第１６回
発展方程式若手セミナー
報告集

１９９４年１２月

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序

第１６回発展方程式若手セミナーは、平成６年度科学研究費補助金（総合研究Ａ）の援助のもとに、平成６年７月３１日より８月３日までの４日間、東京都八王子市にある財団法人大学セミナー・ハウスで行われました。本年は、特別講演に長澤壯之先生（東北大学理学部）をお招きし、離散的勾配流と発展方程式という題目で講演をして頂いた他、２３件の一般講演、及び大学院生による short communications が行われました。

本報告集は、特別講演と一般講演の講演内容を元にまとめたものです。

１６回を迎えた今回のセミナーは、発展方程式及びその関連分野の多彩なテーマによる講演が連日行われ、活発な討論・情報交換が行われました。この報告集が参加された方や関連分野の研究者の方々の研究に貢献できますよう願っております。

このセミナーの準備・運営に当たり多くの方々に協力していただきました。特に、京都大学岩崎敷久先生、大阪大学 田辺広城先生、神戸商船大学 丸尾健二先生、宮崎大学 山田直記先生、神戸商船大学 石井克幸先生、神戸大学 壁谷喜継先生、及び千葉大学大学院生諸氏に厚く御礼申し上げます。また、長澤壯之先生には、忙しい時期にセミナーが重なったにもかかわらず、快く特別講演をお引き受けいただきました。この場をお借り致しまして、深く感謝申し上げる次第です。

平成６年１２月

第１６回発展方程式若手セミナー幹事

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日程

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一般講演

ショートコミュニケーション

離散的勾配流と発展方程式

長澤壯之　東北大学理学部

1　序

ある関数（または写像）のクラス $\mathcal{K}$ で定義された汎関数 J の停留点を求めるという問題は古くからの重要な問題である（[9]）。今、$\mathcal{K}$ がある Banach 空間 X の部分集合で J がその上で Gâteaux 微分可能であるとしよう。Gâteaux 微分（第一変分）を grad$J$ で表そう。すなわち、

$$\langle \mathrm{grad} J(u), \varphi\rangle = \frac{d}{d\varepsilon}J(u+\varepsilon\varphi)\Big|_{\varepsilon=0}$$

とする。停留点は

$$\mathrm{grad} J(u) = 0 \tag{1.1}$$

を満たす $u \in \mathcal{K}$ を求めることになる。(1.1) は Euler-Lagrange 方程式と呼ばれる。(1.1) を直接解いて停留点を見つけることを直接法と呼ぼう。逆にある（微分）方程式が、ある汎関数の Euler-Lagrange 方程式となっているとき、minimizimg sequence や mountain pass の議論によって停留点を捜すことで、もとの方程式の解が求められる。このように、（ある）方程式を解くことと、変分問題を解くことは、表裏一体の関係にあると言ってよい。

方程式が発展方程式である場合にも、このような関係はある。停留点を

$$\begin{cases} u_t = -\dfrac{1}{2}\mathrm{grad} J(u) \quad (t>0), \\ u(0) = u_0 \in \mathcal{K} \end{cases} \tag{1.2}$$

の解の $t\to\infty$ における極限として求める方法は、勾配流の方法、Morse (semi)flow による方法、熱方程式による方法[1]などと呼ばれる。(1.2) は発展方程式の一つである。

逆に (1.2) を変分学的に解く方法を考えよう。以後簡単のため $\mathcal{K} \subset L^2(\Omega)$ （Ω は $\mathbf{R}^m$ 内の有界領域）とする。$h>0$ を固定し、汎関数

$$J_n(u) = \int_\Omega \frac{|u-u_{n-1}|^2}{h}dx + J(u)$$

の $\mathcal{K}$ における minimizer として u_n を帰納的に定義する。上の式の右辺第 1 項の第一変分を求めると、

$$\frac{d}{d\varepsilon}\int_\Omega \frac{|u+\varepsilon\varphi-u_{n-1}|^2}{h}dx\Big|_{\varepsilon=0} = \int_\Omega \frac{2(u-u_{n-1})\varphi}{h}dx = \left\langle \frac{2(u-u_{n-1})}{h}, \varphi\right\rangle_{L^2(\Omega)}$$

[1] J が Dirichlet 積分のとき、(1.1) は Laplace の方程式に、(1.2) は熱方程式になることによる。

となる。従って、u_n が満たす方程式は

$$\frac{u_n - u_{n-1}}{h} = -\frac{1}{2}\mathrm{grad}\,J(u_n) \tag{1.3}$$

となり、(1.2) を時間方向に差分化したものとなる。このことから minimizer の列 $\{u_n\}$ を離散的勾配流 (discrete Morse semiflow) と呼ぶことにする。従って、$h \downarrow 0$ とすると（形式的に）勾配流による方法となり、(1.2) の解が得られる。

(1.2) が汎関数の停留点を見つけるために導入されたものという意味では、(1.3) を（$h > 0$ のまま）用いてもその目的を達することが出来る。すなわち、$n \to \infty$ とするともとの汎関数 J の minimizer に収束する（§ 2）。収束先は、各 u_n は汎関数 J_n の global minimizer であるにも拘らず、J の global minimizer とは限らない。

§ 3 では、$h \downarrow 0$ としたとき、(1.2) の弱解が得られることを示そう。

汎関数

$$I_n(u) = \int_\Omega \frac{|u - 2u_{n-1} + u_{n-2}|^2}{h^2}dx + J(u)$$

を用いれば、t について 2 階の方程式

$$u_{tt} = -\frac{1}{2}\mathrm{grad}\,J(u) \quad (t > 0)$$

の解が同様に構成できる（§ 4）。

更に、離散的勾配流の歴史と発展方程式への応用例（§ 5）、補間法（§ 6）、数値実験例（§ 7）などについて述べる。

2 $n \to \infty$ としたときの挙動

簡単のため、以後は次のような設定で考える。もちろん問題に応じて設定は変える必要がある。

Ω を $\mathbf{R}^m$ 内の有界領域とし、境界 $\partial\Omega$ は滑らかとする。$\mathcal{K}$ は $H^1(\Omega)$ の部分 affine 空間とする。すなわち、ある $u^0 \in \mathcal{K}$ と $H^1(\Omega)$ の閉部分空間 $\mathcal{L}$ が存在して、

$$\mathcal{K} = \{u^0 + v \mid v \in \mathcal{L}\}$$

と書けるとする。$v \in \mathcal{L}$ に対し、

$$\|v\|_{\mathcal{L}} = \|v\|_{H^1}$$

である。また、$\mathcal{K}$ は $H^1(\Omega)$ で弱コンパクトとする。

J を $\mathcal{K}$ 上で定義された Gâteaux 微分可能な汎関数とする。更に、J は、$H^1(\Omega)$ 上 coersive かつ弱位相について下半連続性を持つもつものとする。すなわち、

$$J(u) \geq c\|u\|_{H^1}^2 \quad (u \in \mathcal{K}) \tag{2.1}$$

を満たす正定数 c が存在し、

$$\mathcal{K} \ni u_n \rightharpoonup u \in \mathcal{K} \quad (n \to \infty) \quad \text{ならば} \quad J(u) \leq \liminf_{n\to\infty} J(u_n) \tag{2.2}$$

が成立するとする。$\mathcal{L}^*$ を $\mathcal{L}$ の 形式的な $L^2(\Omega)$-内積に関する双対空間とする。Gâteaux 微分 $\mathrm{grad}\,J(u)$ は $\mathcal{L}^*$ の元である。

$$v_n,\ v \in \mathcal{K} \quad \text{a. e. } t \quad \text{かつ、} \quad v_n \rightharpoonup v \quad \text{in } L^2_{\mathrm{loc}}([0,\infty); H^1(\Omega)) \quad (n \to \infty) \tag{2.3}$$

のとき、

$$\mathrm{grad}\,J(v_{n_j}) \rightharpoonup \mathrm{grad}\,J(v) \quad \text{in } L^2_{\mathrm{loc}}([0,\infty); \mathcal{L}^*) \quad (j \to \infty) \tag{2.4}$$

となる部分列がとれるとする。

例. $\mathcal{K} = \mathcal{L} = H^1_0(\Omega)$, $J(u) = \int_\Omega |\nabla u|^2 dx$ とすると、(2.1) - (2.4) は成立する。

以上の仮定から、minimizing sequence の議論が適用できて、次が分かる。

定理 2.1.

$$J(u) = \inf_{v \in \mathcal{K}} J(v)$$

を満たす $u \in \mathcal{K}$ が存在する。

$h > 0$ を固定する。まず $u_0 \in \mathcal{K}$ を与え、$\mathcal{K}$ 上の汎関数 J_1 を

$$J_1(u) = \int_\Omega \frac{|u - u_0|^2}{h} dx + J(u)$$

で定義する。J_1 も J と同じ性質を持つ事は容易に確かめられる。よって、$\mathcal{K}$ における minimizer として u_1 が定義できる。帰納的に

$$J_n(u) = \int_\Omega \frac{|u - u_{n-1}|^2}{h} dx + J(u)$$

の $\mathcal{K}$ における minimizer として u_n が定義できる。

(1.2) の両辺と $2u_t$ との内積を形式的にとれば、

$$2\int_\Omega |u_t|^2 dx = -\langle \mathrm{grad}\,J(u), u_t \rangle = -\frac{d}{dt} J(u) \tag{2.5}$$

となる。従って、$J(u(t))$ は非増加関数である。このことの時間差分版が次の補題である。

補題 2.1. $\{J(u_n)\} \subset \mathbf{R}_+$, 及び $\{J_n(u_n)\} \subset \mathbf{R}_+$ は、非増大列である。
証明. $J_n(u_n)$ の最小性から、

$$J(u_n) \le J_n(u_n) \le J_n(u_{n-1}) = J(u_{n-1}) \le J_{n-1}(u_{n-1})$$

となる。 □

従って、$\{u_n\} \subset \mathcal{K}$ は $H^1(\Omega)$ の有界集合で、Rellich の定理と $\mathcal{K}$ の弱コンパクト性より

$$u_{n_j} \rightharpoonup u_\infty \quad \text{in} \quad H^1(\Omega), \quad u_{n_j} \to u_\infty \quad \text{in} \quad L^2(\Omega) \quad (j \to \infty) \tag{2.6}$$

なる部分列 $\{u_{n_j}\}$ と $u_\infty \in \mathcal{K}$ が存在する。

定理 2.2. u_∞ によって、$\mathcal{K}$ 上の汎関数 J_∞ を

$$J_\infty(u) = \int_\Omega \frac{|u - u_\infty|^2}{h} dx + J(u)$$

で定義する。u_∞ 自身は、J_∞ の $\mathcal{K}$ での minimizer である。

この定理を示すために、次の補題を示そう。

補題 2.2.

$$\sum_{k=1}^{n} \int_\Omega \frac{|u_k - u_{k-1}|^2}{h} dx + J(u_n) \le J(u_0),$$

特に、

$$\|u_n - u_{n-1}\|_{L^2}^2 \to 0 \quad (n \to \infty)$$

が成立する。

証明. $J_n(u_n)$ の最小性より、$J_n(u_n) \le J_n(u_{n-1}) = J(u_{n-1})$ が成り立つ。これを書き直すと、

$$\int_\Omega \frac{|u_n - u_{n-1}|^2}{h} dx + J(u_n) \le J(u_{n-1})$$

となる。n に関して和をとればよい。 □

注意. (2.5) を t で積分すれば、形式的にエネルギー保存式

$$2\int_0^t \int_\Omega |u_t|^2 dx dt + J(u) = J(u_0)$$

となる。これの時間差分版は、

$$2\sum_{k=1}^{n} \int_\Omega \frac{|u_k - u_{k-1}|^2}{h} dx + J(u_n) = J(u_0)$$

である。補題の主張は第 1 項の係数が 1 である事と、不等式であるという点で弱いものである。但し、J が凸であれば、上の式が不等式で成立する。

補題 2.3 (エネルギー不等式). J が凸であるとすると、

$$2\sum_{k=1}^{n} \int_\Omega \frac{|u_k - u_{k-1}|^2}{h} dx + J(u_n) \le J(u_0)$$

が成立する。

証明. $J_n(u_n)$ の最小性と、J の凸性より $0 \le \theta < 1$ に対し、

$$\int_\Omega \frac{|u_n - u_{n-1}|^2}{h} dx + J(u_n) = J_n(u_n) \le J_n(\theta u_n + (1-\theta)u_{n-1})$$

$$\le \theta^2 \int_\Omega \frac{|u_n - u_{n-1}|^2}{h} dx + \theta J(u_n) + (1-\theta) J(u_{n-1})$$

が成立する。変形して、

$$(1+\theta)\int_\Omega \frac{|u_n - u_{n-1}|^2}{h}dx + J(u_n) \le J(u_{n-1})$$

となる。$\theta \uparrow 1$ として、

$$2\int_\Omega \frac{|u_n - u_{n-1}|^2}{h}dx + J(u_n) \le J(u_{n-1})$$

を得る。n に関して和をとればよい。 □

定理 2.2 の証明.

$$J_\infty(u_\infty) - J_\infty(v) = 3d > 0$$

なる $v \in K$ が存在したとする。(2.6) と補題 2.2 より、

$$\left|J_{n_j}(v) - J_\infty(v)\right| = \left|\int_\Omega \frac{|v - u_{n_j-1}|^2 - |v - u_\infty|^2}{h}dx\right|$$

$$\le \frac{C}{h}\left(\|v\|_{L^2} + \|u_{n_j-1}\|_{L^2} + \|u_\infty\|_{L^2}\right)\left(\|u_{n_j} - u_{n_j-1}\|_{L^2} + \|u_{n_j} - u_\infty\|_{L^2}\right)$$

$$\to 0 \quad (j \to \infty)$$

となる。また、J の下半連続性を用いると、

$$J_\infty(u_\infty) = J(u_\infty) \le \liminf_{j\to\infty} J(u_{n_j}) \le \liminf_{j\to\infty} J_{n_j}(u_{n_j})$$

が得られる。従って、必要ならば部分列を取り直して、

$$\left|J_{n_j}(v) - J_\infty(v)\right| \le d, \quad J_\infty(u_\infty) \le J_{n_j}(u_{n_j}) + d$$

とできる。

一方、$J_{n_j}(u_{n_j})$ の最小性から、

$$J_{n_j}(u_{n_j}) \le J_{n_j}(v) \le J_\infty(v) + d = J_\infty(u_\infty) - 2d \le J_{n_j}(u_{n_j}) - d$$

となり、矛盾が生じる。 □

定理 2.2 より、

$$\operatorname{grad} J_\infty(u_\infty) = 0$$

である。$J_\infty(u)$ の Euler-Lagrange 方程式は、

$${}_{\mathcal{L}^*}\langle \operatorname{grad} J_\infty(u), \varphi\rangle_{\mathcal{L}} = \frac{d}{d\varepsilon}J(u+\varepsilon\varphi)\Big|_{\varepsilon=0} = 2\int_\Omega \frac{\langle u - u_\infty, \varphi\rangle}{h}dx + {}_{\mathcal{L}^*}\langle \operatorname{grad} J(u), \varphi\rangle_{\mathcal{L}}$$

