathbf{R}^d\times\mathbf{R}^d),\quad as\,|\Delta|\to 0. \qquad (22)
$$

更に, 次の評価が成立する. $C>0$ が存在して

$$
|d(t,s,x,y)|\leq C(|t-s|+|x-y|). \qquad (23)
$$

任意の α と β に対して $C_{\alpha\beta}>0$ が存在して

$$
|\partial_x^\alpha\partial_y^\beta d(t,s,x,y)|\leq C_{\alpha\beta}, \qquad (24)
$$

$$
|\partial_x^\alpha\partial_y^\beta r(\hbar,t,s,x,y)|\leq C_{\alpha\beta}\hbar|t-s|^2. \qquad (25)
$$

注. $A(t,x)\equiv 0$ のとき, (1.21) において極限は $1+(t-s)^2 d(t,s,x,y)$ とできる (藤原 [4]).

次の Theorem 3 で $1+(t-s)d(t,s,x,y)$ の明白な公式を与える. $\mathcal{L}$ を Hilbert space $L^2([s,t];\mathrm{R}^d)$ とし, 内積を

$$(u,v)_{\mathcal{L}}=\sum_{k=1}^{d}\int_s^t u_k(\tau)v_k(\tau)d\tau,\ \text{for } u=(u_1,\cdots,u_d),\ v=(v_1,\cdots,v_d)\in\mathcal{L}$$

とする. $\mathcal{H}$を Sobolev space $H_0^1([s,t];\mathrm{R}^d)=\{u|u,\dot{u}\in\mathcal{L},\ u(s)=u(t)=0\}$ として内積を $(u,v)_{\mathcal{H}}=(\dot{u},\dot{v})_{\mathcal{L}}$ で与える.

(6) の一意的な解を

$$q(\tau)=q^0(\tau)+q^1(\tau)\ \text{with}\ q^0(\tau)=\frac{t-\tau}{t-s}y+\frac{\tau-s}{t-s}x, \tag{26}$$

と書く. すると,

$$\ddot{q}^1(\tau)=B(\tau,q(\tau))\dot{q}(\tau)+F(\tau,q(\tau)),\quad s\le\tau\le t\ \text{and}\ q^1(s)=q^1(t)=0. \tag{27}$$

汎関数 $\mathcal{H}\ni u\to S(q^0+u)$ の第一変分は, critical path q^1 で消える. q^1 での第二変分は

$$\mathcal{H}\ni h\to\int_s^t|\frac{d}{d\tau}h(\tau)|^2+{}^t h(\tau)(Yh)(\tau)d\tau, \tag{28}$$

ここで

$$\begin{aligned}(Yh)(\tau)&=B(\tau,q(\tau))\frac{d}{d\tau}h(\tau)+Z(\tau)h(\tau)\\
&=\begin{pmatrix}0 & B_{12}(\tau,q(\tau)) & B_{13}(\tau,q(\tau)) & \cdots & \cdots\\ B_{21}(\tau,q(\tau)) & 0 & B_{23}(\tau,q(\tau)) & \cdots & \cdots\\ \vdots & \vdots & \vdots & \ddots & \cdots\\ B_{d1}(\tau,q(\tau)) & \cdots & \cdots & \cdots & 0\end{pmatrix}\frac{d}{d\tau}\begin{pmatrix}h_1(\tau)\\ h_2(\tau)\\ \vdots\\ h_d(\tau)\end{pmatrix}\\
&\quad+\begin{pmatrix}Z_{11}(\tau) & Z_{12}(\tau) & \cdots\\ Z_{21}(\tau) & Z_{22}(\tau) & \cdots\\ \vdots & \ddots & \cdots\\ \cdots & \cdots & Z_{dd}(\tau)\end{pmatrix}\begin{pmatrix}h_1(\tau)\\ h_2(\tau)\\ \vdots\\ h_d(\tau)\end{pmatrix}.\end{aligned} \tag{29}$$

ここで $Z_{kl}(\tau)$, $k,l=1,\cdots,d$, は,

$$Z_{kl}(\tau)=\sum_{m=1}^{d}(\partial_{(x)_l}B_{km})(\tau,q(\tau))\dot{q}_m(\tau)+(\partial_{(x)_l}F_k)(\tau,q(\tau)), \tag{30}$$

で与えられる掛け算作用素である．作用素 Y は，$\mathcal{L}$ で symmetric で，$x, y \in \mathbf{R}^d$ に対し，$\mathcal{H}$ から $\mathcal{L}$ への有界作用素である．Z は $\mathcal{H}$ から $\mathcal{L}$ への Hilbert-Schmidt 作用素である．

G をディリクレ境界値問題

$$-\ddot{u}(\tau) = f(\tau),\ s \leq \tau \leq t,\ u(s) = u(t) = 0$$

のグリーン作用素とすると，G は $\mathcal{L}$ から $\mathcal{H}$ への Hilbert-Schmidt 作用素であるので，GY は，$\mathcal{H}$ 上の Hilbert-Schmidt 作用素，GZ は，$\mathcal{H}$ 上で Trace class となる．

Theorem 3.

$$1 + (t-s)d(t,s,x,y) \equiv \lim_{|\Delta|\to 0} D(\Delta; t,s,x,y) = \det{}_2(I + GY)e^{\mathrm{Tr}GZ}, \tag{31}$$

ここで，$\det_2$ は regularized determinant で $\mathrm{Tr}GZ$ は GZ の trace である．

注．(31) の $1 + (t-s)d(t,s,x,y)$ の表現は，$A \equiv 0$ の藤原 [5] の結果と異なる．[5] では，GY が trace class なので $1 + (t-s)d(t,s,x,y) = \det(I + GY)$ となるが，ここでは GY が trace class でないので修正する必要がある．

$k(\hbar,t,s,x,y)$ を次のようにおく．

$$k(\hbar,t,s,x,y) = (1 + (t-s)d(t,s,x,y))^{-1/2}(1 + r(\hbar,t,s,x,y)). \tag{32}$$

Theorem 4. $\delta > 0$ が存在して $|t-s| < \delta$ のとき，$K(\Delta;\hbar,t,s,x,y)$ の $|\Delta| \to 0$ の極限

$$\begin{aligned} K(\hbar,t,s,x,y) &= \lim_{|\Delta|\to 0} K(\Delta;\hbar,t,s,x,y) \\ &= \left(\frac{1}{2\pi i\hbar(t-s)}\right)^{d/2} e^{i\hbar^{-1}S(t,s,x,y)}\ k(\hbar,t,s,x,y) \end{aligned} \tag{33}$$

は，シュレーディンガー方程式 の初期値問題 (1)(2) の基本解である．

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競合系の正値解の多重度とその形状について

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1 問題

Lotka-Volterra 型競合系の正値定常解について考える．

$$
(P) \qquad \begin{cases} u'' + u(a - u - bv) = 0 & \text{in } (0,\pi), \\ v'' + v(d - v - cu) = 0 & \text{in } (0,\pi), \\ u'(0) = u'(\pi) = v'(0) = v'(\pi) = 0. \end{cases}
$$

ただし，a, b, c, d は正定数とする．

bistable case について考える．bistable case とは、パラメター $a, b.c, d$ が、$bc - 1 > 0, ac - d > 0, bd - a > 0$ をみたすときである．(P) の定数定常解は $(0,0)$, $(a,0)$, $(0,d)$, $(u^*, v^*) = (\frac{bd-a}{bc-1}, \frac{ac-d}{bc-1})$ の 4 つであるが，(u,v)−平面に図示すると下図 1 のようになる．

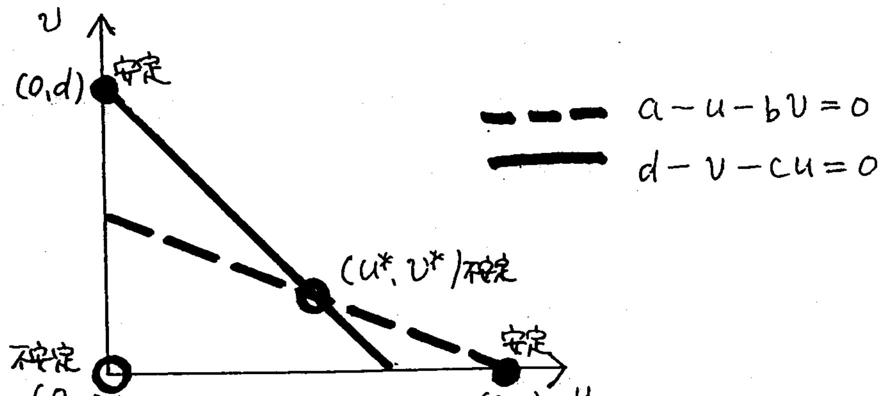
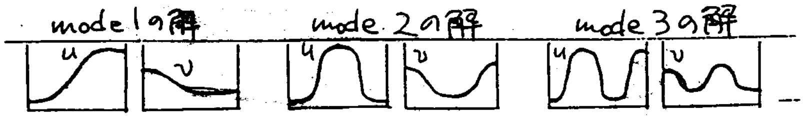


図 1

$(a,0)$, $(0,d)$ が共に安定なので bistable case とよばれる．

2 結果

定義 1 (P) の解 (u,v) が $(0, \frac{\pi}{i})$ 上で共に単調で，$u'(\frac{\pi}{i}) = v'(\frac{\pi}{i}) = 0$ をみたし，これを折り返しによって $[0,\pi]$ 全体に拡張されたものと一致するとき *mode* i の解という．



注．$(u(x),v(x))$ を mode n の解とする．$[0,\frac{\pi}{n}]$ 上で $(u(\frac{\pi}{n}-x),v(\frac{\pi}{n}-x))$ と表されたものを折り返しによって拡張する．これもまた (P) の mode n の解になる．解の単調性より，$(u(x),v(x))$ と $(u(\frac{\pi}{n}-x),v(\frac{\pi}{n}-x))$ は明らかに異なっている．したがって mode n の解を1つみつければ，自動的に2つみつかることになる．

定理 1 (u^*,v^*) の不安定次元が $n(n\geq 2)$ であるならば，mode 1, mode 2, mode 3,mode $n-1$............ の (P) の正値解が少なくとも2個ずつ存在する．

注．(u^*,v^*) は定数解であるので，簡単な計算により不安定次元を調べることができる．b,c を固定し，a,d をパラメターにとり，定理1の条件を ad−平面に描くと図2のようになる．

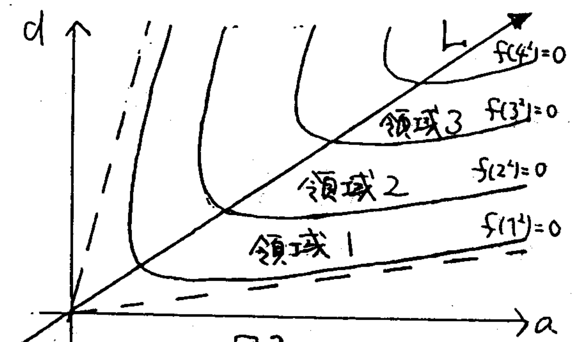


図2

領域n内にパラメター a,d をとると，(u^*,v^*) の不安定次元はn + 1になっている．ここで，

$$f(x)=x^2+\frac{(bd-a)+(ac-d)}{bc-1}x-\frac{(bd-a)(ac-d)}{bc-1}$$

次に非定数定常解が，定数定常解からどのように分岐しているかについて調べる．b,c を固定し，$d=ka$（ただし k は $\frac{1}{b}<k<c$ をみたす定数.）に沿って a を大きくする.（図2中，直線Lに沿って a を大にしたことになる.）

local な分岐理論から，次のことがいえる．Lと $f(i^2)=0$ の交点を $a=\lambda_i$ とする．$a=\lambda_i(i=1,2,3........)$ から mode i の local な分岐解を構成することができる．

定理 2 $a=\lambda_i(i=1,2,3........)$ から分岐した正値解は，mode i の性質を保ったまま *Global* に存在する.（図3）.

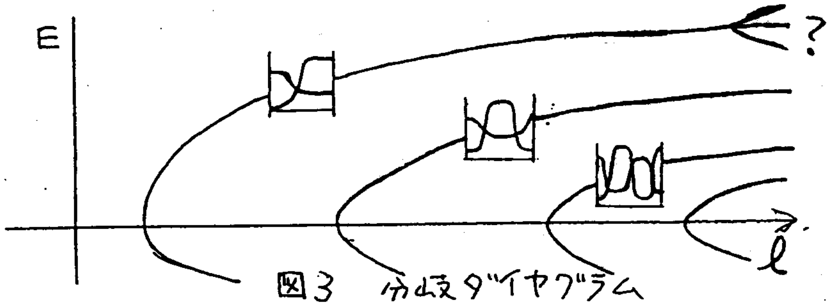


図3　分岐ダイヤグラム

注 観音幸雄先生によって以下のことが示されている.
(u,v) が (P) の正値解であるならば,
(i)(u,v) は定数解
(ii)$\exists j \in \mathrm{N}$, (u,v) は *mode j* の解のどちらかである.

3　証明の方針

（定理１）$E = C^1[0,\pi] \oplus C^1[0,\pi]$ とする. $K = \{u \in C^1[0,\pi], u(x) \geq 0.\}$ とし, Eの cone K^*を次のように定める.

$$K^* = K \oplus (-K).$$

$(u_1,v_1)-(u_2,v_2) \in K^*$ を $(u_1,v_1) \geq_* (u_2,v_2)$ とかく.
Eの部分集合Ωを定める.

$$\Omega := \{(u,v) \in E; (0,d) \leq_* (u,v) \leq_* (a,0), (u',v') \geq_* (0,0)\}$$

Ωは *mode* 1 の解に含まれるが, *mode* 2,*mode* 3,*mode* 4...... の解ではないことに注意する.
E上の写像 A を定める.

$$A(u,v) = (-\Delta + pI)^{-1}\{(p+a-u-bv)u, (p+d-v-cu)v\}, \quad (p \text{ は十分大}).$$

(P) の解を求めることは, $A: E \longrightarrow E$の不動点を求めることと同値である. Ω内の A の不動点をみつければ, *mode* 1 の解をみつけたことになる.（Dancer の有界閉集合上の写像度を使ってみつける.）こうして *mode* 1 の解が得られる. *mode* 2 以上については折り返しを使う.

（定理２）Crandall-Rabinowitz の Global bifurcation の結果を, Dancer の有界閉凸集合上の写像度用いて書き直す.

4 応用

定理 1 の証明は，Lotka-Volterra competition system のみではなく，以下述べる性質をもつ方程式系であれば，そのまま応用することができる．

方程式系 (1) を考える．

$$
(1) \qquad \begin{cases} u'' + f(u,v) = 0 & \text{in } (0,\pi), \\ v'' + g(u,v) = 0 & \text{in } (0,\pi), \\ u'(0) = u'(\pi) = v'(0) = v'(\pi) = 0. \end{cases}
$$

ただし f,g は u,v に関する C^1-級関数で，定数係数のものとする．

方程式系 (1) が次のような性質をみたせば，前章の証明が適用できる．

条件 1 順序保存系であること．

すなわち (i)$f_v \le 0$ かつ $g_u \le 0$ あるいは (ii)$f_v \ge 0$ かつ $g_u \ge 0$.

条件 2 u, v についてのアプリオリ評価があること．

例えば (2) のような cooperative system，あるいは (3) のような cross diffusion system などである．

$$
(2) \qquad \begin{cases} u'' + u(a - u + bv) = 0 & \text{in } (0,\pi), \\ v'' + v(d - v + cu) = 0 & \text{in } (0,\pi), \\ u'(0) = u'(\pi) = v'(0) = v'(\pi) = 0. \end{cases}
$$

$$
(3) \qquad \begin{cases} \frac{d^2}{dx^2}\{(1+\alpha v)u\} + u(a - u - bv) = 0 & \text{in } (0,\pi), \\ \frac{d^2}{dx^2}\{(1+\beta u)v\} + v(d - v - cu) = 0 & \text{in } (0,\pi), \\ u'(0) = u'(\pi) = v'(0) = v'(\pi) = 0, \end{cases}
$$

ただし，bistable case で　$\alpha a < b$ かつ$\beta c < d$ のときに限る．

以上の例よりもさらに非線形項が複雑な形をしている場合を考える．$f(u,v) = 0$ と $g(u,v) = 0$ が，(u,v) 平面上で以下図 3 のような形になる場合を考えてみる．

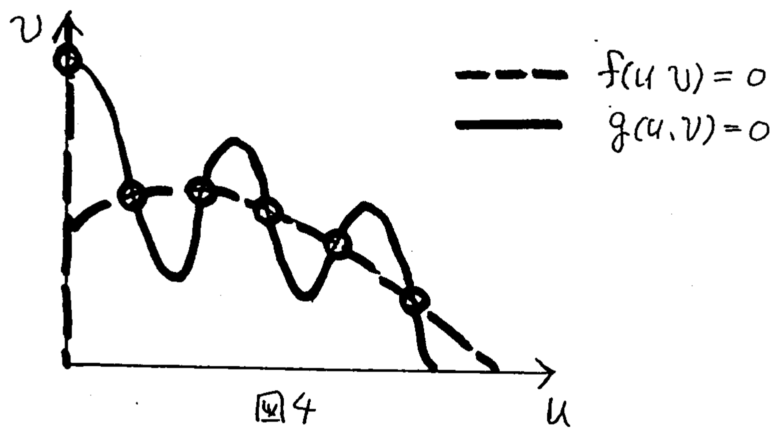


図4

このように方程式系 (1) が複数の正定数解をもつときでも，以下のような手順でさまざまな形状の解をみつけることができる．(ただし正定数解とは u,v 共に正の定数解のことをいい，非負定数解とは u,v 共に非負のの定数解のことをいう.)

手順．非負定数解をすべて書き出し，その不安定次元を調べる．(定数解であるので，不安定次元は簡単な計算により求められる.)

安定な非負定数解の数 = y 個

不安定次元 1 以上 n 以下の正定数解の数 = z 個

であるとする．y − z ≠ 1 ならば，少なくとも 2 つ *mode* n の解が存在する．

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cross-diffusionの効果を入れたある反応拡散系の解の構造についての数値的考察

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数理生態学において, Shigesada-Kawasaki-Teramoto[3] により cross-diffusion の効果を考慮に入れた１つのモデルが１９７９年に提案された. cross-diffusion とは２種類の個体群がある時, 個体群の拡散係数が相手の個体密度に依存するものである. 数学的には準線形方物型方程式系となるものでの解の大域的存在, 一意性, 安定性の問題は容易ではないが, 関連するモデルについて徐々に研究が進展してきた.

最近, 中島 - 山田 ([1],[4]) によってある１つのモデルに対して, 定常解の存在, 非存在に関する色々な十分条件が得られた. 証明は, 無限次元空間における写像度の概念を用いた非線形関数解析的な手法によって証明されている. しかしながら, 得られた条件の具体的な意味と, どこまでが sharp なものであるかは, はっきりと分かっていない.

我々目標は, 数値的なアプローチにより, この十分条件の具体的意味と十分条件の成り立たない場合の解の存在, 非存在の安定性について検討することである.

考える問題は次のような cross-diffusion の効果をいれた反応拡散系

$$(P)\begin{cases} u_t = \Delta[(1+\alpha v)u] + au(1-u-cv), & x\in(0,\pi),\quad t>0,\\ v_t = \Delta[(1+\beta u)v] + bv(1+du-v), & x\in(0,\pi),\quad t>0,\\ u(0,t)=u(\pi,t)=0, v(0,t)=v(\pi,t)=0, & t>0,\\ u(x,0)=u_0(x)\geq 0, v(x,0)=v_0(x)\geq 0, & x\in(0,\pi). \end{cases}$$

ここで α,β は非負定数,a,b,c,d は正定数, さらに u_0,v_0 は与えられた非負関数である. この問題は通常の半線形の prey-predator model

$$(P_0)\begin{cases} u_t = \Delta u + au(1-u-cv), & x\in(0,\pi),\\ v_t = \Delta v + av(1+du-v), & x\in(0,\pi),\\ u=v=0,\\ u(x,0)=u_0(x)\geq 0, v(x,0)=v_0\geq 0, & x\in(0,\pi), \end{cases}$$

に cross-diffusion の効果をつけたものであり, 同じ領域内に生息する２種類の生物の個体数変化をより現実に近いモデルで表したものである.

u を" 餌食" となる個体数密度,v を" 補食者" の個体数密度とする. 補食者 v の増加により 餌食 u の増加率が減少し, その効果が係数 $-c$ に現れ, 餌食 u が増加すれば, 補食者 v も増加する, それが係数 $+d$ に現れている.

問題 (P) のように密度に依存した拡散効果をもった反応拡散方程式は Shigesada-Kawasaki-Teramoto[3] により導入された．拡散項内の $1+\alpha v, 1+\beta u$ は直観的にその個体群が住んでいる各々の場所の住み心地の悪さの指数を意味している（Okubo[2]）．

この領域の境界 $x=0,\pi$ では双方の種にとって生息するには悪い環境にあることを意味している（Dirichlet 境界条件）．

ここで我々は中島 - 山田による解の存在の為の十分条件を具体的に図示しそれに基づき数値実験していく．

まず中島 - 山田による解の存在の為の十分条件を簡単に示す．

<u>定理</u>

$a>1$の時，

$$\begin{cases} \lambda_1\left(-\dfrac{b(1+d\phi_a)}{1+\beta\phi_a}\right)<0 \\ \lambda_1\left(\dfrac{a(c\phi_b-1)}{1+\alpha\phi_b}\right)<0 \end{cases}$$

または，

$$\begin{cases} \lambda_1\left(-\dfrac{b(1+d\phi_a)}{1+\beta\phi_a}\right)>0 \\ \lambda_1\left(\dfrac{a(c\phi_b-1)}{1+\alpha\phi_b}\right)>0 \end{cases}$$

$\Longrightarrow$ 正の解が少なくとも 1 つ存在する．

<u>注意</u>: 非存在のための十分条件

$$\begin{cases} a\leq 1 \\ a>1 \begin{cases} b>1 \quad \beta=0, & \lambda_1\left(\dfrac{a(c\phi_b-1)}{1+\alpha\phi_b}\right)>0 \\ b\leq 1 \begin{cases} \beta=0,\ \ 0<\alpha\leq c, & \lambda_1\left(-\dfrac{b(1+d\phi_a)}{1+\beta\phi_a}\right)>0 \\ \alpha=0,\ \ 0<\beta\leq d, & \lambda_1\left(-\dfrac{b(1+d\phi_a)}{1+\beta\phi_a}\right)>0 \end{cases} \end{cases} \end{cases}$$

ならば解は存在しない．

次に十分条件で用いられている記号の説明をする.

- $\phi_a(x)$ の定義

$$\begin{cases} w_{xx} + aw(1-w) = 0, & x \in (0,\pi), \\ w(0) = w(\pi) = 0. \end{cases}$$

i) $0 \le a \le 1$ の時,
$w \equiv 0$

ii) $a > 1$ の時,
$w > 0 \;\; in \;\; (0,\pi)$ となる解が唯一つ存在する.

これを満たす解を $\phi_a(x)$ とする.

- $\lambda_1\left(\dfrac{-b(d\phi_a+1)}{1+\beta\phi_a}\right), \lambda_1\left(\dfrac{a(c\phi_b-1)}{1+\alpha\phi_b}\right)$ の説明

$$\begin{cases} -w'' + \left(-\dfrac{b(1+d\phi_a)}{1+\beta\phi_a}\right)w = \lambda w & in \;\; (0,\pi) \\ w(0) = 0 \quad w(\pi) = 0 \end{cases}$$

及び

$$\begin{cases} -w'' + \left(\dfrac{a(c\phi_b-1)}{1+\alpha\phi_b}\right)w = \lambda w & in \;\; (0,\pi) \\ w(0) = 0 \quad w(\pi) = 0 \end{cases}$$

の第1固有値を，それぞれ

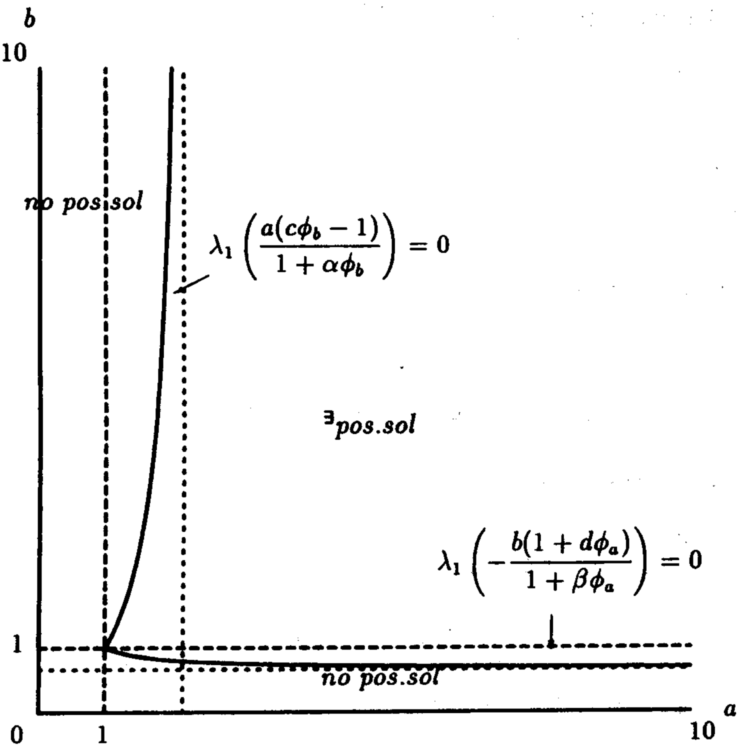
$$\lambda_1\left(-\frac{b(1+d\phi_a)}{1+\beta\phi_a}\right)$$
$$\lambda_1\left(\frac{a(c\phi_b-1)}{1+\alpha\phi_b}\right)$$

とする.

そして、

$$\lambda_1\left(-\frac{b(1+d\phi_a)}{1+\beta\phi_a}\right) = 0$$
$$\lambda_1\left(\frac{a(c\phi_b-1)}{1+\alpha\phi_b}\right) = 0$$

を満たす a, b を変数とした平面が今回, 中島 - 山田の十分条件を具体的に図示したものである. 次に代表的な例を1つを示す. これは全ての領域において解の存在, 非存在がわかっている場合である. 条件としては,$\beta = 0, 0 < \alpha < c$ である.



$\alpha = 0.1$ $c = 0.5$

$\beta = 0.0$ $d = 0.5$

この図に点を取ることにより全てのパラメータが決まり,その場合の方程式を解くことによってそれぞれの領域での解の存在,非存在を調べることが出来る.

それではまず初期値問題 (P) を考えていく.結果だけ述べると上の図の場合は予想通りの結果が得られた.つまり,$^{\exists}pos.sol$ と予想される領域に点を取った場合では u も v も正値解に収束し,*no pos.sol* と予想される場合では u か v のどちらか又は両方が 0 に収束してしまうのである.次に他の条件の場合だが,条件としては,$\alpha = 0, 0 < \beta < d$ や,$\alpha = 0, \beta > d$ や,$\alpha = c > 0, \beta >> d$ がある.これらは全て解の存在,非存在がわかっていない領域を含むものである.結果としては上の例と同じ様な結果が得られた.つまり,$^{\exists}pos.sol$ や *no pos.sol* と予想された領域が図的に対応しているのである.しかし,0 に収束すると予想されるものが完全に収束しきっていない場合もあり疑問が残るので,次に定常問題 (SP) を考えることによりそこで得られた解と比べてどうなのかを見る.又,同時に定常解にどれだけ近づくか,すなわち,安定性についても調べる.

漸近挙動を知るために (P) に対応する定常問題

$$
(SP)\begin{cases}
\Delta[(1+\alpha v)u]+au(1-u-cv)=0, & x\in(0,\pi),\\
\Delta[(1+\beta u)v]+bv(1+du-v)=0, & x\in(0,\pi),\\
u(0)=u(\pi)=0, v(0)=v(\pi)=0, & \\
u\ge 0, v\ge 0, & x\in(0,\pi),
\end{cases}
$$

まず、簡単にする為に

$$
U=(1+\alpha v)u
$$
$$
V=(1+\beta u)v
$$

とし変数変換しておく. よって

$$
(SP)\begin{cases}
U_{xx}+au(1-u-cv)=0, & x\in(0,\pi),\\
V_{xx}+bv(1+du-v)=0, & x\in(0,\pi),\\
U(0)=0, U(\pi)=0, V(0)=0, V(\pi)=0, & \\
U\ge 0, V\ge 0, & x\in(0,\pi),
\end{cases}
$$

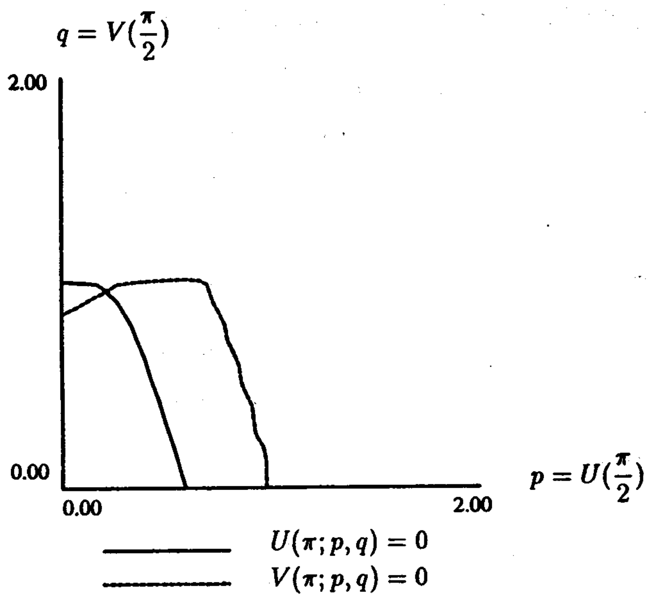
ただし,

$$
u=\frac{2U}{(1+\alpha V-\beta U)+\sqrt{(1+\alpha V-\beta U)^2+4\beta U}}
$$
$$
v=\frac{2V}{(1+\beta U-\alpha V)+\sqrt{(1+\beta U-\alpha V)^2+4\alpha V}}
$$

となる. これを考えていくのだが, 色々と数値計算した結果求める解が, $x=\frac{1}{2}$ で対称になっていることがわかったので次の問題を考えることにする.

$$
(SP)\begin{cases}
U_{xx}+au(1-u-cv)=0, & x\in(\frac{\pi}{2},\pi),\\
V_{xx}+bv(1+du-v)=0, & x\in(\frac{\pi}{2},\pi),\\
U'(\frac{\pi}{2})=0, U(\pi)=0, V'(\frac{\pi}{2})=0, V(\pi)=0, & \\
U\ge 0, V\ge 0, & x\in(\frac{\pi}{2},\pi),
\end{cases}
$$

ここで, $U(\frac{\pi}{2})=p, V(\frac{\pi}{2})=q$ として, $U(\pi)=0$ かつ $V(\pi)=0$ のとなる場合の p,q の組合せを求める. それを求める為に p,q 平面というのを導入するのだが, これは, $U(\pi)=0$ かつ $V(\pi)=0$ となる p,q の組合せの高さ 0 で切った等高線を示した平面である. 次にそれを載せる.



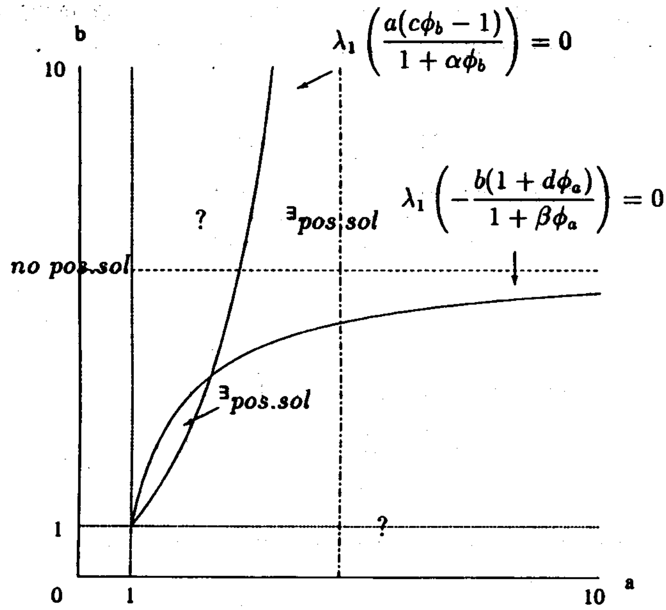
$\alpha = 0.1$ $\beta = 0.0$
$a = 3.0$ $b = 4.0$
$c = 0.5$ $d = 0.5$

図から交点がいくつかあるのがわかるがこの交点が定常解の p, q の組合せにあたる. このうち, 正値定常解は $p > 0, q > 0$ の場合である. この図は先ほど例に示した a, b 平面の条件のパラメータを用いて計算した場合であり, かつ,a, b 平面の正値解があると予想される領域に点を取った場合である.p, q 平面を見ると正値定常解が存在し, かつ, 初期値問題 (P) を解いた時も正値解に収束したから, これは正値解に収束すると言え, また, この解が安定解である. 次に, 正値解が存在しないと予想される領域に点を取った場合も予想通りの結果が得られた. この様にして他の $a-b$ 平面の場合も確かめていくとほぼ予想通りの結果が得られた。その中で $a-b$ 平面が魚型（次ページの図参照）になる場合で $\lambda_1\left(-\frac{b(1+d\phi_a)}{1+\beta\phi_a}\right) > 0$ かつ $\lambda_1\left(\frac{a(c\phi_b-1)}{1+\alpha\phi_b}\right) > 0$ の領域, つまり, 魚の体の部分に当たる領域は解の一意性が保証されないところであり初期値問題 (P) と定常問題 (SP) を解いた時のそれぞれの解は一致しなかった.

以上より実験した a, b の範囲では, 中島 - 山田の十分条件は, ぎりぎりのもので, 必要十分条件になっている様に思われる.

最後にまとめると

- 定常解の構造, 安定性が"数値的"にわかった.
- 中島 - 山田の十分条件をチェックするソフトが出来た.



$\alpha = 0.0$ $\qquad$ $c = 8.0$

$\beta = 5.0$ $\qquad$ $d = 0.0$

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ON THE STABILITY OF VISCOUS SHOCK FRONTS FOR CERTAIN CONSERVATION LAWS IN TWO DIMENSIONAL SPACE

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1. Introduction

We consider the Cauchy problem for the equations of the form

$$u_t + f(u)_x = \mu\Delta u \qquad (t,x,y) \in R_+ \times R^2 \tag{1.1}$$

$$u(0,x,y) = u_0(x,y) \tag{1.2}$$

where f is a smooth function, μ is a positive constant, and

$$u_0(x,y) \to u_\pm \quad \text{as} \quad x \to \pm\infty \quad \text{for any fixed y} \in R \tag{1.3}$$

We investigate the stability of viscous shock front of (1.1) of the form $u(t,x) = \phi(\xi)(\xi = x - st)$ uniformly in y satisfying

$$-s\phi' + f(\phi)' = \mu\phi'' \tag{1.4}$$

$$\phi(\xi) \to u_\pm \quad \text{as} \quad \xi \to \pm\infty \tag{1.5}$$

where the constants $u_\pm$ and s satisfy the Rankine-Hugoniot condition

$$-s(u_+ - u_-) + f(u_+) - f(u_-) = 0 \tag{1.6}$$

and the Generalized shock condition

$$h(\phi) := -s(\phi - u_\pm) + f(\phi) - f(u_\pm)\begin{cases} < 0, & \text{if } u_+ < \phi < u_- \\ > 0, & \text{if } u_+ > \phi > u_-. \end{cases} \tag{1.7}$$

It is noted that condition (1.7) implies

$$f'(u_+) \le s \le f'(u_-) \tag{1.8}$$

and that, especially when f is a convex, the condition(1.7) is equivalent to

$$f'(u_+) < s < f'(u_-) \tag{1.9}$$

which is well-known as Lax's shock condition. If (1.9) holds, we say the shock is "nondegenerate". On the other hand, if at least one of the equalities holds in (1.8), we say the shock is "degenerate".

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Notations.

In what follows, we will $H^l(l \geq 0)$ to denote the usual Sobolev space with norm $\|\cdot\|_l$, and $\|\cdot\|_0 = \|\cdot\|$ will denote the usual L^2 norm.

For the weight function w, L^2_w will denote the space of measurable functions f satisfying $\sqrt{w}f \in L^2$ with the norm

$$|f|_w = (\int_R w(x)|f(x)|^2 dx)^{1/2}$$

When $C^{-1} \leq w(x) \leq C$ for some positive constant C, we note that $L^2_w = L^2$ with $\|\cdot\| \sim |\cdot|_w$.