なので、$u = u_\infty$ を代入して次を得る。

系 2.1. $\operatorname{grad} J(u_\infty) = 0$, すなわち、$u_\infty$ はもとの汎関数 J の停留点である。

注意. u_∞ は J の global minimizer とは限らない ([17])。

補題 2.1 より $\lim_{n\to\infty} J(u_n)$ の存在が分かる。

定理 2.3.

$$\lim_{n\to\infty} J(u_n) = J(u_\infty)$$

が成立する。

証明. (2.2) と (2.6) より、

$$J(u_\infty) \le \liminf_{j\to\infty} J(u_{n_j}) = \lim_{n\to\infty} J(u_n)$$

が分かる。一方、$J_{n_j}(u_{n_j})$ の最小性から

$$J(u_{n_j}) \le J_{n_j}(u_{n_j}) \le J_{n_j}(u_\infty) = \int_\Omega \frac{|u_\infty - u_{n_j-1}|^2}{h} dx + J(u_\infty)$$

となる。(2.6) と補題 2.2 より、$j \to \infty$ として、

$$\lim_{n\to\infty} J(u_n) = \lim_{j\to\infty} J(u_{n_j}) \le J(u_\infty)$$

が得られる。 □

これより直ちに次が分かる。

系 2.2. 別の部分列 $\{u_{n_\nu}\}$ をとると、$\bar{u}_\infty$ への収束が言える。($u_\infty = \bar{u}_\infty$ であるかどうかは、一般には不明であるが、)

$$J(u_\infty) = J(\bar{u}_\infty)$$

が成り立つ。

3 1階の方程式の解の構成

まず、(1.2) の弱解の定義を与えよう。

定義 3.1. u が、(1.2) の弱解であるとは、$u \in L^\infty(0,\infty; H^1(\Omega))$, $u_t \in L^2_{\text{loc}}([0,\infty); \mathcal{L}^*)$, $u(t) \in \mathcal{K}$ a. e. t であり、

$$\text{(3.1)} \qquad \begin{cases} u_t = -\dfrac{1}{2}\text{grad} J(u) \quad (t>0) \quad \text{in } L^2_{\text{loc}}([0,\infty); \mathcal{L}^*), \\ u(0) = u_0 \in \mathcal{K} \end{cases}$$

を満たすことをいう。

$u \in L^\infty(0,\infty; H^1(\Omega)) \subset L^2_{\text{loc}}([0,\infty); H^1(\Omega))$ なので (2.3), (2.4) で $v_n = v = u$ とおくことにより、$\text{grad} J(u) \in L^2_{\text{loc}}([0,\infty); \mathcal{L}^*)$ であることが分かる。

離散的勾配流 $\{u_n\}$ を用いて、半無限区間 $(0,\infty)$ 上の関数 $\bar{u}^h$ と u^h を次の様に定義する。$t \in ((n-1)h, nh]$ に対し、

$$
(3.2) \qquad \begin{cases} \bar{u}^h(t) = u_n, \\ u^h(t) = \dfrac{t-(n-1)h}{h} u_n + \dfrac{nh-t}{h} u_{n-1} \end{cases}
$$

とおく。

補題 3.1. $u \in L^\infty(0,\infty;H^1(\Omega))$, $u_t \in L^2(0,\infty;L^2(\Omega))$ を満たす関数 u が存在して、関数 $\bar{u}^h$ と u^h は、$h \downarrow 0$ としたとき適当な部分列に沿って、

$$
\begin{aligned}
\bar{u}^h &\overset{*}{\rightharpoonup} u && \text{in } L^\infty(0,\infty;H^1(\Omega)), \\
u^h &\overset{*}{\rightharpoonup} u && \text{in } L^\infty(0,\infty;H^1(\Omega)), \\
u^h_t &\rightharpoonup u_t && \text{in } L^2(0,\infty;L^2(\Omega))
\end{aligned}
$$

という意味で u に収束する。

証明. 補題 2.2 は、

$$
(3.3) \qquad \int_0^{nh}\int_\Omega \left|u^h_t\right|^2 dxdt + J(\bar{u}^h(nh)) \le J(u_0)
$$

と書き替えられる。J の coersiveness より、$\{u^h\}$ と $\{\bar{u}^h\}$ は $L^\infty(0,\infty;H^1(\Omega))$ の有界集合、$\{u^h_t\}$ は $L^2(0,\infty;L^2(\Omega))$ の有界集合であることが分かる。従って、適当な部分列に沿って

$$
\begin{aligned}
\bar{u}^h &\overset{*}{\rightharpoonup} u && \text{in } L^\infty(0,\infty;H^1(\Omega))\ , \\
u^h &\overset{*}{\rightharpoonup} v && \text{in } L^\infty(0,\infty;H^1(\Omega))\ , \\
u^h_t &\rightharpoonup w && \text{in } L^2(0,\infty;L^2(\Omega))
\end{aligned}
$$

が分かる。$u_t = w$ は容易に示せる。$u = v$ を示そう。

部分列に沿って、

$$
\bar{u}^h - u^h \rightharpoonup u - v \quad \text{in } L^2_{\mathrm{loc}}([0,\infty);L^2(\Omega))
$$

であることも容易である。そこで

$$
\bar{u}^h - u^h \to 0 \quad \text{in } L^2([0,\infty);L^2(\Omega))
$$

を示せばよい。

$$
\left|u^h - \bar{u}^h\right| \le h\left|u^h_t\right|
$$

なので、(3.3) を用いれば、

$$
\int_0^\infty\int_\Omega \left|u^h - \bar{u}^h\right|^2 dxdt \le h^2\int_0^\infty\int_\Omega \left|u^h_t\right|^2 dxdt \le h^2 J(u_0) \to 0 \quad \text{as } h \downarrow 0
$$

となる。 □

補題 3.2. 必要ならば更に部分列を選んで、

$$\bar{u}^h \to u \quad \text{in } L^2_{\rm loc}([0,\infty);H^1(\Omega)) ,$$

$$u^h \to u \quad \text{in } L^2_{\rm loc}([0,\infty);L^2(\Omega))$$

としてよい。特に、

$$u^h \to u \quad \text{a. e. } \Omega\times(0,\infty)$$

としてよい。

証明. 任意の $T>0$ に対し、$\{\bar{u}^h\}$ は $L^2(0,T;H^1(\Omega))$ の有界集合、$\{u^h\}$ は $H^1(\Omega\times(0,T))$ の有界集合である事による。 □

命題 3.1. $u\in\mathcal{K}$ a. e. t である。

証明. 任意の $t>0$ に対し、$\{u^h(t)\}$ は $H^1(\Omega)$ の有界集合であるので、部分列に沿って、

$$u^h(t)\rightharpoonup\tilde{u} \quad \text{in } H^1(\Omega) \text{ かつ、 a. e. } \Omega$$

であるとしてよい。$\mathcal{K}$ の弱コンパクト性より $\tilde{u}\in\mathcal{K}$ である。補題 3.2 より a. e. t に対し、$\tilde{u}=u(t)$ a. e. Ω である。 □

定理 3.1. u は (1.2) の弱解である。

証明. (1.3) より $(\bar{u}^h,u^h)$ は

$$u^h_t=-\frac{1}{2}\mathrm{grad}\,J(\bar{u}^h) \quad \text{in } \mathcal{L}^*$$

を a. e. t で満たす。補題 3.1, (2.4), 命題 3.1 より、

$$u^h_t\rightharpoonup u_t, \quad \mathrm{grad}\,J(\bar{u}^h)\rightharpoonup \mathrm{grad}\,J(u) \quad \text{in } L^2_{\rm loc}([0,\infty);\mathcal{L}^*)$$

となるので、(3.1) の第 1 式は満たされる。第 2 式を確かめよう。補題 3.2 より 0 に収束する列 $\{t_j\}$ が存在して、

$$\|u^h(t_j)-u(t_j)\|_{L^2}\to 0 \quad (h \text{ の部分列} \downarrow 0)$$

となる。よって、

$$\begin{aligned}
\|u(0)-u_0\|^2_{L^2} &= \|u(0)-u^h(+0)\|^2_{L^2}\\
&=\left\|\int_0^{t_j}(u^h_t-u_t)dt-u^h(t_j)+u(t_j)\right\|^2_{L^2}\\
&\le Ct_j\int_0^{t_j}\left(\|u^h_t\|^2_{L^2}+\|u_t\|^2_{L^2}\right)dt+C\|u(t_j)-u^h(t_j)\|^2_{L^2}\\
&\to 0 \quad (h \text{ の部分列} \downarrow 0,\ j\to\infty)
\end{aligned}$$

となる。ゆえに、$u(0)=u_0$ が成り立つ。 □

4　2 階の方程式の解の構成

t について 2 階の方程式

$$
\text{(4.1)} \qquad \begin{cases} u_{tt} = -\dfrac{1}{2}\operatorname{grad} J(u) \quad (t > 0), \\ u(0) = u_0 \in \mathcal{K}, \\ u_t(0) = v_0 \in \mathcal{L} \end{cases}
$$

を同様の考えに基づいて解いてみよう。

$$u_1 = u_0 + hv_0 \in \mathcal{K}$$

とし、u_n を汎関数

$$I_n(u) = \int_\Omega \frac{|u - 2u_{n-1} + u_{n-2}|^2}{h^2} dx + J(u)$$

の $\mathcal{K}$ における minimizer として定義する。u_n は Euler-Lagrange 方程式

$$
\text{(4.2)} \qquad \frac{u_n - 2u_{n-1} + u_{n-2}}{h^2} = -\frac{1}{2}\operatorname{grad} J(u_n) \quad \text{in } \mathcal{L}^*
$$

を満たす。

(4.1) の第 1 式の両辺と $2u_t$ との内積を形式的に計算することにより、エネルギー保存式

$$\int_\Omega |u_t|^2 dx + J(u) = \int_\Omega |v_0|^2 dx + J(u_0)$$

を得る。J が凸ならば、これに相当する不等式が得られる。

補題 4.1 （エネルギー不等式）. J が凸であるとすると、

$$\int_\Omega \frac{|u_n - u_{n-1}|^2}{h^2} dx + J(u_n) \le \int_\Omega |v_0|^2 dx + J(u_0)$$

が成立する。

証明. $I_n(u_n)$ の最小性と、J の凸性より $0 \le \theta < 1$ に対し、

$$\int_\Omega \frac{|u_n - 2u_{n-1} + u_{n-2}|^2}{h^2} dx + J(u_n) = I_n(u_n) \le I_n(\theta u_n + (1-\theta)u_{n-1})$$

$$\le \int_\Omega \frac{|\theta(u_n - u_{n-1}) - u_{n-1} + u_{n-2}|^2}{h^2} dx + \theta J(u_n) + (1-\theta)J(u_{n-1})$$

が成立する。

$$|u_n - 2u_{n-1} + u_{n-2}|^2 - |\theta(u_n - u_{n-1}) - u_{n-1} + u_{n-2}|^2$$

$$= (1-\theta)(u_n - u_{n-1})\{(1+\theta)(u_n - u_{n-1}) - 2(u_{n-1} - u_{n-2})\}$$

$$= (1-\theta)\{(1+\theta)|u_n - u_{n-1}|^2 - 2(u_n - u_{n-1})(u_{n-1} - u_{n-2})\}$$

$$\ge (1-\theta)\{\theta|u_n - u_{n-1}|^2 - |u_{n-1} - u_{n-2}|^2\}$$

に注意すると、

$$\theta\int_\Omega \frac{|u_n-u_{n-1}|^2}{h^2}dx+J(u_n)\le\int_\Omega \frac{|u_{n-1}-u_{n-2}|^2}{h^2}dx+J(u_{n-1})$$

を得る。$\theta\uparrow 1$ としてから、n に関して和をとればよい。 □

半無限区間 $(0,\infty)$ 上の関数 $\bar{u}^h$ と u^h を前節の様に定義する。

補題 4.2. J が凸であるとする。$u\in L^\infty(0,\infty;H^1(\Omega))$, $u_t\in L^\infty(0,\infty;L^2(\Omega))$ を満たす関数 u が存在して、関数 $\bar{u}^h$ と u^h は、$h\downarrow 0$ としたとき適当な部分列に沿って、

$$\bar{u}^h \stackrel{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,\infty;H^1(\Omega)),$$

$$u^h \stackrel{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,\infty;H^1(\Omega)),$$

$$u_t^h \stackrel{*}{\rightharpoonup} u_t \quad \text{in } L^\infty(0,\infty;L^2(\Omega))$$

という意味で u に収束する。

証明. 補題 4.1 と J の coresiveness より、$\{u^h\}$ と $\bar{u}^h$ は $L^\infty(0,\infty;H^1(\Omega))$ の有界集合、$\{u_t^h\}$ は $L^\infty(0,\infty;L^2(\Omega))$ の有界集合であることが分かる。以下の証明は、補題 3.1 の証明と同様である。 □

次の補題は、§ 3 の対応する部分と同様に示される。

補題 4.3. J が凸であるとすると、補題 3.1, 命題 3.1 の主張が成立する。

ここで、(4.1) の弱解の定義を与えよう。

定義 4.1. u が、(4.1) の弱解であるとは、$u\in L^\infty(0,\infty;H^1(\Omega))$, $u_t\in L^\infty([0,\infty);L^2(\Omega))$, $u(t)\in\mathcal{K}$ a. e. t であり、任意の $\varphi\in C_0^1([0,\infty);\mathcal{L})$ に対し、

$$\begin{cases} -\displaystyle\int_0^\infty {}_{\mathcal{L}^*}\langle u_t,\varphi_t\rangle_{\mathcal{L}}dt+\frac{1}{2}\int_0^\infty {}_{\mathcal{L}^*}\langle \operatorname{grad} J(u),\varphi\rangle_{\mathcal{L}}dt=\int_\Omega v_0\varphi(0)dx, \\ u(0)=u_0\in\mathcal{K} \end{cases}$$

を満たすことをいう。

注意. $u\in L^\infty(0,\infty;H^1(\Omega))$ より $\operatorname{grad} J(u)\in L^\infty_{\mathrm{loc}}([0,\infty);\mathcal{L}^*)$ が従う。

定理 4.1. J が凸であるとすると、u は (4.1) の弱解である。

証明. $n \geq 2$ に対し、Euler-Lagrange 方程式 (4.2) が成立している。φ とのペアリングをとって、$n \geq 2$ に関して和をとると、

$$\int_h^\infty {}_{\mathcal{L}^*}\left\langle \frac{u_t^h(t) - u_t^h(t-h)}{h}, \varphi(t) \right\rangle_{\mathcal{L}} dt + \frac{1}{2}\int_h^\infty {}_{\mathcal{L}^*}\langle \mathrm{grad}\, J(\bar{u}^h), \varphi\rangle_{\mathcal{L}} dt = 0$$

となる。補題 4.2 より、部分列に沿って、

$$\int_h^\infty {}_{\mathcal{L}^*}\left\langle \frac{u_t^h(t) - u_t^h(t-h)}{h}, \varphi(t) \right\rangle_{\mathcal{L}} dt$$

$$= \int_h^\infty {}_{\mathcal{L}^*}\left\langle u_t^h(t), \frac{\varphi(t)}{h} \right\rangle_{\mathcal{L}} dt - \int_h^\infty {}_{\mathcal{L}^*}\left\langle u_t^h(t-h), \frac{\varphi(t)}{h} \right\rangle_{\mathcal{L}} dt$$

$$= \int_h^\infty {}_{\mathcal{L}^*}\left\langle u_t^h(t), \frac{\varphi(t)}{h} \right\rangle_{\mathcal{L}} dt - \int_0^\infty {}_{\mathcal{L}^*}\left\langle u_t^h(t), \frac{\varphi(t+h)}{h} \right\rangle_{\mathcal{L}} dt$$

$$= -\int_h^\infty {}_{\mathcal{L}^*}\left\langle u_t^h(t), \frac{\varphi(t+h) - \varphi(t)}{h} \right\rangle_{\mathcal{L}} dt - \frac{1}{h}\int_0^h {}_{\mathcal{L}^*}\langle u_t^h(t), \varphi(t+h)\rangle_{\mathcal{L}} dt$$

$$\to -\int_0^\infty {}_{\mathcal{L}^*}\langle u_t, \varphi_t\rangle_{\mathcal{L}} dt - \int_\Omega v_0 \varphi(0) dx$$

となる。補題 4.2 - 4.3 を用いれば、§ 3 と同様に、

$$\frac{1}{2}\int_h^\infty {}_{\mathcal{L}^*}\langle \mathrm{grad}\, J(\bar{u}^h), \varphi\rangle_{\mathcal{L}} dt \to \frac{1}{2}\int_0^\infty {}_{\mathcal{L}^*}\langle \mathrm{grad}\, J(u), \varphi\rangle_{\mathcal{L}} dt \quad (h \text{ の部分列} \downarrow 0), \quad u(0) = u_0 \in \mathcal{K}$$

が示せる。 □

5 歴史と応用例

離散的勾配流の考えは、1971 年の Rektorys の論文 [23] 中に見られる。彼は、線形の放物型方程式 (2 階に限らない) の解の構成にこの方法を用いた。しかし、残念ながら少なくとも日本では、この論文はあまり注目されなかったようである。

これとは独立に、菊池は、調和写像に対する勾配流の構成にこの方法を適用することを 1991 年の論文 [11] の中で提案した。

Hildebrandt [10] は、親切にも筆者に、1940 年代の Courant の論文の中に離散的勾配流の考えの萌芽が見られると、教えてくれた。しかし、筆者の勉強不足のために、どの論文のどの部分を指しているのか確認できていない。

このように離散的勾配流は、少なくとも 20 年以上の歴史を持つが、実際に非線形問題に適用されだしたのは、[11] 以降である。以下に述べる応用例は、必ずしも、§ 2 で述べた $\mathcal{K}$ や J の仮定は満たしていない。

まず、Bethuel-Coron-Ghidaglia-Soyeur [3] は、B^3 から S^2 への調和写像に対する勾配流の構成にこの方法を適用した。すなわち、

$$\begin{cases} J(u) = \int_{B^3} |\nabla u|^2 dvol_{B^3}, \\ \mathcal{K} = \{u \in H^1(B^3, S^2) \mid \gamma_{\partial B^3} u = \gamma_{\partial B^3} u_0\} \end{cases}$$

である。$\gamma_{\partial B^3}$ は、∂B^3 への trace operator である。勾配流の構成に限れば、コンパクト多様体 M から S^m への調和写像に対し適用可能であり、Chen [4], Chen-Struwe [5], Rubinstein-Steinberg-Keller [25], Coron [6] 等より容易である。

ついで、筆者と小俣は、Alt-Caffarelli-Friedman [1, 2] によって提出された変分問題

$$\begin{cases} J(u) = \int_\Omega \left(|\nabla u|^2 + Q(x)^2 \chi(\{u > 0\})\right) dx, \\ \mathcal{K} = \{u \in H^1(\Omega) \mid \gamma_S u = \gamma_S u_0,\ \chi(\{u > 0\}) \in L^1(\Omega)\} \end{cases} \tag{5.1}$$

に適用した。ここで、Q は $0 < Q_{\min} \le Q(x) \le Q_{\max} < \infty$ なる Ω 上の可測関数、u_0 は $u_0 \in L^\infty(\Omega)$, $J(u_0) < \infty$ なる関数、S は $\mathcal{H}^{m-1}(S) > 0$ なる $\partial\Omega$ 上の部分集合、γ_S は S 上への trace operator である。[17] では、$n \to \infty$ としたときの挙動の解析をし、[18] では、$h \downarrow 0$ としたときの収束を示した。だだし、この汎関数は $C_0^\infty(\Omega)$ 上ですら連続でなく（従ってもちろん Gâteaux 微分可能でない。凸性もない。)、勾配流が well-defined であるかもよく分からない。最近 Caffarelli が、ある単調性の仮定のもとで勾配流を構成した様であるが、preprint 等を入手していないので確認できていない。勾配流を離散的勾配流の方法で構成するためには、(3.2) のような補間では不十分で、次節で述べるような補間法が必要となるであろう。

Zhou [28] は、elastic-plastic problem

$$\begin{cases} J(u) = \int_\Omega \varphi(\nabla u) dx + \int_\Omega |D^s u| - \int_{\partial\Omega} f u \, d\mathcal{H}^{m-1}, \\ \mathcal{K} = BV(\Omega) \end{cases}$$

に対する evolution problem に離散的勾配流を用いた。ここで、u は $BV(\Omega)$ 関数とし、その gradient measure Du を Lebesgue 分解したとき、Lebesgue 測度に関して絶対連続な部分を $\nabla u \, dx$ とし、特異な部分を $D^s u$ とする。また、

$$\varphi(p) = \begin{cases} \frac{1}{2}|p|^2 & (|p| \le 1), \\ |p| - \frac{1}{2} & (|p| \ge 1) \end{cases}$$

とする。

双曲型方程式への応用として、立川 [27] によって、

$$u_{tt} - \Delta u + |u|^{m-2}u = 0 \tag{5.2}$$

を含む双曲系の初期値境界値問題にこの方法が適用された。この場合、

$$\begin{cases} J(u) = \int_\Omega \left(|\nabla u|^2 + \dfrac{2}{m}|u|^m \right) dx, \\ \\ \mathcal{K} = \{u \in H^1(\Omega) \cap L^m(\Omega) \mid \gamma_{\partial\Omega} u = \gamma_{\partial\Omega} u_0\} \end{cases}$$

とし、§ 4 の議論を行う。(5.2) の左辺に摩擦項 u_t や粘性項 $-\Delta u_t$ が加わった問題は、筆者と立川によって [19, 20] で考察されている。また、[21] では、非柱状領域での問題が考察されている。

厳密には離散的勾配流ではないが、同様の方法で Navier-Stokes 方程式の弱解も構成できる ([16])。

$$\begin{cases} J_n(u) = \int_\Omega \left(\dfrac{|u - u_{n-1}|^2}{h} + |\nabla u|^2 \right) dx + 2\rho \left(\int_\Omega \langle u_{n-1} \cdot \nabla u_{n-1}, u \rangle dx \right) - 2 \int_\Omega \langle f, u \rangle dx, \\ \\ \mathcal{K} = V = \{u \in C_0^\infty \mid \mathrm{div}\ u = 0\} \text{ の } H^1(\Omega) \text{ 位相による完備化} \end{cases}$$

とし、§ 3 と同様の考察をすればよい。ここで、ρ は $\mathbf{R}$ 上の滑らかな関数で $x \geq -1$ のとき $\rho(x) = x$, $x \leq -2$ のとき $\rho(x) = 0$ を満たすものとする。

Fonseca-Katsoulakis [8] の generalized mean curvature evolution の構成法は、精神的には離散的勾配流の考えを用いているといってよいであろう。

勉強不足なため、はっきりしたことは言えないが、小林-田中 [13, 14] とも関連しているのかもしれない。

Regularity に関しては、菊池-三沢 [12], 三沢 [15] がある。

離散的勾配流は、De Giorgi が定義した minimizimg movement の一例 [7, Example 2.1] であり、様々な展開が期待される。

6 補間法

この節以降の結果の証明は、割愛する。(3.2) は、離散的勾配流 $\{u_n\}$ の補間である。しかし、$\bar{u}^h$ は補間として粗いものであるし、(5.1) のように u_n が非線形境界条件をみたさねばならない場合、u^h は同じ境界条件を満たさない。このことから、もっと、変分学的な補間を考える必要が出てくる。新しい補間 $\tilde{u}^h$ は、

- $h \downarrow 0$ のとき $\tilde{u}^h$ や u^h の収束先と同じ関数に収束する。
- $J(\tilde{u}^h(t))$ が非増加関数になる。

を満たすように作るべきであろう (第 2 の要請は、(1.2) の解は、(2.5) より形式的には、$J(u(t))$ が非増加関数になることによる)。

離散的勾配流 $\{u_n\}$ やそれを定義する汎関数 J_n は、h にも依存するので、ここでは、それぞれ $\{u_n^h\}$, J_n^h と書こう。$0 < \theta \leq 1$ に対し、

$$J_n^{\theta h}(u) = \int_\Omega \frac{|u - u_{n-1}^h|^2}{\theta h} dx + J(u)$$

の K における minimizer として $u_n^{\theta h}$ を定義する。$(0, \infty)$ 上の関数 $\tilde{u}^h$ を

$$\tilde{u}^h((n-1+\theta)h) = u_n^{\theta h}$$

で定義する。但し、$n \in \mathbf{N}, 0 < \theta \leq 1$ とする。

$$\tilde{u}^h(nh) = u_n^h$$

なので、$\{u_n^h\}$ の補間になっている。この補間は、上の要請を満たしていることを示せる。

定理 6.1. $h \downarrow 0$ のとき、部分列に沿って、

$$\tilde{u}^h \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0, \infty; H^1(\Omega))$$

が成立する。

$$\mathcal{J}^h(t) = J(\tilde{u}^h(t))$$

で関数 $\mathcal{J}^h$ を定義する。

定理 6.2. $\mathcal{J}^h$ は t について、非増加である。

しかし、上の 2 つの要請は $\bar{u}^h$ も満たしており、これだけでは $\bar{u}^h$ の有効性を主張することはできない。そこで、第 3 の要請を与える。

- 定常流などの特別な場合を除き、$J(\tilde{u}^h(t))$ は狭義減少関数になる。

この要請は (2.5) から妥当と思われるし、$J(\bar{u}^h(t))$ は満たさない。上の $\tilde{u}^h$ は第 3 の要請を満たす。

定理 6.3. ある $t_0 \in [0, \infty]$ が存在して、

$$\begin{cases} \mathcal{J}^h(t) \equiv \mathcal{J}^h(0) = J(u_0) \quad (t \in [0, t_0)), \\ \mathcal{J}^h(t) \text{ は } t \geq t_0 \text{ で狭義減少である。} \end{cases}$$

但し、$t_0 = 0$ のとき $[0, t_0) = \emptyset$ と解釈する。

定義. 定理 6.3 の t_0 を待ち時間（waiting time）と呼ぶ。

定理 6.4. $\operatorname{grad} J(u_0) \neq 0$ とすると、$t_0 = 0$ である。

定理 6.5. J が凸であるとする。このとき、$t_0 = 0$ または $t_0 = \infty$ である。$t_0 = \infty$ となるのは、u_0 が $\mathcal{K}$ における $J(u)$ の minimizer のときに限る。また、このとき、

$$\mathcal{J}^h(t) \equiv J(u_0)$$

である。

J が凸でなければ待ち時間 t_0 はいくらでも大きくなり得る（そのような例が作れるという意味）。そこで、次のようにして、最小離散的勾配流を定義し、それから $\mathcal{J}^h$ を定義すれば、$h \in [0, h] \cup \{\infty\}$ である。

定義. $\mathcal{K}$ の部分集合 $\mathcal{K}_n^h$ を

$$\mathcal{K}_n^h = \left\{ u \in \mathcal{K} \mid J_n^h(u) = \inf_{v \in \mathcal{K}} J_n^h(v) \right\}$$

で定義する。離散的勾配流 $\{u_n^h\}$ は定義から $u_n^h \in \mathcal{K}_n^h$ であるが、それ以上の条件はない。そこで、

$$u_n^h \text{ は } \mathcal{K}_n^h \text{ における } J(u) \text{ の minimizer}$$

となるように選ぶ。帰納的に $\{u_n^h\}$ をこのように選んだものを最小離散的勾配流（minimal discrete Morse semiflow）と呼ぶ。

同様に、

$$\begin{cases} \mathcal{K}_n^{\theta h} = \left\{ u \in \mathcal{K} \mid J_n^{\theta h}(u) = \inf_{v \in \mathcal{K}} J_n^{\theta h}(v) \right\}, \\ \\ u_n^{\theta h} \text{ は } \mathcal{K}_n^{\theta h} \text{ における } J(u) \text{ の minimizer} \end{cases}$$

として $\{u_n^{\theta h}\}$ を定義し、最小離散的勾配流を補間する。

$$\mathcal{J}_{\min}^h(t) = J(u_n^{\theta h}) \quad (t = (n - 1 + \theta)h)$$

とかく。

最小離散的勾配流、及びその補間は定義可能であり、次が成立する。

定理 6.6. $\mathcal{J}_{\min}^h$ の待ち時間 t_0 について、次が成り立つ。

1. u_0 が J_1^h の唯一の minimizer である場合、$t_0 = \infty$ である。

2. 1 以外の場合、$0 \leq t_0 \leq h$ である。

調和写像に対する勾配流 [6, 26] を考えても分かるように、(1.2) の弱解に対して $J(u(t))$ は一般に連続にならず、第 1 種の不連続性があらわれる。従って、$J(\bar{u}^h(t))$ が連続になるように補間しようとする試みはあまり意味がない。むしろ不連続性があらわれて自然であり、その解析が重要である。上の補間でも、一般に第 1 種の不連続性があらわれる。しかし、不連続性は変分学的に特徴づけることができる。

定理 6.7. $t=(n-1+\theta)h$ $(n\in\mathbf{N},\ \theta\in(0,1])$ に対し、

$$\mathcal{J}^h(t-0)=\sup_{u\in\mathcal{K}_n^{\theta h}} J(u)$$

が成り立つ。

定理 6.8. $t=(n-1+\theta)h$ $(n\in\mathbf{N},\ \theta\in(0,1])$ に対し、

$$\mathcal{J}^h(t+0)\geq\inf_{u\in\mathcal{K}_n^{\theta h}} J(u)$$

が成り立つ。更に、$\theta\neq 1$ ならば、

$$\mathcal{J}^h(t+0)=\inf_{u\in\mathcal{K}_n^{\theta h}} J(u)$$

が成り立つ。

注意. $t=nh$ に対し、

$$\mathcal{J}^h(t+0)=\inf_{u\in\mathcal{K}_n^{\theta h}} J(u)$$

は必ずしも成立しない。しかし、$\mathcal{J}^h_{\min}(t)$ については次が成り立つ。

定理 6.9. $t=(n-1+\theta)h \quad (n\in\mathbf{N},\ \theta\in(0,1])$ に対し、

$$\mathcal{J}^h_{\min}(t-0)=\sup_{u\in\mathcal{K}_n^{\theta h}} J(u),\quad \mathcal{J}^h_{\min}(t)=\mathcal{J}^h_{\min}(t+0)=\inf_{u\in\mathcal{K}_n^{\theta h}} J(u)$$