Moreover, by C or c, we denote several positive constants without confusion. For two functions f and g, $f \sim g$ as $x \to a$ means

$$C^{-1}f(x) \leq g(x) \leq Cf(x) \quad \text{in the neighborhood of } a.$$

2.Main Theorems

We begin by stating the existence of viscous shock front. Integrating (1.4) over $(\pm\infty, \xi)$ and noting the Rankine-Hugoniot condition(1.6), we have

$$\mu\phi_\xi = -s(\phi - u_\pm) + f(\phi) - f(u_\pm) \equiv h(\phi) \tag{2.1}$$

We find (2.1) admits smooth solution with shock profile satisfying (1.5) if and only if (1.6),(1.7) hold. So we have

Lemma 1.

(i) *If (1.1),(1.2) admits a travelling wave solution with shock profile $\phi(x-st)$ connecting u_- and u_+, then u_-, u_+ and s must satisfy the Rankine-Hugoniot condition (1.6) and the Generalized shock condition (1.7).*

(ii) *Conversely, suppose that (1.6) and (1.7) hold, then there exists a shock profile $\phi(x-st)$ of (1.1) (1.2) which connects u_- and u_+. $\phi(x-st)$ is unique up to a shift in $\xi = x - st$ and is a monotone function of ξ. Moreover, if $h(\phi) \sim |\phi - u_\pm|^{1+k_\pm}$ as $\phi \to u_\pm$ with $k_\pm \geq 0$, it holds*

$$\begin{aligned}
|\phi - u_\pm| &\sim exp(-c|\xi|) && as\ \xi \to \pm\infty \quad (k_\pm = 0)\\
|\phi - u_+| &\sim |\xi|^{-1/k_+} && as\ \xi \to +\infty \quad (k_+ \neq 0)\\
|\phi - u_-| &\sim |\xi|^{-1/k_-} && as\ \xi \to -\infty \quad (k_- \neq 0).
\end{aligned}$$

It is noted $k_\pm = n$ if $h'(u_\pm) = \cdots = h^{(n)}(u_\pm) = 0$ and $h^{(n+1)}(u_\pm) \neq 0$.

Proof. The proof is given in Kawashima and Matsumura [1].

Following Goodman[3], we reformulate our problem. As an asymptotic state, we consider $\phi = \phi(\xi + \delta(t,y))$. Since $\phi(\xi)$ satisfies (2.1) or (1.4), $\phi(\xi + \delta(t,y))$ satisfies

$$\phi_t - \phi_\xi\delta_t + f(\phi)_\xi = \mu\phi_{\xi\xi}.$$

Hence $u-\phi$ should be satisfied with following equation:

$$(u-\phi)_t - s(u-\phi)_\xi + \{f(u)-f(\phi)\}_\xi = \mu\Delta(u-\phi) + \mu\phi_{\xi\xi}{\delta_y}^2 + \phi_\xi(-\delta_t+\delta_{yy}) \tag{2.2}$$

If the terms proportional to ϕ_ξ cansel and

$$\int_R \{u-\phi\}d\xi\Big|_{t=0} = 0,$$

then we can decompose the solution to $u(t,x,y) = \phi(\xi+\delta(t,y)) + U_\xi(t,\xi,y)$. So (2.2) is equivalent to the following system of U and δ:

$$U_t - \mu\Delta U + h'(\phi)U_\xi = F + \mu\phi_\xi{\delta_y}^2 \tag{2.3}$$

$$\delta_t = \delta_{yy} \tag{2.4}$$

where

$$F = -\{f(\phi+U_\xi) - f(\phi) - f'(\phi)U_\xi\}$$

Since $u(0,x,y) = \phi(x+\delta(0,y)) + U_\xi(0,x,y)$, the initial shift function $\delta_0(y)$ is given as

$$\int_R \{u_0(x,y) - \phi(x+\delta_0(y))\}dx = 0 \tag{2.5}$$

or, explicitely,

$$\delta_0(y) = \frac{1}{u_+ - u_-}\int_R \{u_0(x,y) - \phi(x)\}dx \tag{2.6}$$

Furthermore,

$$U_0(x,y) = \int_{-\infty}^{x} \{u_0(z,y) - \phi(z+\delta_0(y))\}dz \tag{2.7}$$

Here, we assume

$$\delta_0(y) \to \delta_\pm \quad \text{as} \quad y \to \pm\infty \tag{2.8}$$

for some constants $\delta_\pm$.

Goodman[3] has obtained the stability ϕ for weak shock to the following problem:

$$u_t + f(u)_x + g(u)_y = \Delta u \qquad (t,x,y) \in R_+ \times R^2$$

$$u(0,x,y) = u_0(x,y) \to u_\pm \quad \text{as} \quad x \to \pm\infty$$

where f is convex and $\delta_+ = \delta_- = 0$.

Our main aim is to investigate the stability theorem for ϕ when f is nonconvex and $\delta_+ \neq \delta_-$. Main theorems are followings.

Theorem 1 (the case $\delta_+ = \delta_- = 0$).

Let $\phi(\xi)$ be a shock front and that $u_0(x,y) - \phi(x)$ is integrable in x uniformly in y.

(i) *(nondegenerate case $f'(u_+) < s < f'(u_-)$)*

Assume $U_0 \in H^3(R^2)$ and $\delta_0 \in H^2(R)$ if

$$\|U_0\|_3 + \|\delta_0\|_2 \le \varepsilon$$

then the problem (1.1)(1.2) has a unique global solution $u(t,x,y)$ satisfying

$$u - \phi \in C^0([0,\infty); H^2(R^2)) \cap L^2(0,\infty; H^3(R^2)).$$

Moreover, the solution verifies

$$\sup_{R^2} |u(t,x,y) - \phi(x-st)| \to 0 \quad as \quad t \to \infty$$

(ii) *(degenerate case $f'(u_+) = s < f'(u_-)$)*

Assume

$$|h(\phi)| \sim |\phi - u_+|^{1+k_+} \qquad as \quad \phi \to u_+ \quad \text{for finite integer } k_+$$

and that $U_0 \in H^3(R^2) \cap L^2_{<\xi>_+}(R^2)$ and $\delta_0 \in H^2(R)$.
Then there exists a positive constant ε such that if

$$\|U_0\|_3 + |U_0|_{<\xi>_+} + \|\delta_0\|_2 \le \varepsilon$$

then the problem (1.1)(1.2) has a unique global solution $u(t,x,y)$ satisfying

$$u - \phi \in C^0([0,\infty); H^2(R^2) \cap L^2_{<\xi>_+}(R^2)) \cap L^2(0,\infty; H^3(R^2) \cap L^2_{<\xi>_+}(R^2)).$$

Moreover, the solution verifies

$$\sup_{R^2} |u(t,x,y) - \phi(x-st)| \to 0 \quad as \quad t \to \infty$$

where

$$<\xi>_+ = \begin{cases} \sqrt{1+\xi^2}, & \xi \ge 0 \\ 1, & \xi < 0 \end{cases}$$

Next, we consider the case $\delta_+ \neq \delta_-$. Setting $\omega(\frac{y}{\sqrt{t+1}})$ as a similarity solution of (2.4) satisfying $\omega(\pm\infty) = \delta_\pm$, our second theorem is given as follows.

Theorem 2 (the case $\delta_+ \neq \delta_-$).
Let ϕ be a shock front and $\omega(\frac{y}{\sqrt{t+1}})$ be a similarity solution of (2.4) satisfying $\omega(\pm\infty) = \delta_\pm$.

(i) *(nondegenerate case)*
Assume that $U_0 \in H^3(R^2)$ and $\delta_0 - \omega \in H^2(R)$.
Then there exists a positive constant ε such that if

$$\|U_0\|_3 + \|\delta_0 - \omega\|_2 + |\delta_+ - \delta_-| \le \varepsilon$$

then the problem (1.1)(1.2) has a unique global solution $u(t,x,y)$ satisfying

$$u - \phi \in C^0([0,\infty); H^2(R^2)) \cap L^2(0,\infty; H^3(R^2)).$$

Moreover, the solution verifies

$$\sup_{R^2}\left|u(t,x,y) - \phi(x - st + \omega(\frac{y}{\sqrt{t+1}}))\right| \to 0 \quad as \quad t \to \infty$$

(ii) *(degenerate case)*
Assume

$$f'(u_+) = s < f'(u_-) \quad and \quad f''(u_+) \neq 0$$

and $U_0 \in H^3(R^2) \cap L^2_{<\xi>^{\frac{\alpha}{+}}}(R^2)$ for $1 < \alpha < \frac{13}{12}$ and $\delta_0 - \omega \in H^2(R)$. Then there exists a positive constant ε such that if

$$\|U_0\|_3 + |U_0|_{<\xi>^{\frac{\alpha}{+}}} + \|\delta_0 - \omega\|_2 + |\delta_+ - \delta_-| \le \varepsilon$$

then the problem (1.1)(1.2) has a unique global solution $u(t,x,y)$ satisfying

$$u - \phi \in C^0([0,\infty); H^2(R^2) \cap L^2_{<\xi>^{\frac{\alpha}{+}}}(R^2)) \cap L^2(0,\infty; H^3(R^2) \cap L^2_{<\xi>^{\frac{\alpha}{+}}}(R^2)).$$

Moreover, the solution verifies

$$\sup_{R^2}\left|u(t,x,y) - \phi(x - st + \omega(\frac{y}{\sqrt{t+1}}))\right| \to 0 \quad as \quad t \to \infty$$

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Nonexistence of nontrivial solutions of $-\Delta_p u = |u|^{q-2}u$ in unbounded domains

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1 Introduction

次のような非線形楕円型方程式の非自明解の非存在について考える.

$$(\mathrm{E}) \begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) &= |u|^{q-2}u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

ここで Ω は滑らかな境界 $\partial\Omega$ を持つ $\mathbb{R}^N$ の領域, $1 < p, q < \infty$ とする.

$\Delta_p \cdot = \mathrm{div}(|\nabla \cdot|^{p-2}\nabla \cdot)$ は p-Laplacian と呼ばれる作用素で, $p = 2$ の時通常の Laplacian に一致する. Ω が有界領域の場合にはレイリー商 $R(v) = \|\nabla v\|_{L^p}/\|v\|_{L^q}$ を最小化する元が (適当に正規化すれば) (E) の $W_0^{1,p}$ での解となることがわかる.
一般に Ω が非有界領域の場合には, 上に述べたような変分法的な定式化はできない. しかしながら偏微分方程式論的な立場からみて, 非有界領域における非自明解の非存在について論じることは意義のあることであろう. 実際このような試みが過去多数なされてきた.
しかしながら, これらの結果はすべて解のクラスが $C^2(\Omega) \cap C^1(\overline{\Omega})$ に限定されたものである. しかし注意すべきことは, (E) の解は $p \neq 2$ の時には $C^2(\Omega)$ に属さないということである. それゆえ, 説得力のある非存在定理を展開するためには, 我々はより広いクラスで議論をしていく必要がある. この種の試みは, Ω が有界の場合, [4] によりなされた. 本講演の目的は, [4] の有界領域における非存在定理の議論を基に, 非有界領域における [2] より適切なクラスでの非存在定理を研究するところにある.

2 Main Results

今回得られた結果を以下に述べる.

Theorem 1 $p^* := \frac{Np}{N-p}$ *if* $p < N, := \infty$ *if* $p \geq N$, $\Omega = \mathbb{R}^N \setminus \overline{\Omega}_0$ で, Ω_0 は有界 かつ星状領域 (即ち, 適当に原点をとると, $x \cdot \vec{n}(x) \geq 0\ \forall x \in \partial\Omega$ となる) であるとする.

$$Q = \{u \in L^q(\Omega); \nabla u \in (L^p(\Omega))^N, u|_{\partial\Omega} = 0\}.$$

とする時,

1. もし, $q > p^*$ ならば, (E) は $\mathcal{Q}$ に属する非自明解を持たない.

2. もし, $q = p^*$ ならば, (E) は $\mathcal{Q}$ に属する非自明正値解を持たない.

Theorem 2 $\Omega = \mathbb{R}^N$ とする.

$$\mathcal{Q}_0 = \{u \in L^q(\mathbb{R}^N); \nabla u \in (L^p(\Omega))^N, x_i|u|^{q-1} \in L^{p'}_{loc}(\mathbb{R}^N) = 0\}.$$

とする時, もし, $q \neq p^*$ ならば, (E) は $\mathcal{Q}_0$ に属する非自明解を持たない.

Remark $\Omega = \mathbb{R}^N$ の場合は, [7] により, もし $q = p^*$ ならば, (E) は

$$u(x) = [a + b|x|^{\frac{p}{p-1}}]^{\frac{p-N}{p}} \qquad (a, b > 0)$$

という正値解をもつということが分かっている.

3 Pohozaev-type inequality

(E) の古典解に対して Pohozaev の等式を導くことは容易であるが, 変分法的枠組みで考える立場から言えば, 弱解に対しても有効な式を導かなければならない. その場合は Pohozaev の等式の代わりの適切な不等式が期待される.
以下では (E) のあるクラス $(\mathcal{P})$ に属する弱解 u に対する "Pohozaev 型の不等式" を導出する.

Proposition 1 Ω を *Theorem 1* の仮定を満たす領域とする.

$$\mathcal{P} = \mathcal{Q} \cap \{u; |u|^{q-1} \in L^{\frac{p}{p-1}}(\Omega_R)\ \forall R > 0\}$$

ここに, $\Omega_R = \Omega \cap B_R(0)$ とする. この時, $\mathcal{P}$ に属する *(E)* の解 u は以下の *Pohozaev* 型の不等式を満たす.

$$\left(\frac{N}{q} + \frac{p-N}{p}\right) \int_\Omega |u|^q dx + \mathcal{R} \leq 0, \tag{1}$$

ここに,

$$\mathcal{R} = \overline{\lim_{n\to\infty}}\ \overline{\lim_{\varepsilon\to 0}} \frac{\bar{p}-1}{p} \int_{\partial\Omega} (|\nabla w_n^\varepsilon|^2 + \varepsilon)^{\frac{p}{2}} (-x \cdot \vec{n}) dS,\ \bar{p} = \min(p, 2).$$

証明は (E) を近似する方程式を立ててから, compact support をもつ cut-off function をかけ, [4] で用いられた手法を援用することにより示される. 詳しくは, [2] を参照.

Remark $\Omega = \mathbb{R}^N$ の場合は $\partial\Omega = \emptyset$ なので境界積分の項は消え, 以下のように恒等式になる.

Corollary 1 $\Omega = \mathbb{R}^N$ とする. このとき *(E)* の $\mathcal{Q}_0$ に属する全ての解は以下の *Pohozaev* 型の等式を満たす.

$$\left(\frac{N}{q} + \frac{p-N}{p}\right) \int_{\mathbb{R}^N} |u|^q dx = 0 \tag{2}$$

4 On the regularity of weak solutions

前節では Pohozaev 型の不等式を導くのに, $|u|^{q-2}u \in L^{p'}_{loc}(\overline{\Omega})$ という条件を課した. この節では, もっとも適切なクラスである $\mathcal{Q}$ での非存在を論じるためのいくつかの正則性の結果を示す.

Proposition 2 $q < p^*$, もしくは $p < N$ の時は $q = p^*$ とする. u を $u \in L^q(\Omega)$, $\nabla u \in (L^p(\Omega))^N$ を満たす (E) の解であるとすると, $|u|^{q-1}$ は $L^{p'}_{loc}(\overline{\Omega})$ に属する.

Proof いくつかの場合に分けて考える.

1. $p \geq N$ の場合
   任意の有界領域 Ω_R に対して, ノルム $\|u\|_{L^q(\Omega_R)} + \|\nabla u\|_{L^p(\Omega_R)}$ は $W^{1,p}(\Omega_R)$ の norm と同値である, よって $p > N$ (あるいは $p = N$) ならば u は $L^\infty_{loc}(\overline{\Omega})$ (あるいは $L^r_{loc}(\overline{\Omega})$ for all $r \in [1,\infty)$) に属する. ゆえに $|u|^{q-1}$ は $L^{p'}_{loc}(\overline{\Omega})$ となる.

2. $p < N$ の場合
   さらに次の 2 つの場合に分けて考える

   (a) $q < p^*$ の時
   [4] の Lemma 3.1 での "bootstrap method" と同じように証明できる. 定数 $K > 0$ が存在して

$$\|u\|_{L^{p^*}} \leq K\|\nabla u\|_{L^p} \text{ for all } u \in C_0^\infty(\Omega), \tag{3}$$

   [4] の Lemma 3.1 と同じ議論で $\|u\|_{L^{q_k}(\Omega)} \leq C_k$ が結論される. ここに

$$\begin{cases} q_{k+1} = \frac{N}{N-p}q_k^*, q_k^* = q_k - q + p, q_1 = q, \\ C_{k+1} = K^{\frac{p}{q_k^*}}(q_k - q + 1)^{-\frac{1}{q_k^*}}(\frac{q_k^*}{p})^{\frac{p}{q_k^*}}C_k^{\frac{q_k}{q_k^*}}, \\ C_1 = \|u\|_{L^q}. \end{cases}$$

   さらに [4] の Theorem II の証明と同様に定数 $d > 0$ が存在して $C_k \leq e^k$ for all k. ゆえに $u \in L^\infty(\Omega)$ が言える. □

   (b) $q = p^*$ の時
   臨界指数の場合での C^∞ 評価は [3] でなされているが, ここでは次の形の Lemma を用意する.

   Lemma 1 $p < N$ として $u \in \{u; \nabla u \in (L^p(\Omega))^N, u \in L^{p^*}(\Omega), u|_{\partial\Omega} = 0\}$ が (超関数の意味で)

$$(\mathrm{E})^f \begin{cases} -\Delta_p u &= f(x)|u|^{p-2}u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

を満たすとする．ここに，ある $t > \frac{N}{p}$ に対して $f \in L^t(\Omega)$．この時 $u \in L^\infty(\Omega)$ であり，さらに

$$\|u\|_{L^\infty} \leq K^{\frac{\delta}{\delta-1}} \delta^{\frac{\delta}{(\delta-1)^2}} \|f\|_{L^t}^{\frac{1}{p(\delta-1)}} \|\nabla u\|_{L^p}, \tag{4}$$

ここに $\delta = \frac{q(t-1)}{pt} > 1$.