が成り立つ。

これらの定理から、J が凸であれば、次が分かる。

定理 6.10. J が凸であれば、$\mathcal{J}^h(t)\equiv\mathcal{J}^h_{\min}(t)$ であり、かつ連続である。

7 数値実験

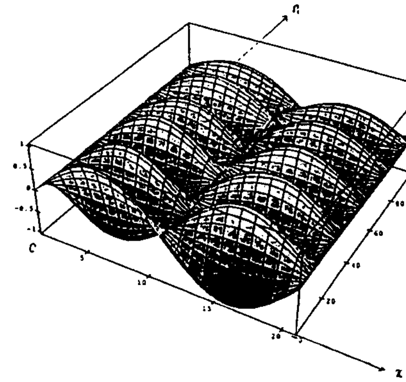
Rektorys の [23] をはじめとする一連の結果は、彼の著書 [24] で集大成されている。それによると、彼が離散的勾配流を考えた背景には、数値解析があったようにみえる。

小俣 [22] は、この方法に基づいて波動方程式を数値的に解く試みをした。次のページの図がそれである。それによると、真の解は減衰しないにもかかわらず、数値解には減衰がみられる。しかも、減衰は h が大きいほど、顕著である。ここでは、それが丸め誤差によるものではなく、この方法特有のものであることが示せる。

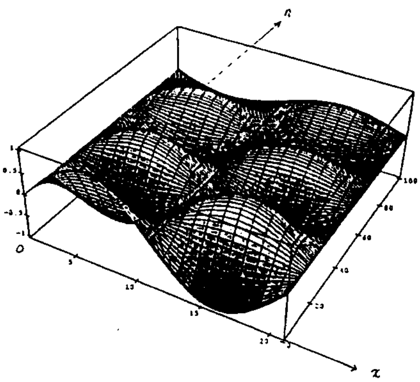
波動方程式の数値解　（小俣 [22] による）

$$\begin{cases} u_{tt} = \Delta u & (0 \le x \le 21,\ t \ge 0), \\ u(0,t) = u(21,t) = 0 & (t \ge 0), \\ u(x,0) = \sin \frac{2}{21}\pi x & (0 \le x \le 21), \\ u_t(x,0) = 0 & (0 \le x \le 21), \end{cases}$$

(A)　$h = 0.1$



(B)　$h = 0.5$



u_n $(n \ge 2)$ を

$$I_n(u) = \int_\Omega \left(\frac{|u - 2u_{n-1} + u_{n-2}|^2}{h^2} + |\nabla u|^2 \right) dx$$

の $H_0^1(\Omega)$ における minimizer とする。ここで、u_0, u_1 は与えられた $L^2(\Omega)$ 関数とする（u_n $(n \ge 2)$ の定義と、以下の議論では $H_0^1(\Omega)$ 関数である必要はない)。

粗く言って、近似解 $\bar{u}_h$, $\hat{u}_h$ は $t \to \infty$ のとき、$e^{-\lambda_1 h t}$ の速さで減衰する。より正確には、以下のようになる。

$\{\varphi_j\}_{j=1}^{\infty}$ を Dirichlet 境界条件下の Laplace operator の固有関数からなる $L^2(\Omega)$ の正規直交系とする。すなわち、

$$\begin{cases} -\Delta \varphi_j = \lambda_j \varphi_j & \text{in} \quad \Omega, \\ \varphi_j = 0 & \text{on} \quad \partial\Omega, \\ \|\varphi_j\|_{L^2} = 1, & \\ 0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots & \end{cases}$$

とする。

定理 7.1. u_0 と u_1 の少なくとも一方は非自明とする。

$$u_\ell = \sum_{j=k_\ell}^{\infty} c_\ell^j \varphi_j \quad (c_\ell^{k_\ell} \neq 0, \quad \ell = 0,\ 1)$$

とおく（$u_\ell = 0$ のとき $k_\ell = \infty$ とする)。

$$k = \min\{k_0, k_1\}$$

で $k \in \mathrm{N}$ を定義する。波動方程式の近似解 $\bar{u}_h$ は、

$$\|\bar{u}_h(t)\|_{L^2} = O\left((1+\lambda_k h^2)^{-\frac{t}{h}}\right) \quad (t \to \infty)$$

を満たす。もう一方の近似解 $\tilde{u}_h$ についても同様である。従って、極限関数は、減衰しない。

熱方程式の近似解について同様の解析をしてみると次の様になる。

定理 7.2. $u_0 \neq 0$ とする。

$$u_0 = \sum_{j=k}^{\infty} c_0^j \varphi_j \quad (c_0^k \neq 0)$$

で $k \in \mathrm{N}$ を定義する。熱方程式の近似解 $\bar{u}_h$ は、

$$\|\bar{u}_h(t)\|_{L^2} = O\left((1+\lambda_k h)^{-\frac{t}{h}}\right) \quad (t \to \infty)$$

を満たす。もう一方の近似解 $\tilde{u}_h$ についても同様である。

従って、近似解と真の解の減衰の速さは同等である。

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Comparison Theorem of Eigenvalues of the NonLocal Cahn-Hilliard Equations

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1 Introduction

In this note we consider the 4-th order parabolic equation (1), which was introduced to describe the dynamics of micro-phase separation of diblock copolymer.

$$(1)\qquad \begin{aligned} u_t &= \Delta\{-\varepsilon^2\Delta u - f(u) + \sigma(-\Delta_N)^{-1}(u - \overline{u_0})\} && \text{in } \Omega,\\ &= \Delta\{-\varepsilon^2\Delta u - f(u)\} - \sigma(u - \overline{u_0}) && \text{in } \Omega,\\ u(x,0) &= u_0(x) && \\ \frac{\partial u}{\partial n} &= \frac{\partial \Delta u}{\partial n} = 0 && \text{on } \partial\Omega,\\ \frac{1}{|\Omega|}\int_\Omega u_0(x)dx &= \overline{u_0}: && \text{a given constant,} \end{aligned}$$

where we define $f(u)$ by $f(u) := -W'(u)$, W is defined below, and n is the unit outward normal to $\partial\Omega$. It is clear that (1) is a conservative equation under the above boundary conditions. This is a gradient flow of the following functional in $H^{-1}(\Omega)$;

$$(2)\qquad \begin{aligned} F_{\varepsilon,\sigma}(u) &:= \int_\Omega \{\frac{\varepsilon^2}{2}|\nabla u|^2 + W(u) + \frac{\sigma}{2}|(-\Delta_N)^{-\frac{1}{2}}(u-\overline{u})|^2\}dx,\\ \overline{u} &:= \frac{1}{|\Omega|}\int_\Omega u dx, \qquad u \in H^1(\Omega), \end{aligned}$$

where Ω is a smooth bounded domain in $\mathbf{R}^n$, $W(u)$ is a double-well potential with global minima $u = \pm 1$, typically of the form $\frac{1}{4}(u^2-1)^2$, ε and σ are positive constants, $H^1(\Omega)$ is the usual Sobolev space of functions with square-integrable derivatives up to 1 in Ω, and $(-\Delta_N)^{-\frac{1}{2}}$ is a fractional power of the Laplace operator under the zero flux boundary condition. (The underlying space for the Laplace operator is the subspace of $L^2(\Omega)$ orthogonal to constants. See Henry [14] for details.) The third term is of nonlocal, since $(-\Delta_N)^{-\frac{1}{2}}$ is, roughly speaking, an integral operator in Ω. Without this term, (2) becomes a well-known functional from which we can derive the Allen-Cahn (non-conserved) and the Cahn-Hilliard (conserved) equations.

The functional (2) was essentially introduced by [21] and [5] (and the references therein) in order to describe the micro-phase separation of diblock copolymer where two different homopolymers are connected and this connectivity is responsible for introducing the long range interaction, i.e., the nonlocal term of (2). The parameter σ is inversely proportional to the square of the total chain length N of the copolymer, and ε represents the interfacial thickness at the bonding point assumed to be sufficiently small, and the average $\overline{u}$ $(-1 < \overline{u} < 1)$ stands for the ratio of components of two homopolymers. The number N is in general quite large, hence σ also becomes very small. In this note we focus on a scaling regime $0 < \varepsilon \ll \sigma \ll 1$. The above micro constraint (connectivity) prevents copolymer from forming a large domain and hence usual coarsening process stops at certain stage of mesoscopic level. Namely, (2) has a potential to have a variety of metastable states (local minimizers) with fine structures, which is not the case for the usual Cahn-Hilliard dynamics, although it has a long and interesting coarsening process. When one tries to minimize the functional (2), one easily see that there is a competition between the first gradient term and the third nonlocal term, assuming that u is close to 1 or -1 off the interface. The first term wants to minimize the area of interface, however the nonlocal term does not become small if u takes 1 or -1 in a large domain. In order to make the third term small, u has to oscillate rapidly around $\overline{u}$ (which increases the area of interface), in other words if $u - \overline{u}$ converges to zero in weak sense in $L^2(\Omega)$, it goes to zero because of the compactness of the operator $(-\Delta_N)^{-\frac{1}{2}}$. Thus there should be an optimal domain size compromizing these two opposite tendencies.

Experimentally and numerically it is well-known in copolymer problems that the final asymptotic states prefer periodic structures such as lamellar, spherical, double-diamond geometries and so on (see, for instance [11], [12], [13], [5]).

Depending on the ratio of two subchains of diblock copolymer (i.e., $\overline{u_0}$), there appears a variety of morphology for the asymptotic steady states of (2). For instance, it is confirmed experimentally (see [12]) that double-diamond structure appears near $\overline{u_0} = 0$ besides lamellar and cylinderical structures, and many other types of morphology could be discovered even for the same values of $\overline{u_0}$. This suggests the coexistence of multiple stable steady states, although the basin of attraction of each morphology changes depending on $\overline{u_0}$. In order to clarify the global morphology diagram, firstly we have to know the solution structure of Morphology Equation, which is discovered in Nishiura and Ohnishi [23], especially $\overline{u_0}$-dependency of each morphology. Secondly we have to investigate stability and basin of attraction of each structure, which is a clue to obtain the global diagram. However it is quite hard to measure the size of each basin, hence it is expected that the modulus of most dangerous spectrum plays the role of measuring the strength of stability instead.

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2 Main Result

The evolutional equation associated with the functional of (2) can be obtaind by taking a gradient operator in some function space. However in order to make the resulting equation a conserved one of local operator form, the usual $L^2(\Omega)$ is not an appropriate space. According to Fife [9], $H^{-1}(\Omega)$ is a nice space for our purpose and the resulting equation becomes (1). On the other hand, if we take a gradient in $L^2(\Omega)$ without taking into account mass conservation, we have the following reaction-diffusion equation of nonlocal type:

$$
(3)\qquad \begin{aligned} u_t &= \varepsilon^2\Delta u + f(u) - \sigma(-\Delta_N)^{-1}(u - \overline{u}(t)), && \text{in } \Omega,\\ u(x,0) &= u_0(x), &&\\ \tfrac{\partial u}{\partial n} &= 0, && \text{on } \partial\Omega, \end{aligned}
$$

where $\overline{u}(t) := \frac{1}{|\Omega|}\int_\Omega u(x,t)dx$.

Let $w(x)$ be a stationary solution of (2), i.e., a solution of

$$
(4)\qquad \begin{aligned} \varepsilon^2\Delta w + f(w) - \sigma(-\Delta_N)^{-1}(w - \overline{u_0}) &= \text{ const. }, && \text{in } \Omega,\\ \tfrac{\partial w}{\partial n} &= 0, && \text{on } \partial\Omega,\\ \tfrac{1}{|\Omega|}\int_\Omega w(x)dx &= \overline{u_0}. && \end{aligned}
$$

We define $q(x)$ as $f'(w(x))$. Then the linearized eigenvalue problem of (2) about $w(x)$ is

$$
(5)\qquad \begin{aligned} -\Delta\{-\varepsilon^2\Delta - q(x)I + \sigma(-\Delta_N)^{-1}\}\phi &= \lambda\phi, && \text{in } \Omega,\\ \tfrac{\partial\phi}{\partial n} = \tfrac{\partial\Delta\phi}{\partial n} &= 0, && \text{on } \partial\Omega,\\ \int_\Omega \phi dx &= 0. && \end{aligned}
$$

Similarly the associated eigenvalue problem with (3) is given by

$$
(6)\qquad \begin{aligned} -\varepsilon^2\Delta\eta - q(x)\eta + \sigma(-\Delta_N)^{-1}(\eta - \overline{\eta}) &= \mu\eta, && \text{in } \Omega,\\ \tfrac{\partial\eta}{\partial n} &= 0, && \text{on } \partial\Omega. \end{aligned}
$$

In order to consider the stability property of steady states of (2), we study the spectral behavior of the associated linearized problem (5) for small ε and σ. However it is quite difficult to treat (5) directly. Hence we take advantage of smallness of ε and σ. Since we are interested in the scaling regime $0 < \varepsilon \ll \sigma \ll 1$, our strategy is as follows: firstly we study the limiting case of $\varepsilon \searrow 0$, and then extract the information on the spectral behavior useful for positive ε. It should be noted that stability in a special class of periodic lattice does NOT imply the stability to the whole class of perturbations. This apparently requires much efforts to controle the spectrum in a rigorous way, because the number of critical eigenvalues, (i.e., those which tend to zero as $\varepsilon \searrow 0$) increases drastically.

Now we state our main result of this note. The following spectral comparison result becomes quite useful to controle the spectrum of (5), which has the same form as Theorem 8 of [4].

Theorem 2.1 *Let $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ be the eigenvalues of the problem (5) and (6), and K and K_0 be the number of the negative eigenvalues of (5) and (6), respectively.*

Then, for any $n = 1, 2, 3, \cdots\cdots$, we have the following:

- *The inequality $\mu_{n+1} \le 0$ implies that $\lambda_n \le \Lambda_1 \mu_{n+1}$.*
- *The inequality $\mu_n \ge 0$ implies that $\lambda_n \ge \Lambda_1 \mu_n$.*
- *$K = K_0 - 1$ or K_0.*

Here Λ_1 has been defined as the first positive eigenvalue of the Laplacian with zero flux (Neumann-0) boundary condition.

The main tool for proving this is the mini-max (and dual) characterization of eigenvalues of our eigenvalue problems. For the non-conserved case, we have the canonical mini-max (and dual) characterization of eigenvalues $\{\mu_n\}_{n=1}^{\infty}$ of (6) as follows:

$$\mu_n = \max_{T_n \in \mathbf{T}_n} \quad \min_{\phi \in T_n^{\perp} \cap H^1(\Omega)} R_0(\phi)$$
$$= \min_{T_n \in \mathbf{T}_n} \quad \max_{\phi \in T_n \cap H^1(\Omega)} R_0(\phi),$$

where

$$R_0(\phi) := \frac{\int_\Omega \{\varepsilon^2 |\nabla\phi|^2 - f'(w(x))\phi^2 + \sigma |(-\Delta_N)^{-\frac{1}{2}}(\phi - \overline{\phi})|^2\} dx}{\int_\Omega \phi^2 dx},$$

$$\mathbf{T}_n := \{T_n \,;\, T_n \text{ is a } n-1\text{-dimensional subspace of } L^2(\Omega)\},$$

and $w(x)$ is a stationary solution (see Courant-Hilbert [8]).