Proof of Lemma $\delta = q(t-1)/pt$ とおく，$t > N/p$ なので $\delta > 1$. 2 つの列 q_k と r_k を

$$q_k = q \cdot \delta^k, q = \frac{Np}{N-p};\ p + r_k - 2 = p \cdot \delta^k, k = 1, 2, \ldots.$$

と定義する．section 3 で定義された cut-off function $g_n(\cdot)$ を用いて，(E)$'$ に $|u_n|^{r_k-2}u_n$ を掛けると次の式が得られる，

$$(r_k - 1) \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla u_n |u_n|^{r_k-2} dx = \int_\Omega |u|^{p-1} |u_n|^{r_k-1} f dx,$$

書き直すと

$$\int_\Omega |\nabla u_n|^p |u_n|^{r_k-2} dx \leq \int_\Omega |u|^{p+r_k-2} |f| dx.$$

Hölder の不等式により

$$\left(\frac{p}{p+r_k-2}\right)^p \int_\Omega |\nabla |u_n|^{\delta^k}|^p dx \leq \|f\|_{L^t} \|u\|_{L^{p\delta^k t'}}^{p\delta^k}.$$

(3) より，もし $u \in L^q(\Omega)$ ならば

$$\|u_n\|_{L^{q_k}} \leq K^{\frac{1}{\delta_k}} \delta^{\frac{k}{\delta_k}} \|f\|_{L^t} \|u\|_{L^{q_{k-1}}}, \tag{5}$$

$q_0 = q$, $u \in L^q(\Omega)$ なので，$k = 1$ の時の (5) 式より $\|u_n\|_{L^{q_1}}$ は有界で，$n \to \infty$ とすることによって，$u \in L^{q_1}(\Omega)$. よって $k = 2$ のときの (5) が成り立つ．これを繰り返すことにより，任意の k に対し，$u \in L^{q_1}(\Omega)$ で，u_n を u としても (5) が成り立つ．ゆえに，

$$\begin{aligned}
\|u_n\|_{L^{q_k}\Omega} &\leq K^{\sum_{j=1}^k \frac{1}{\delta_j}} \delta^{\sum_{j=1}^k \frac{j}{\delta_j}} \|f\|_{L^t}^{\frac{1}{p}\sum_{j=1}^k \frac{1}{\delta_j}} \|u\|_{L^q(\Omega)} \\
&\leq K^{\frac{\delta}{\delta-1}} \delta^{\frac{\delta}{(\delta-1)^2}} \|f\|^{\frac{1}{p(\delta-1)}} \|\nabla u\|_{L^p(\Omega)}.
\end{aligned}$$

$k \to \infty$ とすることにより，(4) 式が得られる． □

この Lemma を $f=|u|^{q-p}, q=\frac{Np}{N-p}$ として適用する.

$$f\in L^t \iff u\in L^s\ (s>q)$$

であるから, ある $s>q$ に対して $u\in L^s(\Omega)$ をいえば良い. 任意の $\alpha>p-1$ を固定して $\beta=p\alpha-(p-1)^2$ とおくと, [3] の Proposition 4.2 により,

$$u\in L^q \Longrightarrow v=u^+\in L^{\frac{\alpha p}{p-1}}$$

がいえる. ゆえに $\frac{\alpha p}{p-1}=q$ と取ると, $\frac{q^2}{p}=q/(\frac{q}{p})>q=\frac{Np}{N-p}$ なので $v\in L^{\frac{q^2}{p}}(\Omega)$. $v=(-u)^+$ とおくと, $u^-\in L^{\frac{q^2}{p}}(\Omega)$ も同様にして言える. □

5 Proof of Theorems

この節では, Theorem 1 と Theorem 2 の証明をする.

Proof of Theorem 1 前節で既に見たように, もし $q\le p^*$ ならば, $\mathcal{Q}$ に属する弱解は, $\mathcal{P}$ に属することがわかるので, $\mathcal{Q}$ に属する任意の弱解は, Pohozaev 型の不等式を満たす. $\mathcal{R}\ge 0$ であるので, $q>p^*$ (super critical) の時は $\|u\|_{L^q}=0$ となり, 定理の前半が証明される.

critical exponent の場合 ($q=p^*$) は, 更に複雑な議論が必要になる. まず最初に, $q=p^*$ であるから $\mathcal{R}=0$ となる. この時, 任意の $\eta>0$ に対して N_0 と $\varepsilon_0>0$ が存在して

$$\int_{\partial\Omega}(|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{p}{2}}(-x\cdot\vec{n})dS<\eta \quad \text{for all } n\ge N_0, \text{ and } \varepsilon\in(0,\varepsilon_0). \tag{6}$$

一方,

$$N|\Omega_0|=\int_{\Omega_0}\operatorname{div}x\,dx=\int_{\partial\Omega_0}x\cdot(-\vec{n}(x))dS=\int_{\partial\Omega}(-x\cdot\vec{n})dS$$

なので, $\rho>0$ と相対連結開集合 $\Gamma_0\subset\partial\Omega$ が存在して $(-x\cdot\vec{n})\ge\rho>0$ on $\overline{\Gamma_0}$. となる. それゆえ (6) より

$$\int_{\Gamma_0}|\nabla w_n^\varepsilon|^p dS<\frac{\eta}{\rho} \quad \text{for all } n\ge N_0, \text{ and } \varepsilon\in(0,\varepsilon_0). \tag{7}$$

$\Gamma_0=\{x_t\}_{0\le t\le 1}$ と parameter 表示すると十分小さい $\ell>0$ と $y_t\in\Omega$ が存在して $\{x_t\}=\partial B_{2\ell}(y_t)\cap\partial\Omega$.
$\Omega_\ell^t=B_{2\ell}(y_t)\setminus\overline{B_\ell(y_t)}$ とおき, Ω_ℓ^t 上の関数 v_t を

$$v_t(x)=\alpha(3\ell-r)^\delta-\alpha\ell^\delta,\ r=|x-y_t|,$$

と定義し, ℓ を更に十分小さくとり, δ を大きくとると, v_t は, 次の式を満たす.

$$\begin{cases} A_\varepsilon v_t+v_t^{q-1}\le 0 & \text{in } \Omega_\ell^t, \\ v_t|_{\partial B_{2\ell}(y_t)}=0, & v_t|_{\partial B_\ell(y_t)}=\alpha(2^\delta-1)\ell^\delta,\ \forall t\in[0,1]. \end{cases} \tag{8}$$

ここで, $(\mathrm{E})_n$ に対する比較定理 ([5, Lemma 3]) により, $\tilde{w}_{n+1}(x) \geq \tilde{w}_n(x)$ in Ω となり, そして $\tilde{w}_n(x) \uparrow u(x)$ for a.e. $x \in \Omega$ が結論される. さらに Harnack の原理 [9] により $w_n(x) > 0$ for all $x \in \Omega_n$. ここで w_n^ε が w_n に一様収束することに注意すると, 十分小さい α が存在して任意の $\varepsilon \in (0, \varepsilon_0)$ と $t \in [0,1]$ に対して, $v|_{\partial B_\ell(y_t)} = \alpha(2^\delta - 1)\ell^\delta \leq w_n^\varepsilon|_{\partial B_\ell(y_t)}$ が成立する. もう一度比較定理を用いて $v(x) \leq w_n^\varepsilon(x)$ for a.e. $x \in \Omega_\ell^t$.
それゆえ $\lambda < 0$ に対して

$$v(x_t + \lambda \vec{n}(x_t)) - v_t(x_t) \leq w_n^\varepsilon(x_t + \lambda \vec{n}(x_t)) - w_n^\varepsilon(x_t).$$

両辺を $\lambda < 0$ で割り $\lambda \to -0$ とすると, 任意の $n \geq n_0$ に対して

$$\frac{\partial w_n^\varepsilon}{\partial n}(x_0) \leq \frac{\partial v_t}{\partial n}(x_t) = -\alpha\delta\ell^{\delta-1} < 0$$

for all $t \in [0,1]$ and $\varepsilon \in (0, \varepsilon_0)$.

これは (7) に矛盾. □

Proof of Theorem 2 Pohozaev 型の等式 (2) より, $\|u\|_{L^q} \neq 0$ ならば $\frac{N}{q} + \frac{p-N}{p} = 0$, つまり $q = p^*$ となり, Theorem 2 が証明される. □

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Static Output Feedback Control of Flexible Structures (Asymptotic Stabilizing Control of 2 nd Order Evolution Equations)

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Abstract
In this report, we propose a theoretical essence of the stabilization of flexible structures by using the static output feedback control.This shows that not only strain outputs but also other outputs are valid in this control method for flexible structures of many boundary conditions including cantilevers, free ends, and other general boundary conditions. Sufficient conditions of structures and observation mechanisms are obtained for the closed loop systems to be asymptotically stable.

1 Introduction

Recently, great attention is paid to control of flexible structures[1][2]. The driving forces behind this interest are that flexible robot arms are required due to their high-speed performance and low energy consumption and that large flexible space structures (including large flexible space manipulators) will be launched with in a decade. In order to suppress vibrations of these structures, many control laws are proposed for stabilizing flexible structures. The dynamic model of flexible structures are described by infinite dimensional models. In most of these methods, feedback controllers are desigined to finite dimensional reduced models which approximate to original infinite dimensional models. It is not easy to implement these controllers in flexible structures, due to uncertainties in design models, to unmodelled dynamics (control and observation spillovers), and to high order of these controllers.

As a simple and robust controller, Luo proposed the direct strain feedback controller (i.e. static strain feedback controller), in order to control vibration of flexible arms.[9][10] For uniform flexible robot arms, he showed that if the strain (the bending moment) signal at the root end of the flexible robot arm is measured and directly fed back, then the vibration can be controlled(damping enhanced). This control strategy is called direct strain feedback.

This is the one of the static output controls. The stabilization of distributed parameters using static output feedback control is introduced by Gressang-Lamont[3] and Sakawa-Matsushita [4][5]. Nambu studied the stabilization using boundary outputs[7][8]. In these researches, parabolic systems are mainly considered and the key to the stabilization is the observability and the pole displacement of the lumped parameter systems which approximate the original distributed parameter systems. Later, the static output feedback stabilization of hyperbolic systems were reported by Sakawa[6]. In all these research, observation operators which generate outputs are bounded. But in the above static strain feedback control, output operators are unbounded. In this problem we show that unbounded output operators are transformed to bounded output operators. This transformation is valid for not only strain output but also other unbounded outputs. The theoretical essence of the static strain output feedback control is extended to other kinds of outputs. Thus static general output feedback control of flexible structures is discussed.

2 Dynamics and Observation Mechanisms of Flexible Structures

Consider a flexible structure and let $u_1(t,x) \in R^q$ $(q = 1,2,3)$ is an elastic displacement at time t, position $x \in \Omega$ where Ω represents a domain of structures and $\Omega \subset R^p$ $(p = 1,2,3)$. In general, the displacement $u_1(t,x)$ of a flexible structure at time t at position x satisfies a partial differential equation

$$\frac{\partial^2 u_1}{\partial t^2} + D_x \frac{\partial u_1}{\partial t} + A_x u_1(t,x) = f_2(t,x) \tag{1}$$

for suitable partial differential operators D and A. For fixed time t, the spatial function $u_1(t,\cdot)$ can be thought as an element of a suitable Hilbert space H. The operator A represents stiffness and D represents damping of flexible structures. $f_2(t,x)$ is a control input.

By introducing $u_2(t,x) = \partial u_1/\partial t$, the dynamics Eq.(1) of the flexible structure becomes

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix} = \begin{pmatrix} u_2(t,x) \\ -A_x u_1(t,x) - D_x u_2(t,x) \end{pmatrix} + \begin{pmatrix} 0 \\ f_2(t,x) \end{pmatrix} \tag{2}$$

Let $u(t,x) = (u_1(t,x),\ u_2(t,x))^T$, $f(t,x) = (0,\ f_2(t,x))^T$, and

$$\begin{aligned} \mathcal{A}_x u(x) &= (u_2(x),\ -A_x u_1(x) - D_x u_2(x))^T \\ & \quad \text{for } u(x) = (u_1(x), u_2(x))^T \end{aligned} \tag{3}$$

Then, Eq.(2) can be rewritten as

$$\frac{\partial u}{\partial t} = \mathcal{A}_x u(t,x) + f(t,x) \tag{4}$$

Suppose we can measure the following data

$$u_{(obs)}(t) = \begin{pmatrix} (c_1(\cdot), Cu_2(t,\cdot))_H \\ \vdots \\ (c_N(\cdot), Cu_2(t,\cdot))_H \end{pmatrix} \tag{5}$$

where $c_k(x)$ is a spatial weighting function. $(k = 1,2,\cdots,N)$ and $(\cdot,\cdot)_H$ is an inner product in a Hilbert space H.

CONDITIONS We suoopse that C^{-1} is a bounded operator on H, CAC^{-1} is a self-adjoint positive definite operator on H, and CDC^{-1} is a positive definite operator on H.

Examles of A and C which satisfies this condition are as follows.

Cantilevers When the flexible structure is a cantilever, the spatial domain Ω of the structure is one-dimensional and Ω becomes an open interval $(0,L)$ where L is a length of the cantilever. Let $H = L^2((0,L);R)$, $A_x = EId^4/dx^4$, $D = \delta A$, and $C_x = d^2/dx^2$, where δ and EI are positive constants and

$$\mathcal{D}(A) = \left\{u \in H^4(0,L); 0 = u(0) = u'(0) = u''(L) = u'''(L)\right\} \tag{6}$$

$$\mathcal{D}(C) = \left\{u \in H^2(0,L); 0 = u(0) = u'(0)\right\} \tag{7}$$

which reflect the boundary conditions of the both ends ($x = 0, L$) of the structure. $H^m(0, L)$ is the Sobolev space of order m on an open interval $(0, L)$ and EI is a positive constant.

Let $c_k(x)$ be a positive continuous function which approximates $\delta(x - a_k)$ ($k = 1, 2, \cdots, N$). The time derivative of the strains (the bending moments) of the structure at $x = a_1, a_2, \cdots, a_N$ are measured. A and CAC^{-1} becomes positive definite self-adjoint operators on H and the above conditions are satisfied. (See Appendix 1.)

Free Beams When the flexible structure is a free beam (a beam whose both ends are free), the spatial domain Ω of the structure is one-dimensional and Ω becomes an open interval $(0, L)$ where L is a length of the beam. Let $H = L^2((0, L); R)$, $A_x = EId^4/dx^4$, $D = \delta A$, and $C_x = d^2/dx^2$, where δ and EI are positive constants and

$$\mathcal{D}(A) = \left\{u \in H^4(0, L); 0 = u''(0) = u'''(0) = u''(L) = u'''(L)\right\} \tag{8}$$

$$\mathcal{D}(C) = \left\{u \in H^2(0, L); 0 = u(0) = u'(0)\right\} \tag{9}$$

which reflect the boundary conditions of the both ends ($x = 0, L$) of the structure. $H^m(0, L)$ is the m-th order Sobolev space on an open interval $(0, L)$ and EI is a positive constant.

Let $c_k(x)$ be a positive continuous function which approximates $\delta(x - a_k)$ ($k = 1, 2, \cdots, N$). The time derivative of the strains (the bending moments) of the structure at $x = a_1, a_2, \cdots, a_N$ are measured. A and CAC^{-1} becomes positive definite self-adjoint operators on H and the above conditions are satisfied. (See Appendix 2.)

General Structures The above conditions are satisfied when A is a self-adjoint positive definite operator on the Hilbert space H, $C = A^\alpha$ ($0 \le \alpha \le 1$ is a constant) and $D = \delta A$ where δ is a positive constant, since $CAC^{-1} = A$. If $C = A^\alpha$ ($0 \le \alpha \le 1$ is a constant), the observed data correspond to time derivatives of the elastic displacements of the structure when $\alpha = 0$, to the time derivative of the strains when $\alpha = 1/2$, and to the shear forces when $\alpha = 3/4$.

3 Static Output Feedback Control of Flexible Structures

Consider a static output feedback control

$$f(t, x) = G(x) u_{obs}(t) \tag{10}$$

where the feedback gain $G(x)$ is

$$G(x) = \begin{pmatrix} 0, \cdots, 0 \\ -g_1(x), \cdots, -g_N(x) \end{pmatrix} \tag{11}$$

For this matrix $G(x)$, we set

$$g_k(x) = \tilde{g}_k \left(C^{-1} c_k\right)(x) \tag{12}$$

where $\tilde{g}_k$ is a positive constant. ($k = 1, 2, \cdots, N$)

The closed loop system becomes

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix} = \tag{13}$$

$$\begin{pmatrix} u_2(t,x) \\ -A_x u_1(t,x) - D_x u_2(t,x) \end{pmatrix} + \begin{pmatrix} 0 \\ \sum_{k=1}^{N} \tilde{g}_k \left(C^{-1}c_k\right)(x)\left(c_k(\cdot), Cu_2(t,\cdot)\right)_H \end{pmatrix}$$

Introduce a new variable $y_i(t) = Cu_i(t)(i = 1,2)$ and $y(t) = (y_1(t), y_2(t))^T$, the closed loop system Eq.(13) is rewritten as follows.

$$\frac{\partial}{\partial t}\begin{pmatrix} y_1(t,\cdot) \\ y_2(t,\cdot) \end{pmatrix} = \tag{14}$$

$$\begin{pmatrix} y_2(t,\cdot) \\ -CAC^{-1}y_1(t,\cdot) - CDC^{-1}y_2(t,\cdot) \end{pmatrix} + \begin{pmatrix} 0 \\ \sum_{k=1}^{N} \tilde{g}_k c_k(\cdot)\left(c_k(\cdot), y_2(t,\cdot)\right)_H \end{pmatrix}$$

We show that this system Eq.(14) is asymptotically stable in the Hilbert space $\mathcal{D}\left((CAC^{-1})^{1/2}\right)\times H$. (This equation looks like the equation in [6]) Let $\langle\cdot,\cdot\rangle$ be an inner product in the Hilbert space $\mathcal{D}\left((CAC^{-1})^{1/2}\right)\times H$, id est

$$\langle u, v\rangle = \left(\left(CAC^{-1}\right)^{1/2}u_1, \left(CAC^{-1}\right)^{1/2}v_1\right)_H + (u_2, v_2)_H \tag{15}$$

$$\text{for } u = (u_1, u_2)^T, v = (v_1, v_2)^T \in \mathcal{D}\left(\left(CAC^{-1}\right)^{1/2}\right) \times H$$

For the solution $y(t) = (y_1(t), y_2(t))^T$ of the closed system Eq.(14), we have

$$\frac{d}{dt}\langle y(t), y(t)\rangle \tag{16}$$

$$= 2\langle\frac{dy}{dt}, y(t)\rangle \tag{17}$$

$$= 2\left(\left(CAC^{-1}\right)^{1/2}y_2(t), \left(CAC^{-1}\right)^{1/2}y_1(t)\right)_H \tag{18}$$

$$-2\left(CAC^{-1}y_1(t), y_2(t)\right)_H - 2\left(CDC^{-1}y_2(t), y_2(t)\right)_H \tag{19}$$

$$-2\left(\sum_{k=1}^{N}\tilde{g}_k c_k(\cdot)\left(c_k, y_2(t)\right)_H, \; y_2(t)\right)_H$$

$$= -2\left(CDC^{-1}y_2(t), y_2(t)\right)_H - 2\sum_{k=1}^{N}\tilde{g}_k\left(c_k, y_2(t)\right)_H^2 \tag{20}$$

$$\leq 0 \tag{21}$$

Therefore, for $\frac{d}{dt}\langle y(t), y(t)\rangle = 0$, it is necessary that $y_2(t) = 0$, since CDC^{-1} is a positive definite operator. This leads to the conclusion that $dy_2/dt = 0$ and $y_1(t) = 0$ from Eq.(14). Thus, $\langle y(t), y(t)\rangle$tends to 0 as $t \to \infty$, Since $u_i(t) = C^{-1}y_i(t)$ $(i = 1,2)$ and C^{-1} is a bounded operator, $u(t) = (u_1(t), u_2(t))^T$ tends to 0 as $t \to \infty$ and the closed loop system Eq.(13) becomes asymptotically stable.

4 Conclusions

In this paper we found the theoretical essence of static output feedback control of flexible structures. This essence shows that not only strain but also other outputs are effective in the stabilization of flexible structures by the static output feedback control, and this control method is applicable for many types of flexible structures including cantilevers and free ends beams and other struciures who has general general boundary conditions. The sufficient conditions (CONDITIONS) of outputs are clarified for the the static output feedback control to be effective.

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Appendix 1

Let A and C be operators on the Hilbert space $H = L^2(0, L)$ defined as follows.

$$A_x = EI\frac{d^4}{dx^4}, \quad \mathcal{D}(A) = \left\{u \in H^4(0,L); 0 = u(0) = u'(0) = u''(L) = u'''(L)\right\} \tag{22}$$

$$C_x = \frac{d^2}{dx^2}, \quad \mathcal{D}(C) = \left\{u \in H^2(0,L); 0 = u(0) = u'(0)\right\} \tag{23}$$

For any $u \in \mathcal{D}(CAC^{-1})$, since

$$\mathcal{D}(C) \ni AC^{-1}u = \frac{d^2u}{dx^2} \tag{24}$$

we have

$$0 = u''(0) = u'''(0) \text{ and } u \in H^4(0, L) \tag{25}$$

and since $u \in \mathcal{D}(AC^{-1})$, we have $C^{-1}u \in \mathcal{D}(A)$, we have

$$0 = u(L) = u'(L) \tag{26}$$

For $u, v \in \mathcal{D}(CAC^{-1})$, by integrating by parts, we have

$$\left(CAC^{-1}u, v\right) = EI\left(\frac{d^4u}{dx^4}, v\right) = EI\left(\frac{d^2u}{dx^2}, \frac{d^2v}{dx^2}\right) = \left(u, CAC^{-1}v\right) \tag{27}$$

Appendix 2

Let A and C be operators on the Hilbert space $H = L^2(0, L)$ defined as follows.

$$A_x = EI\frac{d^4}{dx^4}, \quad \mathcal{D}(A) = \left\{u \in H^4(0, L); 0 = u''(0) = u'''(0) = u''(L) = u'''(L)\right\} \tag{28}$$

$$C_x = \frac{d^2}{dx^2}, \quad \mathcal{D}(C) = \left\{u \in H^2(0, L); 0 = u(0) = u'(0)\right\} \tag{29}$$

For any $u \in \mathcal{D}(CAC^{-1})$, since

$$\mathcal{D}(C) \ni AC^{-1}u = \frac{d^2u}{dx^2} \tag{30}$$

we have

$$0 = u''(0) = u'''(0) \text{ and } u \in H^4(0, L) \tag{31}$$

and since $u \in \mathcal{D}(AC^{-1})$, we have $C^{-1}u \in \mathcal{D}(A)$, we have

$$0 = u(L) = u'(L) \tag{32}$$

For $u, v \in \mathcal{D}(CAC^{-1})$, by integrating by parts, we have

$$\left(CAC^{-1}u, v\right) = EI\left(\frac{d^4u}{dx^4}, v\right) = EI\left(\frac{d^2u}{dx^2}, \frac{d^2v}{dx^2}\right) = \left(u, CAC^{-1}v\right) \tag{33}$$

非線形波動方程式の散乱問題について

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1　導入と結果

次の非線形波動方程式の散乱問題を考える。

$$(1.1) \qquad \Box u = F(u), \quad (t,x) \in \mathbf{R} \times \mathbf{R}^n.$$

ここで、$\Box = \partial_t^2 - \Delta = \partial_t^2 - \sum_{j=1}^n \partial_j^2$, $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ $(j = 1, \cdots, n)$, $F(u) = \lambda|u|^{\rho-1}u$ または$\lambda|u|^\rho$ $(\lambda \in \mathbf{R} \setminus \{0\}, \rho > 1)$ である。以下 u は簡単のため実数値未知関数とする。定理を述べるため記号の説明をする。$< x > := \sqrt{1+|x|^2}$とおく。$H^s(\mathbf{R}^n)$, $W^{s,q}(\mathbf{R}^n)$ で、それぞれ L^2,L^q型のソボレフ空間を表す。$\dot{H}^1(\mathbf{R}^n)$ で、$C_0^\infty(\mathbf{R}^n)$ のセミノルム $\|\nabla u\|_{L^2}$での完備化を表す。$X := \dot{H}^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ とおく。

$\mu = (n^2 + 3n + 2)/2$ とおく。偏微分作用素の集合$\Gamma = \{ \Gamma_j \mid j = 0, \cdots, \mu \} = \{\partial_0, \partial_1, \cdots, \partial_n, L_1, \cdots, L_n, \Omega_{12}, \cdots, \Omega_{n-1n}, L_0\}$ を導入する。ここで$\partial_0 = \partial_t$, $L_j = t\partial_j + x_j\partial_t$, $\Omega_{k\ell} = x_k\partial_\ell - x_\ell\partial_k$ $(1 \le k < \ell \le n)$, $L_0 = t\partial_t + x_1\partial_1 + \cdots + x_n\partial_n$ である。多重指数 $\alpha = (\alpha_0, \cdots, \alpha_\mu)$ に対し、$\Gamma^\alpha := \Gamma_0^{\alpha_0} \cdots \Gamma_\mu^{\alpha_\mu}$ とする。非負整数 Nと p $(1 \le p < \infty)$ に対し、ノルム

$$\|u(t,\cdot)\|_{\Gamma,N,p} := \sum_{|\alpha| \le N} \left(\int_{\mathbf{R}^n} |\Gamma^\alpha u(t,x)|^p dx \right)^{1/p}$$

を定義する。ベクトル $Du := (\partial_t u, \partial_1 u, \cdots, \partial_n u)$ に対し、ノルム

$$\|Du(t,\cdot)\|_{\Gamma,N,p} := \sum_{k=0}^n \|\partial_k u(t,\cdot)\|_{\Gamma,N,p}$$

を定義する。エネルギーノルム

$$\|u(t,\cdot)\|_e := \frac{1}{\sqrt{2}} \sqrt{\|\partial_t u(t,\cdot)\|_{L^2}^2 + \|\nabla u(t,\cdot)\|_{L^2}^2}$$

を定義する。$\mathbf{R}^n_\xi$上で定義された緩増加関数 $H = H(|\xi|)$ に対し、緩増加超関数の空間 $\mathcal{S}'(\mathbf{R}^n)$ 上の作用素 $H(\sqrt{-\Delta})$ を

$$H(\sqrt{-\Delta})v = \mathcal{F}^{-1}[H(|\xi|)\mathcal{F}v] \quad (v \in \mathcal{S}'(\mathbf{R}^n))$$

で定義する。ここで$\mathcal{F}$は$\mathcal{S}'(\mathbf{R}^n)$ 上のフーリエ変換であり、$\mathcal{F}^{-1}$はその逆変換を表す。簡単のため、$\omega := \sqrt{-\Delta}$とおく。以下における議論のため、$L^2(\mathbf{R}^4) \times L^2(\mathbf{R}^4)$ の部分集合

$$\begin{aligned}\Sigma = \{ \ (f,g) \in L^2(\mathbf{R}^4) \times L^2(\mathbf{R}^4) \mid \ & \\ \|(f,g)\|_\Sigma := & \sum_{|\alpha|=0}^{2} \| <\cdot>^{|\alpha|} \partial_x^\alpha f\|_{L^2} + \sum_{|\alpha|=3} \| <\cdot>^2 \partial_x^\alpha f\|_{L^2} \\ & + \sum_{|\alpha|=0}^{1} \| <\cdot>^{|\alpha|+1} \partial_x^\alpha g\|_{L^2} + \sum_{|\alpha|=2} \| <\cdot>^2 \partial_x^\alpha g\|_{L^2} < \infty \ \}\end{aligned}$$

を導入する。ここで$\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_4^{\alpha_4}$ である。$\delta > 0$ に対し$\Sigma_\delta := \{ \ (f,g) \mid \|(f,g)\|_\Sigma < \delta \ \}$ とおく。

Theorem 1 (1.1) において $n = 4, \rho > 2$ とする。νを $2 < \nu < 3$ なる任意の正数とする。λ, ν, ρのみに依存する正数δが存在し、次の $(I), (II)$ が成立する。

(I) 任意の $(f_-, g_-) \in \Sigma_\delta$に対し、(1.1) の解 $u = u(t,x)$ で、

$$u \in \bigcap_{j=0}^{3} BC^j((-\infty,0]\,; H^{3-j}(\mathbf{R}^4)), \tag{1.2}$$

$$\Gamma^\alpha u,\ \Gamma^\alpha \partial_k u \in BC((-\infty,0]\,; L^2(\mathbf{R}^4)) \ \text{ for any } \alpha \text{ with } |\alpha| \le 2, k = 0, \cdots, 4, \tag{1.3}$$

$$\|u(t,\cdot) - u_-(t,\cdot)\|_e \to 0 \quad (t \to -\infty) \tag{1.4}$$

をみたす解が唯一つ存在する。ここで $u_-(t) = (\cos \omega t) f_- + (\omega^{-1} \sin \omega t) g_-$ である。さらに、この解は

$$\begin{aligned}&\sup_{t<0} \|u(t,\cdot)\|_{\Gamma,2,2} + \sup_{t<0} \|Du(t,\cdot)\|_{\Gamma,2,2} \\ &\le C_1 \|(f_-, g_-)\|_\Sigma \ \text{ for some constant } C_1 > 0,\end{aligned} \tag{1.5}$$

$$\|u(t,\cdot) - u_-(t,\cdot)\|_{\Gamma,1,2} = \begin{cases} O(|t|^{-3(\rho-2)/2}) & \text{if } 2 < \rho \le 3, \\ O(|t|^{-3(\rho-2)/\nu}) & \text{if } \rho > 3, \end{cases} \tag{1.6}$$

$$(1.7) \qquad \|u(t,\cdot)-u_-(t,\cdot)\|_{\Gamma,2,2} = O(|t|^{-3(\rho-2)/\nu}),$$

$$(1.8) \qquad \|D\{u(t,\cdot)-u_-(t,\cdot)\}\|_{\Gamma,1,2} = O(|t|^{-3(\rho-1)/2+1}),$$

$$(1.9) \qquad \|D\{u(t,\cdot)-u_-(t,\cdot)\}\|_{\Gamma,2,2} = O(|t|^{-3(\rho-1)/\nu+1})$$

$(t \to -\infty)$ をみたす。

(II) 任意の $(f_+, g_+) \in \Sigma_\delta$に対し、(1.1) の解 $u = u(t,x)$ で

$$(1.10) \qquad u \in \bigcap_{j=0}^{3} BC^j([0,\infty); H^{3-j}(\mathbf{R}^4)),$$

$$(1.11) \qquad \Gamma^\alpha u,\ \Gamma^\alpha \partial_k u \in BC([0,\infty); L^2(\mathbf{R}^4)) \text{ for any } \alpha \text{ with } |\alpha| \le 2,\ k = 0,\cdots,4,$$

$$(1.12) \qquad \|u(t,\cdot)-u_+(t,\cdot)\|_e \to 0 \quad (t \to +\infty)$$

をみたす解が唯一つ存在する。ここで $u_+(t) = (\cos\omega t)f_+ + (\omega^{-1}\sin\omega t)g_+$ である。さらに、この解は

$$(1.13) \qquad \sup_{t>0}\|u(t,\cdot)\|_{\Gamma,2,2} + \sup_{t>0}\|Du(t,\cdot)\|_{\Gamma,2,2} \le C_1\|(f_+, g_+)\|_\Sigma,$$

$$(1.14) \qquad \|u(t,\cdot)-u_+(t,\cdot)\|_{\Gamma,1,2} = \begin{cases} O(t^{-3(\rho-2)/2}) & \text{if } 2 < \rho \le 3, \\ O(t^{-3(\rho-2)/\nu}) & \text{if } \rho > 3, \end{cases}$$

$$(1.15) \qquad \|u(t,\cdot)-u_+(t,\cdot)\|_{\Gamma,2,2} = O(t^{-3(\rho-2)/\nu}),$$

$$(1.16) \qquad \|D\{u(t,\cdot)-u_+(t,\cdot)\}\|_{\Gamma,1,2} = O(t^{-3(\rho-1)/2+1}),$$

$$(1.17) \qquad \|D\{u(t,\cdot)-u_+(t,\cdot)\}\|_{\Gamma,2,2} = O(t^{-3(\rho-1)/\nu+1})$$

$(t \to +\infty)$ をみたす。

Theorem 2 (1.1) において $n = 4$, $\rho > 2$ とする。νを $2 < \nu < 3$ なる任意の正数とする。λ, ν, ρのみに依存する正数εが存在し、次が成立する。

任意の $(f,g) \in \Sigma_\varepsilon$に対し、(1.1) の解 $u = u(t,x)$ で

$$(1.18) \qquad (u(0), \partial_t u(0)) = (f, g),$$

$$(1.19) \qquad u \in \bigcap_{j=0}^{3} BC^j(\mathbf{R}; H^{3-j}(\mathbf{R}^4)),$$

$$(1.20) \qquad \Gamma^\alpha u,\ \Gamma^\alpha \partial_k u \in BC(\mathbf{R}; L^2(\mathbf{R}^4)) \text{ for any } \alpha \text{ with } |\alpha| \le 2,\ k = 0,\cdots,4,$$

をみたす解が唯一つある。この解 u は

$$\sup_{t\in\mathbf{R}}\|u(t,\cdot)\|_{\Gamma,2,2}+\sup_{t\in\mathbf{R}}\|Du(t,\cdot)\|_{\Gamma,2,2} \le C_2\|(f,g)\|_\Sigma \quad \text{for some constant } C_2>0 \tag{1.21}$$

をみたす。さらに $\Box u=0$ の解 u_+, u_- で

$$(u_\pm(0),\partial_t u_\pm(0))\in X, \tag{1.22}$$

$$\|u(t,\cdot)-u_\pm(t,\cdot)\|_e\to 0 \quad (t\to\pm\infty) \tag{1.23}$$

をみたす解が唯一組存在する。この $u_\pm$ は

$$(u_\pm(0),\partial_t u_\pm(0))\in\Sigma, \tag{1.24}$$

$$\|(u_\pm(0),\partial_t u_\pm(0))\|_\Sigma\le C_3\|(f,g)\|_\Sigma \quad \text{for some constant } C_3>0, \tag{1.25}$$

$$\|u(t,\cdot)-u_\pm(t,\cdot)\|_{\Gamma,1,2}=\begin{cases} O(|t|^{-3(\rho-2)/2}) & \text{if } 2<\rho\le 3,\\ O(|t|^{-3(\rho-2)/\nu}) & \text{if } \rho>3,\end{cases} \tag{1.26}$$

$$\|u(t,\cdot)-u_\pm(t,\cdot)\|_{\Gamma,2,2}=O(|t|^{-3(\rho-2)/\nu}), \tag{1.27}$$

$$\|D\{u(t,\cdot)-u_\pm(t,\cdot)\}\|_{\Gamma,1,2}=O(|t|^{-3(\rho-1)/2+1}), \tag{1.28}$$

$$\|D\{u(t,\cdot)-u_\pm(t,\cdot)\}\|_{\Gamma,2,2}=O(|t|^{-3(\rho-1)/\nu+1}) \tag{1.29}$$

$(t\to\pm\infty)$ をみたす。

Theorem 1 の (I), (II) における解 u が

$$(u(0),\partial_t u(0))\in\Sigma,\quad \|(u(0),\partial_t u(0))\|_\Sigma<C_4\delta \quad \text{for some constant } C_4>0$$

をみたすことを、少し工夫すると (1.5), (1.13) から導くことができる。よって

Theorem 3 波動作用素 $W_\pm$

$$W_\pm:(u_\pm(0),\partial_t u_\pm(0))\mapsto(u(0),\partial_t u(0))$$

が、Σ_δから$\Sigma_{C_4\delta}$ の中への写像として定義できる。

$C_4\delta < \varepsilon$となるようδを取り直す。こう取り直しても、Theorem 1 は成立する。よって、Theorem 2 と合わせて

Theorem 4 散乱作用素 S

$$S : (u_-(0), \partial_t u_-(0)) \mapsto (u_+(0), \partial_t u_+(0))$$

が、Σ_δから$\Sigma_{C_3C_4\delta}$ の中への写像として定義できる。

2 ρの lower bound の最適性について

次の積分方程式を考える。

$$u(t) = u_0(t) + \int_0^t \frac{\sin\omega(t-\tau)}{\omega} F(u(\tau))d\tau, \quad t > 0,\ x \in \mathbf{R}^n. \tag{2.30}$$

ここで、$F(u)$ は§1 における $F(u)$ と同じ形をもち、また $u_0(t) = (\cos\omega t)f + (\omega^{-1}\sin\omega t)g$ である。