On the other hand, for the conserved case (5), we encounter a difficulty that the principal part of the differential operator defining the eigenvalue problem

(5) is NOT self-adjoint. Nevertheless, based on the idea of [4], we can rewrite it in the self-adjoint form. This allows us the following mini-max (and dual) characterization of the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$:

$$\begin{aligned}\lambda_n &= \max_{S_n \in \mathbf{S}_n} \quad \min_{\phi \in S_n^{\perp} \cap H^1(\Omega) \cap H_0} R(\phi)\\ &= \min_{S_n \in \mathbf{S}_n} \quad \max_{\phi \in S_n \cap H^1(\Omega) \cap H_0} R(\phi),\end{aligned}$$

where

$$\begin{aligned} R(\phi) &:= \frac{\int_\Omega \{\epsilon^2|\nabla\phi|^2 - f'(w(x))\phi^2 + \sigma|(-\Delta_N)^{-\frac{1}{2}}(\phi)|^2\}dx}{\int_\Omega |(-\Delta_N)^{-\frac{1}{2}}(\phi)|^2 dx},\\ \mathbf{S}_n &:= \{S_n\,;\, S_n \text{ is a } n-1\text{-dimensional subspace of } H_0\},\\ H_0 &:= \{\phi \in L^2(\Omega)\,;\, \textstyle\int_\Omega \phi dx = 0\}.\end{aligned}$$

Using the above two (and two dual) characterizations, we easily see the comparison theorem. Theorem 2.1 is an extension of [4] to the case of $\sigma > 0$. The point is that the additional operator $\sigma(-\Delta_N)^{-1}$ is compact and small, and the structure of the eigenvalues and eigenfunctions of the problem are invariant under small compact perturbations. For details and related topics, see the forthcoming paper, Nishiura and Ohnishi [19].

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EXISTENCE AND NONEXISTENCE OF GLOBAL SOLUTIONS TO FAST DIFFUSIONS

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1. 導入と結果

次の初期値問題の正値解について考える。

$$(1.1) \qquad \begin{cases} \partial_t u = \Delta u^m + u^p & (x,t) \in \mathbf{R}^N \times (0,T) \\ u(x,0) = u_0(x) & x \in \mathbf{R}^N \end{cases}$$

ここで、

$$max\{0, 1 - \frac{2}{N}\} < m < 1 < p$$

$$u_0(x) \geq 0 \quad , u_0(x) \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$$

とする。

このとき、解の存在は、Brezis-Crandall[1] による.

解の存在.

十分小さな $T > 0$ に対して、

$$\exists_1 u(x,t) \in C([0,T] : L^1(\mathbf{R}^N)) \cap L^\infty(\mathbf{R}^N \times [0,T)).$$

ここで、$T = \infty$　の時、u は大域解 (global) であるといい、

$$\|u(\cdot,t)\|_{L^\infty} \to \infty \quad (t \to T)$$

の時、u は有限時間で爆発 (blow up) すると言う。

$p_m^* \equiv m + \frac{2}{N}$ とおく。このとき、つぎの結果が得られる。

定理1. $1 < p \leq p_m^*$ のとき、*(1.1)* の全ての正値解は有限時間で爆発する。

定理2. $p > p_m^*$ のとき、

$$u_0(x) = O(|x|^{-\frac{2}{1-m}}) \quad (|x| \to \infty)$$

なる十分小さな $u_0(x) \in L^\infty(\mathbf{R}^N)$　に対して *(1.1)* の大域解が存在して、

$$\sup_{x \in \mathbf{R}^N} u(x,t) \leq C t^{-\frac{1}{p-1}}, \quad C > 0$$

が成り立つ。

2. 準備

弱解の定義. $u(x,t) \in \mathbf{R}^N \times [0,T)$ が *(1.1)* の弱解であるとは、

(i) 任意の $0 < T' < T$ に対して、

$$u(x,t) \geq 0, u(x,t) \in C([0,T']; L^1(\mathbf{R}^N)) \cap L^\infty(\mathbf{R}^N \times [0,T'])$$

(ii) 任意の有界領域 $G \subset \mathbf{R}^N$, 任意の $0 \leq \tau < T$ と、$\varphi(x,t)|_{\partial G} = 0$ なる非負な $\varphi(x,t) \in C^2(\bar{G} \times [0,T))$ に対して、

(2.1)

$$\int_G u(x,\tau)\varphi(x,\tau)dx - \int_G u(x,0)\varphi(x,0)dx$$
$$= \int_0^\tau \int_G \{u\varphi_t + u^m \Delta\varphi + u^p\varphi\} dxdt - \int_0^\tau \int_{\partial G} u^m \partial_n \varphi dSdt,$$

ここで、n は、境界上の単位法線ベクトル

が成り立つときに言う。

つぎに、比較定理について述べる。

命題 2.1. $u[v]$ を *(1.1)* の *supersolution[subsolution]* とする。このとき、$u(x,0) \geq v(x,0)$ ならば、$\mathbf{R}^N \times [0,T)$ で、$u \geq v$ が成り立つ。

$\varphi(x,t) \in C^2(\mathbf{R}^N \times [0,T))$ を test function として、(2.1) より、

(2.2)
$$\int_{\mathbf{R}^N} u(x,\tau)\varphi(x,\tau)dx - \int_{\mathbf{R}^N} u(x,0)\varphi(x,0)dx$$
$$= \int_0^\tau \int_{\mathbf{R}^N} \{u\varphi_t + u^m\Delta\varphi + u^p\varphi\} dxdt$$

が得られる。ここで、$\varphi = e^{-\epsilon|x|^2}$, $\epsilon > 0$ とおくと、

(2.3)
$$\int_{\mathbf{R}^N} u(x,\tau)e^{-\epsilon|x|^2}dx - \int_{\mathbf{R}^N} u(x,0)e^{-\epsilon|x|^2}dx$$
$$= \int_0^\tau \int_{\mathbf{R}^N} \{(-2N\epsilon + 4\epsilon^2|x|^2)u^m + u^p\}e^{-\epsilon|x|^2} dxdt$$

が成り立つ。

定理 1 の証明は、以下の命題に基づいて行われる。

命題 2.2. u を *(1.1)* の解とする。このとき、任意の $\tau \in (0,T)$ に対して、

(2.4)
$$\int_{\mathbf{R}^N} u(x,\tau)dx - \int_{\mathbf{R}^N} u_0(x)dx = \int_0^\tau \int_{\mathbf{R}^N} u(x,t)^p dxdt$$

が成り立つ。さらに、

$$(2.5)\qquad \int_{\mathbf{R}^N} u(x,t)dx \le e^{a(\tau)t}\int_{\mathbf{R}^N} u_0(x)dx, \quad t\in[0,\tau]$$

が成り立つ。ここで、

$$a(\tau) = \sup_{(x,t)\in\mathbf{R}^N\times(0,\tau)} u(x,t)^{p-1}$$

である。

$J(t),\quad t\ge 0$ を、次のように定義する。

$$J(t) \equiv \left(\int_{\mathbf{R}^N} e^{-\varepsilon|x|^2}dx\right)^{-1}\int_{\mathbf{R}^N} u(x,t)e^{-\varepsilon|x|^2}dx$$

命題 2.3. u_0 を、

$$J(0) > (2N\varepsilon)^{\frac{1}{p-m}}$$

を満たすように十分大とする。
このとき、$T>0$ が存在して、

$$\sup_{x\in\mathbf{R}^N} u(x,t)\to\infty \quad (t\to T)$$

である。

ここで、つぎの初期値問題の解である Barenblatt Solution について述べる。

$$(2.6)\qquad \begin{cases} \partial_t v = \Delta v^m & (x,t)\in\mathbf{R}^N\times(0,\infty) \\ v(x,0) = L\delta(x) & x\in\mathbf{R}^N \end{cases}$$

ここで、$L>0,\quad \delta(x)$ は Dirac の δ- function。

$$l = (m-1+\frac{2}{N})^{-1} = (p_m^*-1)^{-1} > 0$$

とおき、

$$G_m(x) = [A+Bs^2]^{-\frac{1}{1-m}}$$

とおく。
ここで、$B=\frac{(1-m)l}{2mN}>0,\quad A$ は、$\int_{\mathbf{R}^N} G_m(|x|)dx = 1$ を満たす正の数とする。
この解について、次の性質が良く知られている。

命題 2.4. *(2.6)* の解は、

$$(2.7)\qquad E_m(x,t;L) = L(L^{m-1}t)^{-l}G_m(|x|(L^{m-1}t))^{-\frac{l}{N}}$$

で与えられる。
さらに、任意の $k>0$ に対して、

$$(2.8)\qquad k^N E_m(kx,k^{\frac{N}{l}};L) = E_m(x,t;L), \qquad k>0$$

が成り立つ。

3. 定理1の証明

補題 3.1. u を *(1.1)* の大域解とする。
このとき、任意の $t \geq 0, \varepsilon > 0$ に対して、

$$\int_{\mathbf{R}^N} u(x,t)e^{-\varepsilon|x|^2}dx \leq C(N)\varepsilon^{-\frac{N}{2}+\frac{1}{p-m}} \tag{3.1}$$

が成り立つ。
ここで、$C(N) = \pi^{\frac{N}{2}}(2N)^{\frac{1}{p-m}}$ である。

補題 3.2. u を *(1.1)* の大域解とする。
$1 < p < p_m^*$ のとき、任意の $t \geq 0$ に対して、

$$\int_{\mathbf{R}^N} u(x,t)dx = 0 \tag{3.2}$$

が成り立ち、
$p = p_m^*$ のとき、任意の $t \geq 0$ に対して、

$$\int_{\mathbf{R}^N} u(x,t)dx \leq C(N) \tag{3.3}$$

が成り立つ。

補題 3.3. u を *(1.1)* の大域解とする。
$p = p_m^*$ のとき、任意の $t > 0$ に対して、

$$\int_0^t \int_{\mathbf{R}^N} u(x,\tau)^p dx d\tau \leq C(N) \tag{3.4}$$

が成り立つ。

つぎの補題は、Friedman-Kamin[2] による。

補題 3.4. $u_0(x) \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ とし、v を、初期値問題、

$$\begin{cases} \partial_t v = \Delta v^m & (x,t) \in \mathbf{R}^N \times (0,\infty) \\ v(x,0) = u_0(x) & x \in \mathbf{R}^N \end{cases} \tag{3.5}$$

の解とする。
このとき、$v_k(x,t) = k^N v(kx, k^{\frac{N}{T}}t)$ とおくと、v_k は、

$$\begin{cases} \partial_t v = \Delta v^m & (x,t) \in \mathbf{R}^N \times (0,\infty) \\ v(x,0) = k^N u_0(kx) & x \in \mathbf{R}^N \end{cases} \tag{3.6}$$

を満たし、v_k は $\mathbf{R}^N \times (0,\infty)$ で局所一様に $E_m(x,t;L)$ に収束する；

$$v_k(x,t) \to E_m(x,t;L) \quad (k \to \infty) \tag{3.7}$$

ここで、$L = \int_{\mathbf{R}^N} u_0(x)dx$ である。

定理 *1* の証明. $u_0(x) \equiv 0$ に対して、初期値問題 (1.1) が非自明な大域解を持つと仮定して矛盾を導く。

$1 < p < p_m^*$ の場合は、補題 3.2 より $u(x,t) \equiv 0$ となる。よって、$p = p_m^*$ のときを考える。

$$(3.8) \qquad u_k(x,t) = k^N u(kx, k^{\frac{N}{l}} t)$$

とおくと、$u_k(x,t)$ は、次の初期値問題、

$$(3.9) \qquad \begin{cases} \partial_t u_k = \Delta(u_k)^m + u_k^p & (x,t) \in \mathbf{R}^N \times (0,\infty) \\ u_k(x,0) = k^N u_0(kN) & x \in \mathbf{R}^N \end{cases}$$

の解である。

u_k は大域解なので、補題 3.3 より、

$$(3.10) \qquad \int_\delta^t \int_{\mathbf{R}^N} u_k(x,\tau)^p dx d\tau \le C(N), \quad t > 0$$

が成り立つ。

定義より、$\mathbf{R}^N \times (0,\infty)$ で $v_k(x,t) \le u_k(x,t)$ である。したがって、補題 3.4 と、Fatou の補題より、任意の $0 < \delta < t$ に対して、

$$(3.11) \qquad \int_\delta^t \int_{\mathbf{R}^N} E_m(x,\tau;L)^p dx d\tau \le \liminf_{k\to\infty} \int_\delta^t \int_{\mathbf{R}^N} v_k(x,\tau)^p dx d\tau \le C(N)$$

が成り立つ。ところで、命題 2.4 より、

(3.12)
$$\int_\delta^t \int_{\mathbf{R}^N} E_m(x,\tau;L)^p dx d\tau = L^p \int_\delta^t (L^{m-1}\tau)^{-pl}(L^{m-1}\tau)^l d\tau \int_{\mathbf{R}^N} G_m(|x|)^p dx$$

が成り立つ。

$p = p_m^*$ なので、$-l(p-1) = -1$ である。よって、上式の右辺は $\delta \to 0$ または、$t \to \infty$ としたとき、∞ に発散する。

これは、(3.11) に矛盾である。よって、定理 1 が証明された。 □

4. 定理 2 の証明

$p > p_m^*$ とする。定理 2 は、次の形の supersolution を構成して証明する。

$$Z(x,t) = (t+t_0)^{-\alpha}[a + b|x|^2(t+t_0)^{-2\nu\alpha}]^{-\frac{1}{1-m}}$$

ここで、$t_0, a, b, \alpha > 0$ である。

$X = a + b|x|^2(t+t_0)^{-2\nu\alpha}$ とおく。Z は、$\partial_t Z - \Delta Z^m \ge Z^p$ をみたすので、

$$(4.1) \qquad X^{-\frac{1}{1-m}-1}\{-\alpha a(t+t_0)^{-\alpha-1} + \frac{2mNab}{1-m}(t+t_0)^{-m+2\nu)\alpha}$$

$$-\alpha(1-\frac{2\nu}{1-m})b|x|^2(t+t_0)^{-(1+2\nu)\alpha-1}+\frac{2mb^2}{1-m}(N-\frac{2}{1-m})|x|^2(t+t_0)^{-(m+4\nu)\alpha}\}$$

$$\geq X^{-\frac{1}{1-m}-1}(t+t_0)^{-\alpha p}[a+b|x|^2(t+t_0)^{-2\nu\alpha}]^{-\frac{p+m-2}{1-m}}.$$

が成り立つ。ここで、$\alpha+1=(m+2\nu)\alpha=\alpha p$,i,e.,$\alpha=\frac{1}{p-1}, \nu=\frac{p-m}{2}$ とおくと、

$$(\text{左辺}) = X^{-\frac{1}{1-m}-1}(t+t_0)^{-\alpha p}\{-\frac{a}{p-1}+\frac{2mNab}{1-m}+\frac{1}{1-m}X$$

$$+\frac{2mb}{1-m}(N-\frac{2}{1-m})X-\frac{a}{1-m}-\frac{2mab}{1-m}(N-\frac{2}{1-m})\}.$$

である。よって、(4.1) より、

$$F(X)\equiv X^{-\frac{p+m-2}{1-m}}-\frac{1}{1-m}\{1+2mb(N-\frac{2}{1-m})\}X$$

$$+\frac{a}{1-m}\{\frac{p-m}{p-1}-\frac{4mb}{1-m}\}\leq 0.$$

を得る。上の不等式は、$F(a)\leq 0, F'(X)\leq 0 \quad \text{in} X\geq a$ を満たす a,b に対して成り立つ。$F(a)\leq 0$ より

(4.2) $$0<a^{-\frac{p-1}{1-m}}\leq\frac{1}{1-m}\{2mNb-\frac{1-m}{p-1}\}$$

が得られ、$F'(X)\leq 0$ より

(4.3) $$F'(X)=-\frac{p+m-2}{1-m}X^{-\frac{p-1}{1-m}}-\frac{1}{1-m}\{1-\frac{2mbN(p_m^*-1)}{1-m}\}\leq 0 \quad \text{in} X\geq a$$

が得られる。

$p+m-2\geq 0$ のとき、(4.3) より、

(4.4) $$b\leq\frac{1-m}{2mN(p_m^*-1)}$$

$p+m-2<0$ のとき、(4.3) より、

(4.5) $$0<a^{-\frac{p-1}{1-m}}\leq\frac{-(1-m)+2mbN(p_m^*-1)}{(1-m)(p+m-2)}$$

が得られる。

これらの結果より、つぎの補題を得る。

補題 4.1.