Proposition 2.1 $4 \leq n \leq 6$ ならば $1 < \rho \leq (n+3)/(n-1)$ とし、$n \geq 7$ ならば $1 < \rho \leq 1 + 2(n+1)/(n^2 - 3n - 2)$ とする。任意の正数 kを 1 つとり固定する。この時、$\mathrm{supp} f \cup \mathrm{supp} g \subseteq \{\ x \in \mathbf{R}^n \mid |x| \leq k\ \}$ なる任意の $(f, g) \in H^2(\mathbf{R}^n) \times H^1(\mathbf{R}^n)$ に対し、T ($T > 0$) が $\|f\|_{H^2} + \|g\|_{H^1}$, k, n, λ, ρに依存してとれて、(2.1) の解 $u = u(t, x)$ で

$$u \in C([0,T]; L^q(\mathbf{R}^n)) \qquad \left(q = \frac{2(n+1)}{n-1}\right), \tag{2.31}$$

$$\mathrm{supp}\, u(t) \subseteq \{\ x \in \mathbf{R}^n \mid |x| \leq t + k\ \} \tag{2.32}$$

をみたす解が唯一つ存在する。さらに、この解は

$$u \in \bigcap_{j=0}^{1} C^j([0,T]; H^{1-j}(\mathbf{R}^n)) \bigcap L^\infty(0, T; W^{1,q}(\mathbf{R}^n)) \tag{2.33}$$

をみたす。

証明は、Sideris [4] 中の定理 1 の証明に従えばよい。Proposition 2.1 は (2.1) の $t>0$ での時間局所存在定理を述べたものである。全く同様にして、(2.1) の $t<0$ での局所存在も証明できる。

さて$\rho_0(n)=(n+1+\sqrt{n^2+10n-7})/2(n-1)$ とおく。なお $1<\rho_0(n)<(n+3)/(n-1)$ ($\forall n \geq 2$), $\rho_0(4)=2$ なる関係がある。[4] の定理 2 によれば、$F(u)=\lambda|u|^\rho$, $\lambda>0$, $1<\rho<\rho_0(4)$ の時、もし (f,g) が、さらに

$$\int_{\mathbf{R}^4}|x|^{-1/2}f(x)dx>0,\quad \int_{\mathbf{R}^4}|x|^{1/2}g(x)dx>0$$

をみたすと、ある T_0 $(0<T_0<\infty)$ に対し

$$\lim_{t\to T_0-0}\|u(t,\cdot)\|_{L^q}=\infty$$

が、Proposition 2.1 における局所解 u について成立してしまう。$\|u(t)\|_{L^2}\leq\|f\|_{L^2}+t\sup_{0<s<t}\|\partial_t u(s)\|_{L^2}$ と、qの取り方より $H^1(\mathbf{R}^4)\hookrightarrow L^q(\mathbf{R}^4)$ に注意する。従って、$1<\rho<\rho_0(4)=2$ の時、(2.1) の時間大域解で

$$u\in C(\mathbf{R};\dot{H}^1(\mathbf{R}^4)),\ \partial_t u\in C(\mathbf{R};L^2(\mathbf{R}^4))$$

をみたし、かつ

$$\|u(t,\cdot)-u_\pm(t,\cdot)\|_e\to 0\quad (t\to\pm\infty)$$

を free wave equation の適当な解 u_+, u_-に対しみたすような解が存在しないことがあることがわかる。$n=4$, $1<\rho<2$ で、f, gが適当になめらかで $\mathrm{supp}f\cap\mathrm{supp}g$がコンパクトの時は微分方程式 (1.1) を $(u(0),\partial_t u(0))=(f,g)$ の下で解く初期値問題と積分方程式 (2.1) の適当な意味での同値性を示すことができる。従って Theorem 2 におけるρの下限 $2(=\rho_0(4))$ は最適である。

3 従来の結果との比較

データが小さいときの (1.1) に対する散乱問題は Strauss [5] により初めて論じられた。[5] では一般次元 ($n \geq 2$) で問題を扱っている。[5] の定理 14 によれば、$n = 4$ の時 $(3 + 2\sqrt{3})/3(= 2.154\cdots) < \rho \leq 7/3$ ならば、散乱作用素が $(\dot{H}^1(\mathbf{R}^4) \cap \dot{W}^{1,1+1/\rho}(\mathbf{R}^4)) \times (L^2(\mathbf{R}^4) \cap L^{1+1/\rho}(\mathbf{R}^4))$ の 0 の近傍から $(\dot{H}^1(\mathbf{R}^4) \cap L^{\rho+1}(\mathbf{R}^4)) \times (L^2(\mathbf{R}^4) \cap \dot{W}^{-1,\rho+1}(\mathbf{R}^4))$ の中への写像として定義できる。ここで$\dot{W}^{1,p}(\mathbf{R}^4)$ は $C_0^\infty(\mathbf{R}^4)$ の $\|\nabla u\|_{L^p}$での完備化であり、$\dot{W}^{-1,p'}(\mathbf{R}^4)$ $(1/p + 1/p' = 1)$ は、その相対空間である。この下限は、後に Mochizuki & Motai [3] により 2.086$\cdots$に改良されている。ただし、[3] でのデータのクラスは、[5] のそれより狭いようである。Theorems 1 − 4 は、[3], [5] での下限を改善し、かつ漸近完全性を示したものである。$n = 2$ または 3 の時も$\rho > \rho_0(n)$ の仮定の下で小さいデータに対し散乱作用素が定義できることが示されている（例えば [2], [6]）。しかし、そこでの問題設定はここでのそれより多少制限されたものであることを注意しておく。最後に、空間Xにおける散乱理論については Ginibre & Velo [1] の命題 3.3 を見られたい。

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熱方程式の自己相似解について

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In this talk, we will study structure of positive solutions to the following initial value problem

(IVP)
$$\begin{cases} u_{rr}+\dfrac{n-1}{r}u_r+\dfrac{r}{2}u_r+\dfrac{1}{p-1}u+|u|^{p-1}u=0, & r>0, \\ u(0)=\alpha \quad (0<\alpha<\infty), \end{cases}$$

where $n\geq 3$ and $p>1$. In [HaW], Haraux and Weissler have shown non-uniqueness of solution to semilinear heat equation,

(1)
$$\psi_t=\Delta\psi+|\psi|^{p-1}\psi,\ (t,x)\in(0,\infty)\times\mathbf{R}^n.$$

In the proof, they have used the properties of solutions to (IVP). When we discuss the following function, which is called a self-similar solution,

$$\psi(t,x):=t^{-\frac{1}{p-1}}u\left(\frac{x}{\sqrt{t}}\right),$$

it can be seen that ψ satisfies (1) if and only if $u(y):=u(x/\sqrt{t})$ satisfies

(2)
$$\Delta u+\frac{1}{2}y\cdot\nabla u+\frac{1}{p-1}u+|u|^{p-1}u=0,\ y\in\mathbf{R}^n.$$

Moreover, if we set $r=|y|$, then $u=u(r)$ satisfies the equation of (IVP). Haraux and Weissler have obtained the following result on (IVP).

THEOREM 1. ([HaW]) If $1+2/n<p<(n+2)/(n-2)$, then there exists a positive number α_* such that $u(r;\alpha_*)$, which is a solution of (IVP) starting from $u(0)=\alpha_*$, satisfies the following conditions:

(i) $u(r;\alpha_*)>0$ for $r\geq 0$.

(ii) $\lim_{r\to\infty}r^{2/(p-1)}u(r;\alpha_*)=0$.

(iii) For all $m>0$, $\lim_{r\to\infty}r^m u(r;\alpha_*)=0$ and $\lim_{r\to\infty}r^m u_r(r;\alpha_*)=0$.

In view of Theorem 1, we can see that there exists a positive solution which decays rapidly as $r \to \infty$ in case $1+2/n < p < (n+2)/(n-2)$. Moreover, using above result, they have shown

<u>THEOREM 2.</u> ([HaW]) If $1+2/n < p < (n+2)/(n-2)$, then there exists a solution to (1) satisfying the following properties:

(i) $\psi(t,x) > 0$ for all $(t,x) \in (0,\infty) \times \mathrm{R}^n$.

(ii) If $1 \le q < n(p-1)/2$, then $\lim_{t\to 0} \|\psi(t,\cdot)\|_{L^q} = 0$.

In order to prove Theorem 2, put

$$\psi(t,x;\alpha.) := t^{-\frac{1}{p-1}} u(r;\alpha.),$$

where $u(r;\alpha.)$ is the solution to (IVP) obtained in Theorem 1. Noting $r = |x|/\sqrt{t}$, we can see $\psi(t,x;\alpha.) > 0$ for all $(t,x) \in (0,\infty) \times \mathrm{R}^n$ from (i) of Theorem 1. Moreover, in view of (ii) of Theorem 1 since $\|u(\cdot\ ;\alpha.)\|_{L^q} < \infty$ for all $q \ge 1$,

$$\|\psi(t,\cdot\ ;\alpha.)\|_{L^q} = t^{-\frac{1}{p-1}+\frac{n}{2q}} \|u(\cdot\ ;\alpha.)\|_{L^q} \to 0 \text{ as } t \to 0.$$

Therefore, it is sufficient to take $\psi(t,x;\alpha.)$ as a solution of (1).

In view of Theorem 2, initial value problem of heat equation

$$\begin{cases} \psi_t = \Delta\psi + |\psi|^{p-1}\psi, & (t,x) \in (0,\infty) \times \mathrm{R}^n, \\ \psi(t,\cdot) \to 0 \text{ as } t \to 0 & \text{in certain } \mathrm{L}^q(\mathrm{R}^n), \end{cases}$$

has at least three solutions, i.e., trivial solution, $\psi(t,x;\alpha.)$ and $-\psi(t,x;\alpha.)$, which is also a solution in view of the form of (1). Thus non-uniqueness of solutions to (1) has been shown.

As already mentioned, Harawx and Weissler have shown the existence of positive solution to (IVP) which decays rapidly as $r \to \infty$. Our aim of this talk is to get the uniqueness of this solution, i.e., to prove the uniqueness of $\alpha. \in (0,\infty)$ which satisfies conditions of Theorem 1.

Moreover, we want to investigate the behaviour of $u(r)$ for each $\alpha \in (0,\infty)$ completely. In order to make this problem clear, we will classify the solutions to (IVP). For each $\alpha \in (0,\infty)$, (IVP) has a unique solution $u(r) \in C^2([0,\infty))$ with $u_r(r) = 0$, which is denoted by $u(r;\alpha)$. Furthermore, starting from initial value α, $u(\cdot\,;\alpha)$ decreases as long as positive. So at first, we want to know whether $u(\cdot\,;\alpha)$ has a zero or not in $[0,\infty)$. Furthermore, if $u(\cdot\,;\alpha)$ does not have a zero, i.e., $u(\cdot\,;\alpha) > 0$ in $[0,\infty)$, then we also want to get the asymptotic behaviour as $r \to \infty$. In this direction, Peletier Terman and Weissler [PTW] have obtained the following estimates.

<u>THEOREM 3.</u> ([PTW]) Set $S := \lim_{r\to\infty} r^{2/(p-1)} u(r;\alpha)$. Then for all $\alpha \neq 0$, S exists and is finite. Moreover,

(i) If $S = 0$, then there exists some constant $R \neq 0$ such that

(3) $$u(r;\alpha) = R r^{\frac{2}{p-1}-n} \exp\left(-\frac{r^2}{4}\right)\left\{1 + O(r^{-2})\right\} \quad \text{as } r \to \infty.$$

(ii) If $S \neq 0$, then

(4) $$u(r;\alpha) = S r^{-\frac{2}{p-1}} + o\left(r^{-\frac{2}{p-1}}\right) \quad \text{as } r \to \infty.$$

Theorem 3 says that the asymptotic behaviour of solutions to (IVP) is either (3) or (4). Now we will classify solutions of (IVP) as follows:

(i) $u(r;\alpha)$ is a crossing solution. $\underset{\text{def}}{\Leftrightarrow}$ $u(\cdot\,;\alpha)$ has a zero in $(0,\infty)$, i.e., there exists some $z \in (0,\infty)$ such that $u(z;\alpha) = 0$.

(ii) $u(r;\alpha)$ is a rapidly decaying solution. $\underset{\text{def}}{\Leftrightarrow}$ $u(\cdot\,;\alpha) > 0$ in $[0,\infty)$. Moreover, $u(r;\alpha)$ satisfies (3) with $R > 0$.

(iii) $u(r;\alpha)$ is a slowly decaying solution. $\underset{\text{def}}{\Leftrightarrow}$ $u(\cdot\,;\alpha) > 0$ in $[0,\infty)$. Moreover, $u(r;\alpha)$ satisfies (4) with $S > 0$.

In view of above classification, we want to decide completely whether $u(r;\alpha)$ is a crossing solution, a rapidly decaying solution or a slowly decaying solution for each initial value α. To our problem, we will summarize results in [HaW] as follows.

THEOREM 4. ([HaW]) On the structure of solutions to (IVP),

(i) If $1 < p \le 1 + 2/n$, then $u(r;\alpha)$ is a crossing solution for every $\alpha > 0$.

(ii) If $p \ge (n+2)/(n-2)$, then $u(r;\alpha)$ is a slowly decaying solution for every $\alpha > 0$.

(iii) If $1 + 2/n < p < (n+2)/(n-2)$, then there exists a rapidly decaying solution.

In [HaW], they could not obtain complete structure on case $1 + 2/n < p < (n+2)/(n-2)$. But they have given the following conjecture:

Conjecture by Haraux and Weissler [HaW]

There exists a unique positive number α_* such that $u(r;\alpha_*)$ is a rapidly decaying solution. Moreover, $u(r;\alpha)$ is a crossing solution for every $\alpha \in (\alpha_*, \infty)$ and $u(r;\alpha)$ is a slowly decaying solution for every $\alpha \in (0, \alpha_*)$.

Recently, Yanagida [Ya] shows that above conjecture is right in case $1 + 2/n < p \le n/(n-2)$. In addition, if $n/(n-2) \le p < (n+2)/(n-2)$, as a joint work with Claus Dohmen (University of Bonn) we also get

THEOREM A. ([DHi]) Suppose $n \ge 3$ and $n/(n-2) \le p < (n+2)/(n-2)$. Then the conjecture by Haraux and Weissler is right.

This thorem is proved by using the structure thorem by Yanagida and Yotsutani (see [YaYo] or [Yo]). Thus we can conclude the complete structure of solutions to (IVP) for $n \ge 3$ and $p > 1$.

Moreover, in (p,α)-plane we will define the following domains:

$$\begin{cases} D_C := \{(p,\alpha)\in(1,\infty)\times(0,\infty) \mid u(r;\alpha) \text{ is a crossing solution}\}, \\ D_R := \{(p,\alpha)\in(1,\infty)\times(0,\infty) \mid u(r;\alpha) \text{ is a rapidly decaying solution}\}, \\ D_S := \{(p,\alpha)\in(1,\infty)\times(0,\infty) \mid u(r;\alpha) \text{ is a slowly decaying solution}\}. \end{cases}$$

From above definition, we want to investigate the relation of D_C, D_R and D_S in (p,α)-plane.

To this problem, as a joint work with Eiji Yanagida (Tokyo Institute of Technology) we have

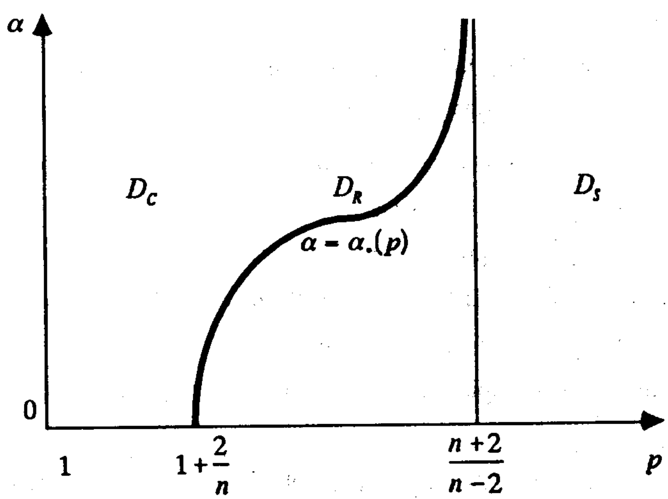
<u>THEOREM B.</u> ([HiYa]) For $1+2/n<p<(n+2)/(n-2)$, D_R is unique C^1-class curve in (p,α)-plane. If we define D_R by

$$\alpha=\alpha_*(p) \quad \text{for } p\in\left(1+\frac{2}{n},\frac{n+2}{n-2}\right),$$

then $\alpha_*(p)$ satisfies

$$\alpha_*(p)\to 0 \text{ as } p\to 1+\frac{2}{n} \text{ and } \alpha_*(p)\to\infty \text{ as } p\to\frac{n+2}{n-2}.$$

Moreover, in (p,α)-plane, domain D_C is the left side of curve D_R and domain D_S is the right side of curve D_R. (See Fig.1.)



(Fig.1)

Sketch of proof. In view of Theorem A, we have already known that for each $p \in (1+2/n,(n+2)/(n-2))$ there exists $\alpha_*(p)$ uniquely. In order to obtain that $\alpha = \alpha_*(p)$ is a C^1-class curve in $(1+2/n,(n+2)/(n-2))$, we will carry out the following steps:

I. Using implicit function theorem (IFT), we will show that there exists unique C^1-class branch at the neighbourhood of $(p,\alpha)=(1+2/n,0)$.

II. Moreover, using IFT again, we will show that this branch can be extended to $p=(n+2)/(n-2)$.

III. Finally, we will prove $\alpha_*(p) \to \infty$ as $p \to (n+2)/(n-2)$.

Furthermore, it follows from Theorem A that the left side and the right side of D_R are D_C and D_S, respectively.

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ASYMPTOTIC BEHAVIOR OF BLOWUP SOLUTIONS OF A PARABOLIC EQUATION WITH THE P-LAPLACIAN

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1. INTRODUCTION AND RESULTS

In this paper we mainly consider the blowup problem for the following initial boundary value problem:

$$\text{(1.1)} \qquad \begin{cases} u_t = \Delta_p u + |u|^{q-2}u, & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t \geq 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where p, $q > 2$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. Especially, we here study the case when $p = q$.