(i) $p+m-2\geq 0$ のとき、

$$\frac{1-m}{2mN(p-1)}<b\leq\frac{1-m}{2mN(p_m^*-1)}$$

とおく。このとき、a を十分大きく選ぶと、*(4.2)* と *(4.4)* が成り立つ。

(ii) $p+m-2<0$ のとき、

$$\frac{1-m}{2mN(p-1)}<b<\frac{1-m}{2mN(p_m^*-1)}$$

とおく。このとき、a を十分大きく選ぶと、*(4.2)* と *(4.5)* が成り立つ。

注意. *(ii)* は、

$$\max\{0, 1-\frac{2}{N}\} < m < 1-\frac{1}{N}, \quad p_m^* < p < 2-m$$

のとき起こり、*(i)* は、他の $\{m,p\}$ に対して起こる。

定理2の証明. $t_0 > 0$ とし、$\{a,b\}$ を、補題 4.1 の条件を満たす正の数とする。このとき、

$$Z(x,t) = (t+t_0)^{-\frac{1}{p-1}}[a+b|x|^2(t+t_0)^{-\frac{p-m}{p-1}}]^{-\frac{1}{1-m}}$$

は、(1.1) の supersolution となる。

$$u_0(x) \le C(1+|x|^2)^{-\frac{1}{1-m}}$$

とおく。$t_0 > 0, C > 0$ を、

$$0 < t_0 \le (a^{-1}b)^{\frac{p-1}{p-m}}, \quad C \le (b^{-1}t_0)^{\frac{1}{1-m}}$$

を満たすように十分小さくとると、

$$0 \le u_0(x) \le Z(x,0), \quad x \in \mathbf{R}^N$$

を得る。u を (1.1) の解とすると、比較定理により、

$$u(x,t) \le Z(x,t)$$

を得る。これと、補題 2.1 の (2.5) より大域解の存在が言えて、定理2の decay order が得られる。 □

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ON THE CRITICAL POWER AND DECAY TO THE CAUCHY PROBLEM FOR WAVE EQUATIONS

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1. Introduction and Statements of Results. In this note we consider the initial value problem to semilinear wave equations:

$$(1.1)\qquad \begin{aligned} &u_{tt} - \Delta_x u = F(u) \quad \text{in} \quad \mathbb{R}^n_x \times [0,\infty), \\ &u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \quad \text{for} \quad x \in \mathbb{R}^n, \end{aligned}$$

where u is a real valued function and $\phi \in C^{[n/2]+2}(\mathbb{R}^n)$, $\psi \in C^{[n/2]+1}(\mathbb{R}^n)$. To begin with, we mention two typical examples of the nonlinear term $F(u)$. First one is $F(u) = -u|u|^{p-1}$ with $p > 1$. In this case it is partially proved that if

$$(1.2)\qquad 1 < p \leq \frac{n+2}{n-2},$$

the initial value problem (1.1) adomits a unique global solution even though the initial data are large. (See also [8], [10], [15] and [20]).

The other one is $F(u) = |u|^p$ with $p > 1$. It is known that if the initial data are large in certain sense, the solution to the initial value problem (1.1) blows up in finite time. (See e.g. [5]). Therefore it is necessary to exist a global solution that the initial data are sufficiently small. However under this smallness assumption on the initial data, F. John [9] obtained such a remarkable result that if $1 < p < p_0(n)$, any nontrivial solution blows up in finite time and that if $p > p_0(n)$, then a unique global solution exists, when $n = 3$ and the initial data are compactly supported. Here $p_0(n)$ is the positive root of the quadratic equation:

$$(1.3)\qquad (n-1)p^2 - (n+1)p - 2 = 0.$$

When $2 \leq n \leq 4$, we know the criticality of the value $p_0(n)$ in the above sence. (See for instance [6], [7], and [25]). Moreover when $n = 2, 3$, J. Schaeffer [18] proved blow-up of soltions if $p = p_0(n)$. T.C. Sideris [19] showed that there is no global solutioin if $1 < p < p_0(n)$ in general space dimensions. On the other hand, when $n \geq 5$, only for the large values of p, we know some global existence results established by [3], [4], [16] and [17].

We now tern our attention to the decay rate of the initial data, because it plays an important role to determine a global behavior of the solutions as well as the power p. Let the initial data satisfy either

$$(1.4) \qquad \sum_{|\alpha| \leq [n/2]+2} |\partial_x^\alpha \phi(x)| + \sum_{|\alpha| \leq [n/2]+1} |\partial_x^\alpha \psi(x)| \leq \varepsilon (1 + |x|)^{-(\kappa+1)}$$

or

$$(1.5) \qquad \phi \equiv 0, \quad \psi(x) \geq \varepsilon (1 + |x|)^{-(\kappa+1)},$$

where κ and ε are positive parameters. When $n = 3$, F. Asakura [2] proved that if $\kappa > \kappa_0 = 2/(p-1)$ and $p > p_0(n)$, a unique global solution exists by assuming (1.4) with ε sufficiently small. Moreover it is proved that if $\kappa < \kappa_0$ and (1.5) holds, any classical solution blows up in finite time. This result extended to two space dimensional case. And for the critiacal case $\kappa = \kappa_0$, the global existence results obtained when $n = 2, 3$. (See [1], [14] and [22-24]). Moreover when $n \geq 4$, H. Takamura [21] proved the blow-up result if $0 < \kappa < \kappa_0$. Furthermore we know the following upper bound of the lifespan, which is defined precisely below:

$$(1.6) \qquad T_\varepsilon \leq C \varepsilon^{-(p-1)/(2-(p-1)\kappa)}, \quad C = C(n, p, \kappa) > 0 \quad \text{if} \quad 0 < \kappa < \kappa_0.$$

The aim of this note is to research the case where $\kappa \geq \kappa_0$, provided the initial data are radially symmetric and the space dimensions is odd. In what follows, we consider the case $F(u) = |u|^p$ for the sake of simplicity. Then our problem is written as

$$(1.7\text{a}) \qquad u_{tt} - u_{rr} - \frac{n-1}{r} u_r = |u|^p \quad \text{in} \quad \Omega = (0, \infty) \times [0, T),$$

$$(1.7\text{b}) \qquad u(r, 0) = f(r), \quad u_t(r, 0) = g(r) \quad \text{for} \quad r > 0,$$

where $n = 2m + 3$ with m a positive integer and T is a positive constant. We assume that $f \in C^{k+1}([0,\infty))$, $g \in C^k([0,\infty))$ and that

$$(1.8)_k \qquad \sum_{j=0}^{k+1} |f^{(j)}(r)|\langle r\rangle^{\kappa+j} + \sum_{j=0}^{k} |g^{(j)}(r)|\langle r\rangle^{1+\kappa+j} \leq \varepsilon,$$

where κ, ε are positive parameters and we have set $\langle r\rangle = \sqrt{1+r^2}$.

We denote the life span of solutions by T_ε:

$$T_\varepsilon = \sup\{T > 0 : \text{ there exists uniquely a solution to (1.7) belonging to } C^2(\Omega)\}.$$

If the initial data can be extended on the whole line as even functions such that

$$r^{2m+1}f(r) \in C^{m+1}(\mathbb{R}) \quad \text{and} \quad r^{2m+1}g(r) \in C^m(\mathbb{R}),$$

we get

Theorem 0 ([11]). *Suppose that* $(1.8)_m$ *and*

$$(1.9) \qquad p > \max(p_0(n), m).$$

Then there is a positive number $\varepsilon_0 = \varepsilon_0(n,p,\kappa)$ *such that for* $0 < \varepsilon \leq \varepsilon_0$

$$(1.10) \qquad \begin{aligned} &T_\varepsilon = \infty && \textit{if} \quad \kappa \geq \kappa_0 = 2/(p-1), \\ &T_\varepsilon \geq C\varepsilon^{-(p-1)/(2-(p-1)\kappa)} && \textit{if} \quad 0 < \kappa < \kappa_0, \end{aligned}$$

where $C = C(n,p,\kappa) > 0$.

Remarks. a) The solution obtained in the above theorem is cotinously extended at $r = 0$.

b) When $n = 5$, (1.9) implies $p > p_0(n)$, because $1 < p_0(5) < 2$.

However the theorem can not deal with the case where $n \geq 7$. Therefore our main interest lies in the next theorem.

Main Theorem. *Suppose that* $(1.8)_1$ *and*

$$(1.11) \qquad p > p_0(n) \quad \text{and} \quad p < \frac{m+2}{m}.$$

Then there is a positive number $\varepsilon_1 = \varepsilon_1(n, p, \kappa)$ such that for $0 < \varepsilon \le \varepsilon_1$, (1.10) holds.

Remark. The latter condition needs to control the singularlity of the solution at $r = 0$. But it is wider than the upper restriction in (1.2).

In this note we denote various constants depending only on n, p and κ by C_0, C_1 and so on.

2. Sketch of the proof of Main Theorem. First we consider a linear wave equation:

$$u_{tt} - u_{rr} - \frac{n-1}{r} u_r = 0 \quad \text{in} \quad \Omega. \tag{2.1}$$

We put

$$u^0(r,t) = \int_{|t-r|}^{t+r} g(\lambda) K(\lambda, r, t) d\lambda + \partial_t \int_{|t-r|}^{t+r} f(\lambda) K(\lambda, r, t) d\lambda, \quad r \neq 0, \tag{2.2}$$

where we have set

$$K(\lambda, r, t) = \frac{(-1)^m}{2m!} \left(\frac{\lambda}{r}\right)^{2m+1} \left(\frac{\partial}{\partial \lambda} \frac{1}{2\lambda}\right)^m \phi^m(\lambda, r, t),$$
$$\phi(\lambda, r, t) = (\lambda - t + r)(t + r - \lambda).$$

Proposition 2.1 ([13]). *We suppose $f \in C^2([0,\infty))$ and $g \in C^1([0,\infty))$. Then u^0 belongs to $C^2(\Omega)$ and satisfies (2.1) and (1.7b).*

Proposition 2.2. *We suppose $(1.8)_1$. Then we have for $(r,t) \in \Omega$*

$$|\partial^\alpha_{r,t} u^0(r,t)| \le C_0 \varepsilon r^{1-m-|\alpha|} \langle r \rangle^{|\alpha|-1} \Psi_\kappa(r,t), \quad |\alpha| \le 1, \tag{2.3}$$

where we have set

$$\Psi_\kappa(r,t) = \begin{cases} \langle t+r \rangle^{-\kappa+m} & \text{if } 0 < \kappa < m+1, \\ \langle t+r \rangle^{-1} \left(1 + \log \dfrac{\langle t+r \rangle}{\langle t-r \rangle}\right) & \text{if } \kappa = m+1, \\ \langle t+r \rangle^{-1} \langle t-r \rangle^{-\kappa+m+1} & \text{if } \kappa > m+1. \end{cases}$$

Next we consider the following integral equation associated with the problem (1.7):

$$u(r,t) = u^0(r,t) + L(u)(r,t) \quad \text{in} \quad \Omega, \tag{2.4}$$

where u^0 is given by (2.2) and we have set

$$L(u)(r,t) = \int_0^t w(r,t,\tau)d\tau,$$

$$w(r,t,\tau) = \int_{|\lambda_-|}^{\lambda_+} G(\lambda,\tau)K(\lambda,r,t-\tau)d\lambda$$

with $\lambda_\pm = t - \tau \pm r$ and $G(\lambda,\tau) = |u(\lambda,\tau)|^p$.

Here we introduce a Banach space X on which we will construct a solution of (2.4):

$$X = \{u(r,t) \in C^{1,0}(\Omega): \ \|u\| < \infty\},$$

where the norm $\|u\|$ is defined by

$$\|u\| = \sum_{i=0}^{1} \sup_{(r,t)\in\Omega} |r^{m-1+i}\partial_r^i u(r,t)| \langle r\rangle^{1-i} \Psi_\kappa^{-1}(r,t)$$

where Ψ_κ is defined in (2.3).

Note that without loss of generality we may assume

$$\kappa \le (m+1)p - 1. \tag{2.5}$$

Proposition 2.3 ([13]). *We suppose that (1.11) and (2.5) hold and that u belongs to X. Then we have $L(u)(r,t) \in C^2(\Omega)$. Moreover $L(u)$ satisfies zero initial data and*

$$(\partial_t^2 - \partial_r^2 - \frac{n-1}{r}\partial_r)L(u)(r,t) = G(r,t) \quad in \quad D'(\Omega).$$

Proposition 2.4. *Let the hypotheses of the preceding proposition be fulfilled. Then we have for $(r,t) \in \Omega$*

$$\|L(u)\| \le C_1 \|u\|^p \Phi_\kappa(T+r), \tag{2.6}$$

where we have set $\Phi_\kappa(s) = \max(1, \langle s\rangle^{2-(p-1)\kappa})$ for $s \in \mathbb{R}$.

We now introduce an auxiliary norm $|||u|||$ for $u \in X$ by

$$|||u||| = \sup_{(r,t)\in\Omega} |r^m u(r,t)| \Psi_\kappa^{-1}(r,t).$$

Proposition 2.5. *Suppose (1.11) and (2.5) hold. Let u, v belong to X. Then we have*

$$|||L(u) - L(v)||| \le C_2 M \Phi_\kappa(T+r), \tag{2.7}$$

$$\|L(u) - L(v)\| \le (C_3 N_1 + C_4 N_2)\Phi_\kappa(T+r), \tag{2.8}$$

where we have set $M = |||u-v|||(\|u\|^{p-1} + \|v\|^{p-1})$, $N_1 = \|u-v\|(\|u\|^{p-1} + \|v\|^{p-1})$ *and* $N_2 = |||u-v|||^{p-1}(\|u\|^m + \|v\|^m)$. *Moreover if we further assume* $0 < \kappa < \kappa_0$, *we then obtained (2.7) and (2.8) with* $\Phi_\kappa(T+r)$ *replaced by* $\langle T\rangle^{2-(p-1)\kappa}$.

Let $u \in X$ be a solution to the integral equation (2.4). Then it is also a solution to the initial value problem (1.7) by Propositions 2.1 and 2.3. Moreover the uniqueness for such a solution that $u(r,t) \in C^1(\Omega) \cap C^2(\Omega \backslash \{t = r\})$ and $\partial^\alpha_{r,t} u(r,t) = O(r^{-m})$ as $r \to 0$ with $|\alpha| = 1$ is shown in [13].

First we consider the case where $\kappa \ge \kappa_0$. We define a sequence of functions $\{u_k\}_{k=0}^\infty$ by

$$u_{k+1} = u_0 + L(u_k) \quad \text{for} \quad k \ge 0, \quad u_0 = u^0,$$

where u^0 is given by (2.2). By (2.3) we have $\|u_0\| \le C_0\varepsilon$. Since $\kappa \ge \kappa_0$, we have (2.6), (2.7) and (2.8) with $\Phi_\kappa(T+r)$ replaced by 1. Let ε_2 be so small that

$$2C_0\varepsilon_2 \le 1 \quad \text{and} \quad 2^{p+2}C_5(C_0\varepsilon_2)^{p-1} \le 1 \tag{2.9}$$

with $C_5 = \sum_{i=1}^4 C_i$. Then we find a solution $u \in X$ to the integral equation (2.4) for $0 < \varepsilon \le \varepsilon_2$ and arbitraly $T > 0$ by following [9], p.257-p.259. (See also [11], Section 5).

Next we treat the case where $0 < \kappa < \kappa_0$. We take a positive number ε_1 satisfying

$$\varepsilon_1 \le \varepsilon_2 \quad \text{and} \quad 2^{p+2}C_5(\sqrt{2})^{2-(p-1)\kappa}(C_0\varepsilon_1)^{p-1} < 1.$$

Since $2-(p-1)\kappa > 0$ by $0 < \kappa < \kappa_0$, there is a number $t_\varepsilon > 1$ uniqely such that

$$2^{p+2}C_5(\sqrt{2}t_\varepsilon)^{2-(p-1)\kappa}(C_0\varepsilon)^{p-1} = 1 \quad \text{for} \quad 0 < \varepsilon \le \varepsilon_1. \tag{2.10}$$

Then considering (2.9), for $0 < \varepsilon \le \varepsilon_1$ and $0 < T \le t_\varepsilon$ we have

$$2C_0\varepsilon \le 1 \quad \text{and} \quad 2^{p+2}C_5\langle T\rangle^{2-(p-1)\kappa}(C_0\varepsilon)^{p-1} \le 1,$$

because $\langle T\rangle \le \sqrt{2}\max(T,1)$. Hence for $0 < \varepsilon \le \varepsilon_1$ we get a local solution by iteration and obtain (1.10) by (2.10), because $T_\varepsilon \ge t_\varepsilon$. □

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THE ASYMPTOTIC BEHAVIOUR OF RADIAL SOLUTIONS TO QUASI–LINEAR WAVE EQUATIONS IN TWO SPACE DIMENSIONS NEAR THE BLOWING UP POINT

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1. Introduction.

Consider the Cauchy problem:

$$u_{tt} - c^2(u_t, u_r)(u_{rr} + \frac{1}{r}u_r) = \frac{1}{r}u_r G(u_t, u_r), \quad (r,t) \in (0,\infty) \times (0, T_\varepsilon), \tag{1.1}$$

$$u(r,0) = \varepsilon f(r), \quad u_t(r,0) = \varepsilon g(r), \quad r \in (0,\infty), \tag{1.2}$$

where

$$\begin{aligned} c(u_t, u_r) &= 1 + \frac{a_1}{2}u_t^2 + \frac{a_2}{2}u_t u_r + \frac{a_3}{2}u_r^2 + O(|u_t|^3 + |u_r|^3), \\ G(u_t, u_r) &= O(u_r^2 + u_t^2), \end{aligned}$$

near $u_t = u_r = 0$. Equation (1.1) is a radially symmetric form of quasi-linear wave equation in two space dimensions which involves the equation of vibrating membrane. In [4], we obtained the following blow up result:

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log(1 + T_\varepsilon) \le \frac{1}{H},$$

where T_ε is the lifespan of the radial solution of the Cauchy problem (1.1), (1.2) and H is a constant depending only on f, g and $\partial^2 c(0,0)$. More precisely, the blow up occurs as follows. If we set

$$w(r,t) = \frac{c(u_t, u_r)v_{rr} - v_{rt}}{2c(u_t, u_r)} \quad \text{with} \quad v(r,t) = r^{\frac{1}{2}}u(r,t),$$

then we find that

$$|w(r,t)| \longrightarrow \infty \quad \text{as} \quad \varepsilon^2 \log(1+t) \to \frac{1}{H}$$

along a pseudo-characteristic curve for sufficiently small ε.

In this paper, we investigate the asymptotic behaviour of $w(r,t)$ when $\varepsilon^2\log(1+t)$ tends to $\frac{1}{H}$.

2. Statement of Results.

As we did in [3], we assume $f, g \in C_0^\infty(\mathrm{R}^2)$, $|f|+|g| \not\equiv 0$ and $f(r) = g(r) = 0$ for $r = M$. Moreover we assume $a_1 - a_2 + a_3 = a \neq 0$ which means (1.1) does not satisfy the *null-condition*. Then we can define a positive constant H by

$$\begin{aligned} H &= \max_{\rho\in \mathrm{R}}(-a\mathcal{F}'(\rho)\mathcal{F}''(\rho)) \\ &= -\,a\mathcal{F}'(\rho_0)\mathcal{F}''(\rho_0), \end{aligned}$$

where $\mathcal{F}(\rho)$ is the Friedlander radiation field which is constructed by f and g (see [4]). We introduce a variable $s = \varepsilon^2\log(1+t)$ and we write $t = t_X$ when $s = X$, *i.e.*,

$$X = \varepsilon^2\log(1+t_X).$$

To state our results, we have to recall the facts which are obtained in [4].

Firstly, for any $B > H$ we consider the Burgers equation:

$$U_{\rho s} + \frac{a}{2}(U_\rho)^2 U_{\rho\rho} = 0, \quad (\rho, s) \in \mathrm{R}\times[0, \frac{1}{B}],$$

$$U_\rho(\rho, 0) = \mathcal{F}'(\rho), \qquad \rho\in\mathrm{R},$$

then, there exists an $\varepsilon(B) > 0$ such that the Cauchy problem (1.1), (1.2) has a smooth solution in $0 \le t \le t_{\frac{1}{B}}$ and the following holds.

$$\begin{aligned} |\partial_r^l\partial_t^m u(r, t_{\frac{1}{B}}) - \varepsilon r^{-\frac{1}{2}}(-1)^m\partial_\rho^{l+m}U(r - t_{\frac{1}{B}}, \frac{1}{B})| &\le C_{l,m,B}\,\varepsilon^{\frac{5}{4}} r^{-\frac{1}{2}} \\ \text{for}\quad r - t_{\frac{1}{B}} > -\frac{1}{3\varepsilon} \quad &\text{and} \quad l+m\neq 0 \end{aligned} \tag{2.1}$$

for $\varepsilon < \varepsilon(B)$. Moreover, U satisfies

$$\begin{aligned} U(\rho(s), s) &= \mathcal{F}'(\rho_0), \\ U_{\rho\rho}(\rho(s), s) &= \frac{\mathcal{F}''(\rho_0)}{1 + a\mathcal{F}'(\rho_0)\mathcal{F}''(\rho_0)s} = \frac{\mathcal{F}''(\rho)}{1 - Hs}, \end{aligned} \tag{2.2}$$

for $0 \le s \le \frac{1}{B}$ along the curve Λ_{ρ_0} defined by

$$\frac{d\rho}{ds} = \frac{a}{2}(U_\rho)^2 \quad \text{for} \quad s \ge 0, \quad \rho = \rho_0 \quad \text{for} \quad s = 0.$$

These facts are proved in section 3 of [4] by using the energy inequality and the Klainerman inequality.

Secondly, we define a pseudo-characteristic curve Z by

$$\frac{dr}{dt} = c(u_t, u_r) \quad \text{for} \quad t \geq t_{\frac{1}{B}}, \quad r = \rho(\frac{1}{B}) + t_{\frac{1}{B}} \quad \text{for} \quad t = t_{\frac{1}{B}}$$

and a function w by

$$w(r,t) = \frac{cv_{rr} - v_{rt}}{2c} \qquad \text{with} \qquad v(r,t) = r^{\frac{1}{2}}u(r,t).$$

Then, for any $A < H$ there exists an $\bar{\varepsilon}(A) > 0$ such that if $\varepsilon < \bar{\varepsilon}(A)$, then w should satisfy

$$w'(t) = \alpha_0(t)w(t)^2 + \alpha_1(t)w(t) + \alpha_2(t) \quad \text{for} \quad t_{\frac{1}{B}} \leq t \leq t_{\frac{1}{A}}, \tag{2.3}$$

$$w(t_{\frac{1}{B}}) = \varepsilon U_{\rho\rho}(\rho(\frac{1}{B}), \frac{1}{B}) + O(\varepsilon^{\frac{5}{4}}), \tag{2.4}$$

where

$$w(t) = w(r(t), t) \qquad \text{for} \qquad (r(t), t) \in Z$$

and

$$\begin{aligned} \alpha_0(t) &= -\,a\varepsilon\mathcal{F}'(\rho_0)(1+t)^{-1} + O(\varepsilon^{\frac{5}{4}}(1+t)^{-1}), \\ \alpha_1(t) &= O(\varepsilon^4(1+t)^{-1} + \varepsilon^2(1+t)^{-2}), \\ \alpha_2(t) &= O(\varepsilon(1+t)^{-2}), \end{aligned} \tag{2.5}$$

as long as u exists. Here $X = O(Y)$ means $|X| \leq CY$ with constant C depending only on B, f, g, ρ_0, a and M. This fact is proved in section 4 and 5 of [4] by using (2.1), (2.2) and *a priori* estimats of u.

Now we state our results.

Theorem. *For any $\delta > 0$ there exists an $\varepsilon_\delta > 0$ such that $w(t)$ is well-defind in $t_{\frac{1}{B}} \leq t \leq t_{\frac{1}{H}-\delta}$ for $\varepsilon < \varepsilon_\delta$ and at the point $t = t_{\frac{1}{H}-\delta}$,*

$$\lim_{\varepsilon \to 0}(\frac{1}{H} - \varepsilon^2 \log(1+t))\frac{w(t)}{\varepsilon} = \frac{1}{H}\mathcal{F}''(\rho_0)$$

holds.

However, since we are interested in the behaviour of w when $\varepsilon^2 \log(1+t)$ tends to $\frac{1}{H}$, we reduce the above result into

Corollary.

$$\lim_{\varepsilon \to 0,\ \varepsilon^2 \log(1+t) \to \frac{1}{H}} (\frac{1}{H} - \varepsilon^2 \log(1+t))\frac{w(t)}{\varepsilon} = \frac{1}{H}\mathcal{F}''(\rho_0).$$

In three space dimensions, for the radial solution of the Cauchy problem:

$$u_{tt} - c^2(u_t)(u_{rr} + \frac{2}{r}u_r) = 0,$$
$$u(r,0) = \varepsilon f(r), \quad u_t(r,0) = \varepsilon g(r),$$

with $c(u_t) = 1 + au_t + O(u_t^2)$ and $a \neq 0$, F. John [5] and L. Hörmander [2] have shown a blow up result

$$\limsup_{\varepsilon\to 0} \varepsilon \log(1 + T_\varepsilon) \le \frac{1}{\max(a\mathcal{F}''(\rho))}.$$

In this case, if we set $H = \max(a\mathcal{F}''(\rho)) = a\mathcal{F}''(\rho_0)$, we also expect

$$\lim_{\varepsilon\to 0,\ \varepsilon\log(1+t)\to\frac{1}{H}} (\frac{1}{H} - \varepsilon \log(1 + t))\frac{w(t)}{\varepsilon} = \frac{1}{H}\mathcal{F}''(\rho_0),$$

which would be obtained in parallel.

For the non radially symmetric case, S. Alinhac [1] studies the Cauchy problem

$$\partial_t^2 u - \Delta u = \sum_{i,j,k=0}^{2} g_{ij}^k \partial_k u \partial_{ij}^2 u, \quad (x,t) \in \mathbb{R}^2 \times (0, T_\varepsilon),$$
$$u(x,0) = u^0(x;\varepsilon), \quad u_t(x,0) = u^1(x;\varepsilon), \qquad x \in \mathbb{R}^2,$$

where $\partial_0 = \partial_t$ and g_{ij}^k are constants. Note that this problem differs from ours in the power of $\partial_k u$. If u^0, u^1 and g_{ij}^k satisfy the *non degenerate* condition (ND), he finds the *asymptotic lifespan* T_ε^a which satisfies the following: For any $N \in \mathbb{N}$, there exists an $\varepsilon_N > 0$ such that if $\varepsilon < \varepsilon_N$, then

$$T_\varepsilon > T_\varepsilon^a - \varepsilon^N$$

and

$$\frac{1}{C} \le (T_\varepsilon^a - t)\|\partial^2 u(t)\|_{L_x^\infty} \le C \qquad \text{for} \qquad \frac{C}{\varepsilon^2} \le t \le T_\varepsilon^a - \varepsilon^N$$

holds for some constant C. Since he estimates $\partial^2 u$ not along a pseud-characteristic curve but in whole space $\mathbb{R}^2$, it seems difficult to determine the constant C.

In the rest of this paper, we concentrate on the proof of Theorem.

3. Proof of Theorem.

In [3], we have proved that there exists an $\varepsilon_1(\delta) > 0$ such that for $\varepsilon < \varepsilon_1$ the Cauchy problem (1.1), (1.2) has a smooth solution u in $0 \le t \le t_{\frac{1}{H}-\delta}$ and therefore $w(t)$ is well-defined in $t_{\frac{1}{B}} \le t \le t_{\frac{1}{H}-\delta}$. Thus we have only to prove that for any $\eta > 0$ there exists an $\varepsilon_0(\delta, \eta) > 0$ such that

$$|(\frac{1}{H} - s)\frac{w(t)}{\varepsilon} - \frac{1}{H}\mathcal{F}''(\rho_0)| < \eta$$

for $\varepsilon < \varepsilon_0$ and $s = \frac{1}{H} - \delta$. If we take $\frac{1}{A} = \frac{1}{H} + \delta$ in the argument in section 2, there exist an $\varepsilon_2(\delta) > 0$ such that if $\varepsilon < \varepsilon_2$, $w(t)$ should satisfy the ordinary differential equation (2.3), (2.4) in $t_{\frac{1}{B}} \le t \le t_{\frac{1}{H}+\delta}$ as long as u exists. Thus we find that for $\varepsilon < \min(\varepsilon_1, \varepsilon_2)$ the ordinary differential equation (2.3), (2.4) make sence in $t_{\frac{1}{B}} \le t \le t_{\frac{1}{H}-\delta}$.