As for the existence and non-existence of global solutions of (1.1), the following results are well known (see [14],[9],[5],[11]):
(i) When $p > q$, (1.1) has a global solution for any $u_0 \in W_0^{1,p}$.
(ii) When $p < q$, for sufficiently small initial function $u_0 \in W_0^{1,p}$, (1.1) has a global solution, and if u_0 is large enough, the solution blows up in a finite time.
(iii) When $p = q$, put $\lambda_1 = \inf\{\|\nabla u\|_p^p/\|u\|_p^p : u \in W_0^{1,p} \setminus \{0\}\}$. If $\lambda_1 \geq 1$, (1.1) has a global solution for any $u_0 \in W_0^{1,p}$.
Here, $W_0^{1,p} \equiv W_0^{1,p}(\Omega)$ denotes the usual Sobolev space with the norm $\|u\|_{W_0^{1,p}} = \|\nabla u\|_p$, and $\|\cdot\|_p$ denotes the $L^p(\Omega)$ norm.

From the above results, we see that the case $p = q$ is critical for the existence of blowup solutions of (1.1). For the critical exponents of other equations and their role, we refer to the survey paper by Levine [8]. Here, we should note that little is known about the case when $p = q$

and $\lambda_1 < 1$. So, in what follows, we study (1.1) with the case when $p = q > 2$, that is, we consider the following problem:

$$\text{(P)} \qquad \begin{cases} u_t = \Delta_p u + |u|^{p-2}u, & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t \geq 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Our first purpose in this paper is to derive sufficient conditions on blowing up of solutions of (P) (Theorems B and C). The second purpose is to study the asymptotic behavior of solutions of (P). Here, we note that we consider not only the asymptotic behavior of blowup solutions but also that of global solutions. In both cases, we show that each solution of (P) behaves asymptotically like a self-similar solution of (P). First, we derive blowup rate and decay rate of solutions of (P) for each case (Theorem D). Next, we investigate the asymptotic profile of both blowup and global solutions of (P) near the maximal existence time (Theorem E). These results for the case $p > 2$ in (P) may be regarded as a natural extension of the linear case $p = 2$ in (P).

To be more precise, we here recall the local existence results for (P). The local existence of strong solutions of (P) is already studied by many authors (see [5],[7],[10],[12]). Here, a function $u(x,t)$ is said to be a strong solution of (P) in $[0,T]$ if (i) $u \in C([0,T]; W_0^{1,p}(\Omega))$, (ii) u_t, $\Delta_p u$ and $|u|^{p-2}u \in L^2(0,T; L^2(\Omega))$, and (iii) u satisfies (P). Assume that $p > 2$, and $2(p-1) \leq Np/(N-p)$ if $p < N$. Then, for any $u_0 \in W_0^{1,p}$, there exists a positive number T such that (P) has a strong solution in $[0,T]$. Moreover, let T^* be the maximal existence time of the strong solution $u(t)$ of (P). Then, if $T^* < \infty$, it follows together with (1.6) below that

$$\lim_{t \to T^*} \|u(t)\|_2 = \lim_{t \to T^*} \|\nabla u(t)\|_p = \infty.$$

Furthermore, if we put $E(u) = \|\nabla u\|_p^p - \|u\|_p^p$, we have

$$\partial_t \|u(t)\|_2^2 = -2E(u(t)) \qquad \text{a.e. in} \quad [0,T^*), \tag{1.2}$$

$$\partial_t E(u(t)) = -p\|u_t(t)\|_2^2 \qquad \text{a.e. in} \quad [0,T^*). \tag{1.3}$$

We note that $E(\lambda u) = \lambda^p E(u)$ holds for any $\lambda > 0$ and $u \in W_0^{1,p}$, which is a special feature in the critical case. Our main idea in this paper is to introduce the Rayleigh type quotient $E(u)/\|u\|_2^p$. The following lemma is important in this paper.

Lemma A. *Assume that $u_0 \in W_0^{1,p} \setminus \{0\}$, and let $u(t)$ be a strong solution of (P) in $[0,T^*)$. Then, we have*

$$\partial_t \frac{E(u(t))}{\|u(t)\|_2^p} \le 0 \qquad \text{a.e. in} \quad [0,T^*).$$

Lemma A follows immediately from (1.2) and (1.3), but it plays an essential role in the proofs of the following theorems. We should mention that a similar result to Lemma A is obtained by Berryman and Holland [1] for the fast diffusion $(u^{q-1})_t = \Delta u$ with $q > 2$. In [1] they study the asymptotic behavior of finite time extinction solutions of it.

First, we derive two sufficient conditions that the solution of (P) blows up in a finite time.

Theorem B. *Let $p > 2$ and $\lambda_1 < 1$. Assume that $u_0 \in W_0^{1,p}$ satisfies $E(u_0) < 0$. Then, the strong solution of (P) blows up in a finite time.*

Theorem C. *Let $p > 2$ and $\lambda_1 < 1$. Assume that $u_0 \in W_0^{1,p} \setminus \{0\}$ is non-negative in Ω. Then, the strong solution of (P) blows up in a finite time.*

Here, we recall that $\lambda_1 = \inf\{\|\nabla u\|_p^p/\|u\|_p^p : u \in W_0^{1,p} \setminus \{0\}\}$, and if $\lambda_1 \ge 1$, every strong solution of (P) exists globally in time. Theorems B and C supplement the known results by many authors concerning the existence and non-existence of global solutions of (1.1) by giving information about the case of $p = q > 2$. In [2] Galaktionov showed a similar result to Theorem C for $u_t = \Delta u^m + u^m$ with $m > 1$ by using the so-called Kaplan method [6]. We should mention that this method is not applicable to our problem (P), and our proof of Theorem C is quite different from that of [2].

Next, we consider the asymptotic behavior of strong solutions of (P). We begin with deriving blowup rate and decay rate of strong solutions of (P).

Theorem D. *Assume $p > 2$ and $u_0 \in W_0^{1,p} \setminus \{0\}$. Let T^* be the maximal existence time of the strong solution $u(t)$ of (P). Put $\gamma_* = \lim_{t\to T^*}[E(u(t))/\|u(t)\|_2^p]$.*

(*i*) *If $T^* < \infty$, we have $\gamma_* < 0$ and*

$$\lim_{t\to T^*}[-\gamma_*(p-2)(T^*-t)]^{\frac{1}{p-2}}\|u(t)\|_2 = 1. \tag{1.4}$$

(*ii*) *If $T^* = \infty$ and $\gamma_* > 0$, we have*

$$\lim_{t\to\infty}[\gamma_*(p-2)t]^{\frac{1}{p-2}}\|u(t)\|_2 = 1. \tag{1.5}$$

Remark 1.1. Put $\gamma_1 = \inf\{E(u)/\|u\|_2^p : u \in W_0^{1,p} \setminus \{0\}\}$. Then, we see that $\gamma_1 > -\infty$. In fact, by the Gagliardo-Nirenberg and the Young inequalities, there exist positive constants $\alpha \in (0,p)$, C_1 and C_2 such that

$$\|u\|_p^p \le C_1 \|u\|_2^{p-\alpha} \|\nabla u\|_p^\alpha \le \frac{1}{2}\|\nabla u\|_p^p + C_2\|u\|_2^p, \qquad u \in W_0^{1,p},$$

from which we have

$$\|\nabla u\|_p^p \le 2E(u) + 2C_2\|u\|_2^p, \qquad u \in W_0^{1,p}, \tag{1.6}$$

and we have $\gamma_1 \ge -C_2$. So, it follows from Lemma A and this fact that the limit $\gamma_* = \lim_{t\to T^*}[E(u(t))/\|u(t)\|_2^p]$ exists and $\gamma_* \ge \gamma_1$ holds for any strong solution $u(t)$ of (P). We also note that from Theorem B, if $T^* = \infty$, we have $\gamma_* \ge 0$. Moreover, we see that $\gamma_1 < 0$ [resp. $\gamma_1 = 0$, $\gamma_1 > 0$] if and only if $\lambda_1 < 1$ [resp. $\lambda_1 = 1$, $\lambda_1 > 1$].

Remark 1.2. A function $u(x,t) = v(t)w(x)$ of variable separation type is called a self-similar solution of (P) with $u_0(x) = v(0)w(x)$ if v and $w \in W_0^{1,p}$ satisfy

$$v_t = -\gamma|v|^{p-2}v \qquad \text{in} \quad \mathbf{R}, \tag{1.7}$$

$$-\Delta_p w - |w|^{p-2}w = \gamma w \qquad \text{in} \quad \mathcal{D}'(\Omega) \tag{1.8}$$

for some $\gamma \in \mathbf{R}$. From Theorem D, we see that the blowup rate and the decay rate of general strong solutions of (P) in Theorem D are the same as those of the self-similar solutions of (P).

Remark 1.3. In the case when $p < q$ in (1.1), the decay rate of small global solutions of (1.1) is given by H. Ishii [5]. However, it seems that in [5] there are no results for blowup rate of solutions of (1.1) when $2 < p < q$. For the semilinear case $p = 2 < q$, see Giga and Kohn [3] and references therein.

The following theorem states that the asymptotic profiles of solutions of (P) are given by the solutions of (1.8).

Theorem E. *Assume that $p > 2$ and $u_0 \in W_0^{1,p}\setminus\{0\}$. Let $T^* \in (0,\infty]$ be the maximal existence time of the strong solution $u(t)$ of (P). Then, for any sequence $\{t_j\}$ satisfying $t_j \to T^*$, there exist a subsequence $\{t_{j'}\}$ of $\{t_j\}$ and $w \in W_0^{1,p}$ such that*

$$\frac{u(t_{j'})}{\|u(t_{j'})\|_2} \to w \qquad \textit{in} \quad W_0^{1,p}, \tag{1.9}$$

$$-\Delta_p w - |w|^{p-2}w = \gamma_* w \qquad \textit{in} \quad \mathcal{D}'(\Omega), \qquad \|w\|_2 = 1, \tag{1.10}$$

where $\gamma_* = \lim_{t\to T^*}[E(u(t))/\|u(t)\|_2^p]$.

Remark 1.4. It is natural to ask in Theorem E whether the limit $u(t)/\|u(t)\|_2$ exists or not in $W_0^{1,p}$ as $t \to T^*$. At the present, we do not know the answer, even if the solution $u(t)$ of (P) is non-negative. Of course, if non-negative solution $w \in W_0^{1,p}$ of (1.10) is unique, then it follows immediately from Theorem E that $u(t)/\|u(t)\|_2 \to w$ in $W_0^{1,p}$ as $t \to T^*$ for any non-negative and non-zero solution $u(t)$ of (P). However, as we show in Section 2 for the case $N = 1$, non-negative solution of (1.10) is not unique in general.

2. Eigenvalue Problem (1.10) for $N = 1$

In this section, we consider the eigenvalue problem (1.10) for the case $N = 1$. Especially, we are interested in the set of all non-negative solutions of (1.10) with $\gamma_* < 0$, which is related to the asymptotic profiles of non-negative blowup solutions of (P).

First, we consider the following boundary value problem:

$$\left\{\begin{array}{ll} -(|u'|^{p-2}u'(x))' - |u|^{p-2}u(x) = -u(x), & x \in \Omega, \\ u \in W_0^{1,p}(\Omega), \qquad u(x) \geq 0, \not\equiv 0, & x \in \Omega. \end{array}\right. \tag{2.1}$$

Here, the symbol $'$ denotes the differentiation with respect to x. Let S_l be the set of all solutions of (3.1) for $\Omega = (-l, l)$. Then, the structure of S_l is determined as follows.

Proposition 2.1. *Let l_p be the positive number such that*

$$\lambda_1(-l_p, l_p) = \inf\left\{\frac{\|u'\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(-l_p, l_p),\ u \neq 0\right\} = 1,$$

and $m_p = pl_p/(p-2)$.

1. *If $l \leq l_p$, then S_l is empty.*
2. *If $l_p < l \leq m_p$, then there exists a unique positive solution Φ_l of (3.1) and $S_l = \{\Phi_l\}$.*
3. *If $l > m_p$, then $S_l = \bigcup_{k=1}^{[l/m_p]} S_l^k$, where $[l/m_p]$ denotes the largest integer not exceeding l/m_p, and $S_l^k = \{\sum_{j=1}^k \Phi_{m_p}(\cdot - y_j) : -l \leq y_1 - m_p,\ y_j + 2m_p \leq y_{j+1},\ j = 1, \cdots, k-1,\ y_k + m_p \leq l\}$.*

As a corollary to Proposition 2.1, we have the main result in this section.

Theorem 2.2. *Let $\gamma < 0$ and $\Sigma(\gamma)$ be the set of all solutions of*

$$\left\{\begin{array}{ll} -(|u'|^{p-2}u'(x))' - |u|^{p-2}u(x) = \gamma u(x), & x \in (-l, l), \\ u \in W_0^{1,p}(-l, l), \qquad \|u\|_2 = 1, \qquad u(x) \geq 0, & x \in (-l, l). \end{array}\right.$$

(1) *When $l \leq l_p$, $\Sigma(\gamma)$ is empty for any $\gamma < 0$.*

(2) *When $l_p < l \leq m_p$, let $\gamma_1 = E(\Phi_l)/\|\Phi_l\|_2^p$. Then $\gamma_1 < 0$ and $\Sigma(\gamma_1) = \{\tilde{\Phi}_l\}$, where $\tilde{\Phi}_l = \Phi_l/\|\Phi_l\|_2$, and $\Sigma(\gamma)$ is empty if $\gamma \neq \gamma_1$.*

(3) *When $l > m_p$, for $k = 1, 2, \cdots, [l/m_p]$, let $\gamma_k = k^{1-p/2}E(\Phi_{m_p})/\|\Phi_{m_p}\|_2^p$. Then $\gamma_1 < \gamma_2 < \cdots < \gamma_{[l/m_p]} < 0$ and $\Sigma(\gamma_k) = \{\sum_{j=1}^k \tilde{\Phi}_{m_p}(\cdot - y_j) : -l \leq y_1 - m_p,\ y_j + 2m_p \leq y_{j+1},\ j = 1, \cdots, k-1,\ y_k + m_p \leq l\}$, and $\Sigma(\gamma)$ is empty if $\gamma \notin \{\gamma_1, \gamma_2, \cdots, \gamma_{[l/m_p]}\}$.*

Theorem 2.2 follows immediately from Proposition 2.1. We note that γ_1 defined in Remark 1.1 coincides with that in Theorem 2.2 in this case. In order to prove Proposition 2.1, we consider the following initial value problem:

$$
\begin{cases} (|u'|^{p-2}u'(x))' = u(x) - |u|^{p-2}u(x), & x > 0, \\ u(0) = \alpha > 0, \quad u'(0) = 0. \end{cases} \tag{2.2}
$$

Lemma 2.3. *Let $\alpha_p = (p/2)^{1/(p-2)}$ and $F(s) = (p/(p-1))(|s|^2/2 - |s|^p/p)$, and let $x_\alpha = \infty$ if $\alpha < \alpha_p$, and $x_\alpha = \int_0^\alpha [F(s) - F(\alpha)]^{-1/p} ds$ if $\alpha \geq \alpha_p$. For $\alpha > 0$, there exists a unique solution φ_α of (2.2) in $(0, x_\alpha)$, and φ_α is positive in $(0, x_\alpha)$. Moreover, when $\alpha \geq \alpha_p$, $x_\alpha < \infty$ and φ_α satisfies $\varphi_\alpha(x_\alpha) = 0$, $\varphi'_\alpha(x_\alpha) < 0$ if $\alpha > \alpha_p$, and $\varphi'_\alpha(x_\alpha) = 0$ if $\alpha = \alpha_p$.*

Remark 2.4. By an elementary computation, we see that x_α is strictly decreasing with respect to $\alpha \geq \alpha_p$. It is known that $l_p = (p-1)^{1/p}B(1/p, 1-1/p)/p = [\pi(p-1)^{1/p}]/[p\sin(\pi/p)]$, where $B(\cdot,\cdot)$ is the beta function. Another elementary calculation yields $\lim_{\alpha\to\infty} x_\alpha = l_p$ and $x_{\alpha_p} = m_p$.

Proposition 2.1 follows from Lemma 2.3 and Remark 2.4. In particular, Φ_l is given by

$$
\Phi_l(x) = \begin{cases} \varphi_{\alpha(l)}(x), & \text{for } \ 0 \leq x \leq l, \\ \varphi_{\alpha(l)}(-x), & \text{for } \ -l \leq x < 0, \end{cases}
$$

where $\alpha(l) \in [\alpha_p, \infty)$ is the unique number such that $l = x_{\alpha(l)}$.

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半線形楕円型方程式の球対称解の構造と数値計算

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半線形楕円型方程式

$$\Delta u + K(|x|)u^p = 0 \quad \text{in } \mathbf{R}^n, \qquad (\text{ただし，} p > 1,\ n > 2)$$

の正値球対称な解の大域的構造について報告する．

常微分方程式

$$(\mathrm{E}) \qquad \begin{cases} u_{rr} + \dfrac{n-1}{r}u_r + K(r)u_+^p = 0, & r > 0, \\ u(0) = \alpha > 0\ , \end{cases}$$

で原点の値 α を動かしたときの解の様子を調べる．ただし，$u_+ = \max\{u, 0\}$．

まず，与えられた関数 $K(r)$ に次の条件を導入する．

$$(\mathrm{K}) \qquad \begin{cases} K(r) \ \text{ is continuous on } \ (0,\infty), \\ K(r) \geq 0 \text{ and } K(r) \not\equiv 0 \text{ on } (0,\infty), \\ rK(r) \in L^1(0,1). \end{cases}$$

仮定(K)のもとで，初期値問題(E)は一意的な大域解

$$u = u(r;\alpha) \ \in \ C([0,\infty)) \ \cap C^2((0,\infty)),$$

をもつ(特に，$K(r) \ \in \ C([0,\infty))$ のときは，$u(r) \ \in \ C^2([0,\infty))$，$u_r(0) = 0$)．さらに解は，次の三つのタイプのいずれかになる．

(i) $u(r)$ が零点をもつもの．(crossing solution とよぶ)

(ii) $r \to \infty$ とすると $r^{n-2}u(r)$ が有限の極限値をもつもの．(rapidly decaying solution とよぶ)

(iii) $r \to \infty$ とすると $r^{n-2}u(r)$ が発散するもの．(slowly decaying solution とよぶ)

とくに，$p = (n+2)/(n-2)$，$K(r) \equiv 1$ のとき，(E)式は次の厳密解で与えられる．

$$u(r;\alpha) = \left\{1 + \frac{\alpha^{4/(n-2)}}{n(n-2)}r^2\right\}^{-(n-2)/2}$$

この解は正であり，すべての初期値に対して，rapidly decaying solution である．

それでは，$K(r) = 1$ に摂動を加えたものを考えたときの解全体の構造について調べる．これに関連する既知の結果を次に述べる．

[Ding-Ni] (1985)

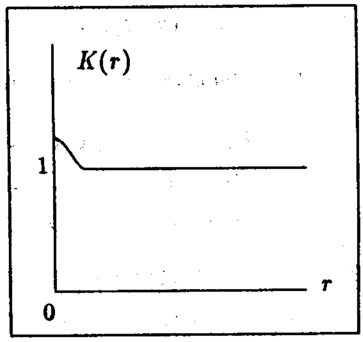
$p=(n+2)/(n-2)$ とする．

$K(r)$ は，$(0,\infty)$ において非増加であり，

$$K_r(r)\not\equiv 0,$$

と仮定する．

このとき解の構造はすべての初期値に対して，slowly decaying solution となる．

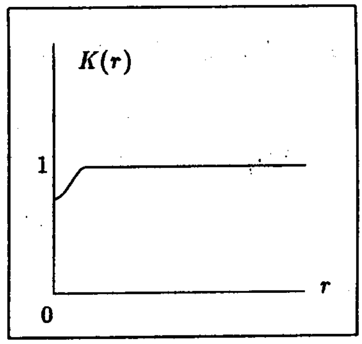


$K(r)$ は，$(0,\infty)$ において非減少であり，

$$K_r(r)\not\equiv 0,$$

と仮定する．

このとき解の構造はすべての初期値に対して，crossing solution となる．



[Yanagida-Yotsutani] (1993)

$p=(n+2)/(n-2)$ とする．$R\in(0,\infty)$ に対して

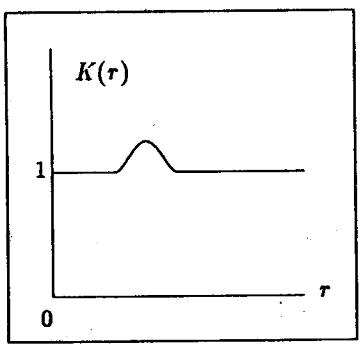
$K(r)$ は $(0,\ R)$ 上で非減少，

$K(r)$ は $(R,\ \infty)$ 上で非増加，

であり，

$$\lim_{r\to\infty}K(r)=\lim_{r\to 0}K(r),$$

かつ，$(0,\infty)$ 上で $K(r)\geq 0$，$K(r)\not\equiv$ 定数であると仮定する．



このとき解の構造は以下の条件を満たす α_* が唯一つ存在する．

$$\begin{cases} \alpha>\alpha_* \text{ のとき } u(r;\alpha) \text{ は crossing solution} \\ \alpha=\alpha_* \text{ のとき } u(r;\alpha) \text{ は rapidly decaying solution} \\ \alpha<\alpha_* \text{ のとき } u(r;\alpha) \text{ は slowly decaying solution} \end{cases}$$

$K(r)=1$ にほんのわずかな摂動を与えることによって解の構造が不連続に変化しているように見える．既知の結果では，指数 $p=(n+2)/(n-2)$ に固定したものを考えていたが，指数 p をパラメータとして動かすことを考える．よって，(E) 式において，原点での値 α と指数 p の両方を動かしたときの解の様子を調べる．そこで，$p-\alpha$ 平面における解の構造を調べるにあたり，

$$D := \{(p,\ \alpha);\ p > 1,\ \alpha > 0\},$$

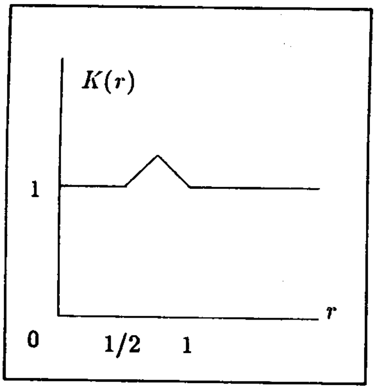
とし，D の部分集合 D_C, D_R, D_S を次のように定める．

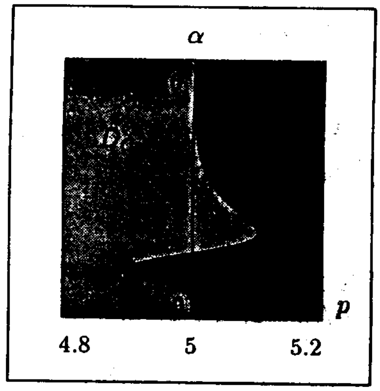
$$D_C := \{(p,\ \alpha) \in D;\ u(r) \text{ is a crossing solution}\},$$
$$D_R := \{(p,\ \alpha) \in D;\ u(r) \text{ is a rapidly decaying solution}\},$$
$$D_S := \{(p,\ \alpha) \in D;\ u(r) \text{ is a slowly decaying solution}\}.$$

目的　$K(r) = 1$ に摂動を与えたものを考えたとき，$p - \alpha$ 平面における D_C, D_R, D_S の形状を調べる．具体的に $K(r)$ を与えて，$p - \alpha$ 平面の数値計算の結果を示す．

例 1.　$n = 3$ とし，$K(r)$ を次のようにおく．($\varepsilon = 1$)

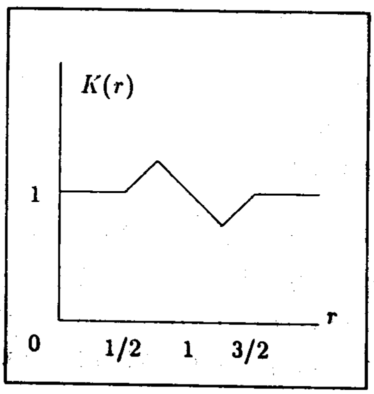
$$K(r) = 1 + \varepsilon\eta(r), \qquad \eta(r) = \left(\frac{1}{4} - \left|r - \frac{3}{4}\right|\right)\chi_{[\frac{1}{2},1]}(r),$$

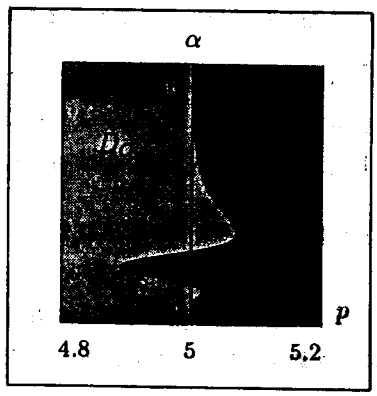




例 2.　$n = 3$ とし，$K(r)$ を次のようにおく．($\varepsilon = 1$)

$$K(r) = 1 + \varepsilon(\eta(r) - \eta_1(r)), \qquad \eta(r) = \left(\frac{1}{4} - \left|r - \frac{3}{4}\right|\right)\chi_{[\frac{1}{2},1]}(r),$$
$$\eta_1(r) = \left(\frac{1}{4} - \left|r - \frac{5}{4}\right|\right)\chi_{[1,\frac{3}{2}]}(r),$$





例1の $p-\alpha$ 平面の数値計算の結果を見ると，$n=3$ の場合に，$p=(n+2)/(n-2)$ における既知の結果と一致していることがわかる．$p=(n+2)/(n-2)$ に固定したもので考えていると，わずかな摂動によって解の構造が不連続に変化しているように見えたが，指数 p をパラメータとして動かすことにより，解の構造が自然な形で連続的に変化していることがわかった．

今，$p-\alpha$ 平面では開集合 D_C，D_S があり，その境界が D_R に一致していた．この D_R に着目してみることにする．

摂動 ε を十分小にしたとき，つまり $K(r)=1$ に対する摂動がわずかな場合の解の構造に関する興味深い結果が得られたので示す．

<u>**定理**</u>　$\eta(r)$ は compact support をもつ関数とし，$K(r)$ は

$$K(r)=1+\varepsilon\eta(r),$$

で与えられるとする．このとき ε が十分小であれば，D_R は，なめらかな曲線であり，

$$p \fallingdotseq \frac{n+2}{n-2}+\frac{2}{n-2}\cdot\frac{\displaystyle\int_0^\infty r^n\eta_r(r)\left\{1+\frac{\alpha^{4/(n-2)}}{n(n-2)}r^2\right\}^{-n}dr}{\displaystyle\int_0^\infty r^{n-1}\left\{1+\frac{\alpha^{4/(n-2)}}{n(n-2)}r^2\right\}^{-n}dr}\varepsilon,$$

と近似できる．

厳密な証明にはかなりの準備を必要とするので，直観的な証明をつける．準備として，補題 1，補題 2 を列挙する．

<u>**補題 1.**</u>　(Pohozaev 恒等式)

u が (E) 式 を満たすとき，次の Pohozaev 恒等式が成り立つ．

$$\frac{d}{dr}P(r;u)\equiv\frac{1}{p+1}\left\{rK_r(r)-\frac{(n-2)p-(n+2)}{2}K(r)\right\}r^{n-1}u_+(r)^{p+1},$$

ただし，

$$P(r;u)=\frac{1}{2}r^{n-1}u_r(r)\{ru_r(r)+(n-2)u(r)\}+\frac{1}{p+1}r^nK(r)u_+(r)^{p+1}.$$

<u>**補題 2.**</u>　(解の特徴付け)

(a)　$u=u(r;\alpha)$ を crossing solution とし，$z(\alpha)$ を $u(r;\alpha)$ の零点とすれば，$r\in[z(\alpha),\infty)$ に対して，$P(r;u)>0$ が成り立つ．

(b) $u = u(r;\alpha)$ が rapidly decaying solution であれば，点列 $\{\bar{r}_i\}$ で $i \to \infty$ のとき，$\bar{r}_i \to \infty$ および $P(\bar{r}_i;u) \to 0$ を満たすものが存在する．

(c) $u = u(r;\alpha)$ が slowly decaying solution であれば，点列 $\{\hat{r}_i\}$ で $i \to \infty$ のとき，$\hat{r}_i \to \infty$ および $P(\hat{r}_i;u) < 0$ を満たすものが存在する．

定理の直観的証明 $K^\varepsilon(r)$ に対応する解を u^ε とおく．補題1，2より，$u^\varepsilon(r;\alpha)$ が rapidly decaying solution ならば，

$$\lim_{r\to\infty} P(r;u^\varepsilon(r;\alpha)) = 0,$$

となる．Pohozaev 恒等式より，

$$\int_0^\infty \left\{ rK_r^\varepsilon(r) - \frac{(n-2)p-(n+2)}{2}K^\varepsilon(r)\right\} r^{n-1}(u_+^\varepsilon)^{p+1}dr = 0,$$

書き直すと，

$$\int_0^\infty \left\{ \varepsilon r\eta_r(r) - \tfrac{n-2}{2}\left(p - \tfrac{n+2}{n-2}\right)(1+\varepsilon\eta(r))\right\} r^{n-1}(u_+^\varepsilon)^{p+1}dr = 0.$$

両辺を $(p,\ \varepsilon) = \left(\frac{n+2}{n-2},\ 0\right)$ のまわりでテーラー展開する．

$$\begin{aligned}
&\int_0^\infty \left\{ \varepsilon r\eta_r(r) - \tfrac{n-2}{2}\left(p - \tfrac{n+2}{n-2}\right)(1+\varepsilon\eta(r))\right\} r^{n-1}(u_+^\varepsilon)^{p+1}dr\Big|_{\substack{p=\frac{n+2}{n-2}\\ \varepsilon=0}} \\
&+ \tfrac{\partial}{\partial p}\left\{\left(-\tfrac{n-2}{2}\left(p-\tfrac{n+2}{n-2}\right)\right)\cdot\int_0^\infty r^{n-1}(u_+^\varepsilon)^{p+1}dr\right\}\Big|_{\substack{p=\frac{n+2}{n-2}\\ \varepsilon=0}}\cdot\left(p-\tfrac{n+2}{n-2}\right) \\
&+ \tfrac{\partial}{\partial\varepsilon}\left\{\varepsilon\cdot\int_0^\infty r^n\eta_r(r)(u_+^\varepsilon)^{p+1}dr\right\}\Big|_{\substack{p=\frac{n+2}{n-2}\\ \varepsilon=0}}\cdot\varepsilon + h.o.t. = 0.
\end{aligned}$$

整理すると，

$$\begin{aligned}
&\left\{-\tfrac{n-2}{2}\cdot\int_0^\infty r^{n-1}(u_+^\varepsilon)^{p+1}dr\right\}\Big|_{\substack{p=\frac{n+2}{n-2}\\ \varepsilon=0}}\cdot\left(p-\tfrac{n+2}{n-2}\right) \\
&+\left\{\int_0^\infty r^n\eta_r(r)(u_+^\varepsilon)^{p+1}dr\right\}\Big|_{\substack{p=\frac{n+2}{n-2}\\ \varepsilon=0}}\cdot\varepsilon \quad + h.o.t. = 0.
\end{aligned}$$

今，$(p,\ \varepsilon) = \left(\frac{n+2}{n-2},\ 0\right)$ の近くで考えているので，

$$-\tfrac{n-2}{2}\int_0^\infty r^{n-1}(u^0)^{2n/(n-2)}dr\cdot\left(p-\tfrac{n+2}{n-2}\right) + \int_0^\infty r^n\eta_r(r)(u^0)^{2n/(n-2)}dr\cdot\varepsilon + h.o.t. = 0.$$

書き直すと,

$$p \doteqdot \frac{n+2}{n-2}+\frac{2}{n-2}\cdot\frac{\int_0^\infty r^n\eta_r(r)(u^0)^{2n/(n-2)}dr}{\int_0^\infty r^{n-1}(u^0)^{2n/(n-2)}dr}\varepsilon,$$

ただし, u^0 は

$$\begin{cases} (u^0)_{rr}+\dfrac{n-1}{r}(u^0)_r+(u^0)^{(n+2)/(n-2)}=0, & r>0,\\ u^0(0)=\alpha>0, \end{cases}$$

の解である. 実は

$$u^0(r;\alpha)=\alpha\left\{1+\frac{\alpha^{4/(n-2)}}{n(n-2)}r^2\right\}^{-(n-2)/2}$$

で与えられるので, u^0 を代入すると,

$$p \doteqdot \frac{n+2}{n-2}+\frac{2}{n-2}\cdot\frac{\int_0^\infty r^n\eta_r(r)\left\{1+\frac{\alpha^{4/(n-2)}}{n(n-2)}r^2\right\}^{-n}dr}{\int_0^\infty r^{n-1}\left\{1+\frac{\alpha^{4/(n-2)}}{n(n-2)}r^2\right\}^{-n}dr}\varepsilon,$$

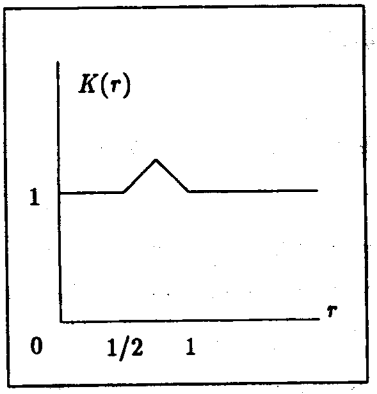
が得られる.

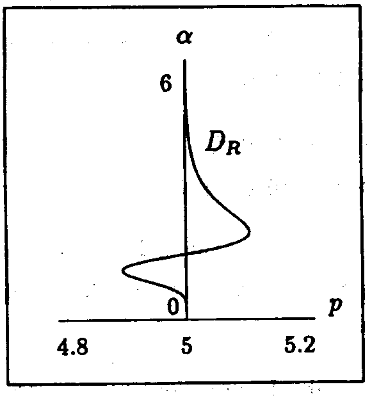
この推論の途中にいろいろ不完全なところがあるので直観的な証明である. 数学的証明は, [MYY2] を見て下さい.

与えられた $K(r)$ に対する D_R の具体的な近似式を求める.

<u>例 1'.</u> $n=3$ とし, $K(r)$ を次のようにおく. $(\varepsilon=1)$

$$K(r)=1+\varepsilon\eta(r), \qquad \eta(r)=\left(\frac{1}{4}-\left|r-\frac{3}{4}\right|\right)\chi_{[\frac{1}{2},1]}(r),$$

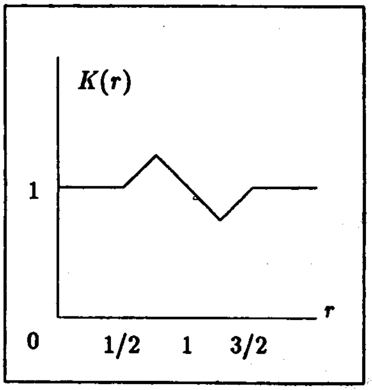


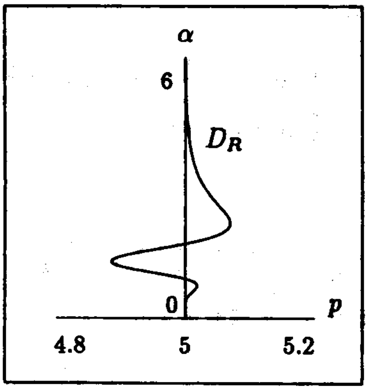


$$p \doteqdot 5+\frac{8\sqrt{3}}{\pi}\alpha^6\frac{(-15840-2928\alpha^4+577\alpha^8+78\alpha^{12})}{(3+\alpha^4)^2(12+\alpha^4)^2(16+3\alpha^4)^2}\varepsilon.$$

例 2'. $n=3$ とし，$K(r)$ を次のようにおく．($\varepsilon = 1$)

$$K(r) = 1 + \varepsilon(\eta(r) - \eta_1(r)), \qquad \begin{aligned} \eta(r) &= \left(\frac{1}{4} - \left|r - \frac{3}{4}\right|\right) \chi_{[\frac{1}{2},1]}(r), \\ \eta_1(r) &= \left(\frac{1}{4} - \left|r - \frac{5}{4}\right|\right) \chi_{[1,\frac{3}{2}]}(r), \end{aligned}$$





$$p \fallingdotseq 5 + \frac{1536\sqrt{3}}{\pi}\alpha^6 \frac{(589824 - 331776\alpha^4 - 698368\alpha^8 - 148224\alpha^{12} + 10044\alpha^{16} + 2025\alpha^{20})}{(12+\alpha^4)^2(4+3\alpha^4)^2(16+3\alpha^4)^2(48+25\alpha^4)^2}\varepsilon.$$

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