Now the following lemma is useful.

Lemma. *Let $w(t)$ be a solution of the ordinary differential equation*

$$w'(t) = \alpha_0(t)w(t)^2 + \alpha_1(t)w(t) + \alpha_2(t) \quad \text{for} \quad t_0 \le t \le T$$

and assume

$$\alpha_0(t) \ge 0 \qquad \text{for} \qquad t_0 \le t \le T,$$
$$w(t_0) > K$$

where

$$K = \int_{t_0}^{T} |\alpha_2(t)| \exp(-\int_{t_0}^{t} \alpha_1(\tau)d\tau)dt.$$

Then $w(t)$ satisfies

$$w(t)\exp(-\int_{t_0}^{t}\alpha_1(\tau)d\tau) \ge \frac{w(t_0) - K}{1 - (w(t_0) - K)\int_{t_0}^{t}\alpha_0(\tau)\exp(\int_{t_0}^{\tau}\alpha_1(\xi)d\xi)d\tau}$$

and

$$w(t)\exp(-\int_{t_0}^{t}\alpha_1(\tau)d\tau) \le \frac{w(t_0) + K}{1 - (w(t_0) + K)\int_{t_0}^{t}\alpha_0(\tau)\exp(\int_{t_0}^{\tau}\alpha_1(\xi)d\xi)d\tau}.$$

Proof of Lemma. At first we consider the case $\alpha_1(t) \equiv 0$. Let $w_1(t)$ be a solution of

$$w_1'(t) = \alpha_0(t)(w_1(t) - K)^2, \tag{3.1}$$

$$w_1(t_0) = w(t_0) \tag{3.2}$$

and set

$$w_2(t) = \int_{t_0}^{t} |\alpha_2(\tau)|d\tau.$$

Since $\alpha_0(t) \ge 0$, we find that

$$w_1(t) \ge w(t_0) > K = w_2(T) \ge w_2(t)$$

and that

$$\begin{aligned}(w_1(t) - w_2(t))' =& \alpha_0(t)(w_1(t) - K)^2 - |\alpha_2(t)| \\ \le& \alpha_0(t)(w_1(t) - w_2(t))^2 + \alpha_2(t), \\ w_1(t_0) - w_2(t_0) =& w(t_0).\end{aligned}$$

Thus the usual comparison theorem leads

$$w_1(t) - w_2(t) \le w(t). \tag{3.3}$$

By solving the ordinary differential equation (3.1), (3.2), $w_1(t)$ is represented by

$$w_1(t) = K + \frac{w(t_0) - K}{1 - (w(t_0) - K)\int_{t_0}^{t}\alpha_0(\tau)d\tau}.$$

Substituting this equality into (3.3), we find

$$\begin{aligned} w(t) &\geq K + \frac{w(t_0) - K}{1 - (w(t_0) - K)\int_{t_0}^{t} \alpha_0(\tau)d\tau} - w_2(t) \\ &\geq \frac{w(t_0) - K}{1 - (w(t_0) - K)\int_{t_0}^{t} \alpha_0(\tau)d\tau}. \end{aligned}$$

On the other hand, if we let $w_3(t)$ be a solution of

$$w_3'(t) = \alpha_0(t)(w_3(t) + K)^2,$$

$$w_3(t_0) = w(t_0),$$

then we find

$$\begin{aligned} (w_3(t) + w_2(t))' =& \alpha_0(t)(w_3(t) + K)^2 + |\alpha_2(t)| \\ \geq& \alpha_0(t)(w_3(t) + w_2(t))^2 + \alpha_2(t), \\ w_3(t_0) + w_2(t_0) =& w(t_0). \end{aligned}$$

Thus we obtain

$$w_3(t) + w_2(t) \geq w(t).$$

Since $w_3(t)$ is represented by

$$w_3(t) = -K + \frac{w(t_0) + K}{1 - (w(t_0) + K)\int_{t_0}^{t} \alpha_0(\tau)d\tau},$$

we obtain

$$\begin{aligned} w(t) \leq& -K + \frac{w(t_0) + K}{1 - (w(t_0) + K)\int_{t_0}^{t} \alpha_0(\tau)d\tau} + w_2(t) \\ \leq& \frac{w(t_0) + K}{1 - (w(t_0) + K)\int_{t_0}^{t} \alpha_0(\tau)d\tau}. \end{aligned}$$

For the general case, setting

$$W(t) = w(t)\exp(-\int_{t_0}^{t} \alpha_1(\tau)d\tau)$$

and applying the result we have just proved to $W(t)$, we obtain the inequalities we wanted.

Now we want to apply Lemma to (2.3), (2.4) as $t_0 = t_{\frac{1}{B}}$ and $T = t_{\frac{1}{H}-\delta}$. By (2.5), we have

$$\begin{aligned} \exp(\pm\int_{t_{\frac{1}{B}}}^{t} \alpha_1(\tau)d\tau) &= \exp(O(\int_{t_{\frac{1}{B}}}^{t} \varepsilon^4(1+\tau)^{-1}d\tau)) \\ &= \exp(O(\varepsilon^4 \log(1+t)) + O(\varepsilon^4 \log(1+t_{\frac{1}{B}}))) \\ &= \exp(O(\varepsilon^2)) = 1 + O(\varepsilon^2) \quad \text{for} \quad t_{\frac{1}{B}} \leq t \leq t_{\frac{1}{H}-\delta}, \end{aligned}$$

$$\begin{aligned}K =& \int_{t_{\frac{1}{B}}}^{t_{\frac{1}{H}-\delta}} |\alpha_2(t)| \exp(-\int_{t_{\frac{1}{B}}}^{t} \alpha_1(\tau)d\tau)dt \\ =& O((1+\varepsilon^2)\varepsilon \int_{t_{\frac{1}{B}}}^{t_{\frac{1}{H}-\delta}} (1+t)^{-2}dt) \\ =& O(\varepsilon(1+t_{\frac{1}{B}})^{-1}) + O(\varepsilon(1+t_{\frac{1}{H}-\delta})^{-1}) \\ =& O(\varepsilon^3),\end{aligned} \tag{3.4}$$

$$\begin{aligned}&\int_{t_{\frac{1}{B}}}^{t} \alpha_0(\tau)\exp(\int_{t_{\frac{1}{B}}}^{\tau} \alpha_1(\xi)d\xi)d\tau \\ =&(1+O(\varepsilon^2))(-a\varepsilon\mathcal{F}'(\rho_0)+O(\varepsilon^{\frac{5}{4}}))\int_{t_{\frac{1}{B}}}^{t}(1+\tau)^{-1}d\tau \\ =&(-a\varepsilon\mathcal{F}'(\rho_0)+O(\varepsilon^{\frac{5}{4}}))(\log(1+t)-\log(1+t_{\frac{1}{B}})) \\ &\text{for} \quad t_{\frac{1}{B}} \le t \le t_{\frac{1}{H}-\delta}.\end{aligned}$$

Since $H > 0$, $-a\mathcal{F}'(\rho_0)$ and $\mathcal{F}''(\rho_0)$ have the same sign. Without loss of generality, we can assume that both are positive and then it follows from (2.4) and (3.4) that there exists an $\varepsilon_3 > 0$ such that

$$w(t_{\frac{1}{B}}) > K$$

and

$$\alpha_0(t) = 0$$

hold for $\varepsilon < \varepsilon_3$. Thus we can apply Lemma and obtain

$$\begin{aligned}&(1+C\varepsilon^2)w(t) \\ \ge& \frac{w(t_{\frac{1}{B}}) - C\varepsilon^3}{1-(w(t_{\frac{1}{B}})-C\varepsilon^3)(-a\varepsilon\mathcal{F}'(\rho_0)U_{\rho\rho}(\frac{1}{B})-C\varepsilon^{\frac{5}{4}})(\log(1+t)-\log(1+t_{\frac{1}{B}}))} \\ =& \frac{\varepsilon U_{\rho\rho}(\frac{1}{B})-C\varepsilon^{\frac{5}{4}}}{1-(-a\mathcal{F}'(\rho_0)U_{\rho\rho}(\frac{1}{B})-C\varepsilon^{\frac{1}{4}})(s-\frac{1}{B})} \quad \text{for} \quad \frac{1}{B} \le s \le \frac{1}{H}-\delta,\end{aligned}$$

where $U_{\rho\rho}(\frac{1}{B}) = U_{\rho\rho}(\rho(\frac{1}{B}), \frac{1}{B})$ and C is a constant depending only on B, f, g, ρ_0, a and M and it varies from line to line. By (2.4), we get

$$\begin{aligned}\frac{w(t)}{\varepsilon} \ge& (1-C\varepsilon^2)\frac{U_{\rho\rho}(\frac{1}{B})-C\varepsilon^{\frac{1}{4}}}{1-(-a\mathcal{F}'(\rho_0)U_{\rho\rho}(\frac{1}{B})-C\varepsilon^{\frac{1}{4}})(s-\frac{1}{B})} \\ =& \frac{U_{\rho\rho}(\frac{1}{B})-C\varepsilon^{\frac{1}{4}}}{1-\frac{s-\frac{1}{B}}{\frac{1}{H}-\frac{1}{B}}+C\varepsilon^{\frac{1}{4}}} \\ =& \frac{\frac{1}{H}\mathcal{F}''(\rho_0)-C\varepsilon^{\frac{1}{4}}}{\frac{1}{H}-s+C\varepsilon^{\frac{1}{4}}} \quad \text{for} \quad \frac{1}{B} \le s \le \frac{1}{H}-\delta.\end{aligned}$$

If we set $s = \frac{1}{H} - \delta$, we have

$$\begin{aligned}(\frac{1}{H} - s)\frac{w(t)}{\varepsilon} \geq & (\frac{1}{H}\mathcal{F}''(\rho_0) - C\varepsilon^{\frac{1}{4}})\frac{\frac{1}{H} - s}{\frac{1}{H} - s + C\varepsilon^{\frac{1}{4}}} \\ = & \frac{1}{H}\mathcal{F}''(\rho_0)\frac{\delta}{\delta + C\varepsilon^{\frac{1}{4}}} - \frac{C\delta\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}} \\ = & \frac{1}{H}\mathcal{F}''(\rho_0) - \frac{C\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}} - \frac{C\delta\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}}.\end{aligned}$$

There exists an $\varepsilon_4(\delta, \eta) > 0$ such that if $\varepsilon < \varepsilon_4$, then

$$\frac{C\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}} + \frac{C\delta\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}} < \eta,$$

i.e.,

$$(\frac{1}{H} - s)\frac{w(t)}{\varepsilon} - \frac{1}{H}\mathcal{F}''(\rho_0) > -\eta$$

holds. Similarly, using the other inequality in Lemma we find that there exists an $\varepsilon_5(\delta, \eta) > 0$ such that if $\varepsilon < \varepsilon_5$, then

$$(\frac{1}{H} - s)\frac{w(t)}{\varepsilon} - \frac{1}{H}\mathcal{F}''(\rho_0) < \eta$$

holds. Thus if we take $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5)$, we find that

$$|(\frac{1}{H} - s)\frac{w(t)}{\varepsilon} - \frac{1}{H}\mathcal{F}''(\rho_0)| < \eta$$

holds for $\varepsilon < \varepsilon_0$ and this completes the proof of Theorem.

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Periodic solutions to multi-phase Stefan problems with nonlinear dynamic boundary conditions

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Introduction

In this paper, we consider periodic stabilities for the solutions to Stefan problems with periodic condition in time in the enthalpy formulation with nonlinear dynamic boundary conditions: Find functions $u = u(t,x)$ on $R \times \Omega$ and $V = V(t,x)$ on $R \times \Gamma$ satisfying that

$$u_t - \Delta\beta(u) = 0 \quad \text{in } R \times \Omega, \tag{0.1}$$

$$\begin{cases} -\dfrac{\partial\beta(u)}{\partial\nu} = g(t,x,V) + \rho(V)_t \\ \beta(u) = V \end{cases} \quad \text{on } R \times \Gamma, \tag{0.2}$$

where Ω is a bounded domain in $R^N (N \geq 2)$ with smooth boundary $\Gamma = \partial\Omega$; $\beta : R \to R$ and $\rho : R \to R$ are given nondecreasing functions; $g = g(t,x,\xi) : R \times \Gamma \times R \to R$ is a given function which is nondecreasing in $\xi \in R$ for a.e. $(t,x) \in R \times \Gamma$; $(\partial/\partial\nu)$ denotes the outward normal derivative on Γ. We denote by $SP = SP(\beta,\rho,g)$ the above system (0.1) $\sim$ (0.2).

By many authors initial-boundary value problems for (0.1) with usual boundary conditions have been studied. In case the flux condition is of the form $-\partial\beta(u)/\partial\nu = g(t,x,\beta(u))$, the problem was uniquely solved in the variational sense by Visintin [16], Niezgodka, Pawlow & Visintin [14] and Niezgodka & Pawlow [13]. Also, some interesting results dealing with the boundary condition (0.2) are found in Cannon [9], Hintermann [12] and Grobbelaar & Dalsen [10]. Recently, boundary conditions similar to (0.2) were discussed by Primicerio & Rodrigues [15] and in Aiki [1, 2] for one-dimensional Stefan problems with dynamic boundary conditions the local in time existence and uniqueness of classical solutions were shown. In particular, for the function $\rho(r) = r$ the existence, uniqueness and behavior for the solution to the initial-boundary value problems for $SP(\beta,\rho,g)$ and periodic solutions for $SP(\beta,\rho,g)$ are already studied in Aiki [6, 5, 4]. The purpose of this paper is to establish the existence, the order property and the asymptotic stability of periodic solutions under the periodic condition in time, that is, $g(t,x,\xi) = g(t+T,x,\xi)$ a.e. on $R \times \Gamma \times R$ for some positive constant T. In papers by Haraux & Kenmochi [11], Aiki, Kenmochi & Shinoda [7, 8] and Aiki [5, 4] similar questions to those above were discussed.

1. Statements of main results

Throughout this paper we assume that the functions $\beta : R \to R$ and $\rho : R \to R$ satisfy the following conditions $\{(\beta 1), (\beta 2)\}$ and (ρ), respectively:

(β1) β is non-decreasing and Lipschitz continuous on R with $\beta(0) = 0$.

(β2) There are some positive constants L_β, l_β such that

$$|\beta(r)| \geq L_\beta |r| - l_\beta \quad \text{for all } r \in R.$$

(ρ) ρ is increasing and bi-Lipschitz continous on R with $\rho(0) = 0$.

Also, let $g = g(t, x, \xi) : R \times \Gamma \times R \to R$ be a function and suppose that:

(g1) $g(t, x, \xi)$ is non-decreasing in $\xi \in R$ for a.e. $(t, x) \in R \times \Gamma$;

(g2) for any $\xi \in R$, $g(\cdot, \cdot, \xi) \in L^2_{loc}(R; L^2(\Gamma))$;

(g3) $g(t, x, \xi)$ is locally Lipschitz continuous in ξ uniformly with respect to $(t, x) \in R \times \Gamma$, that is, for each $M > 0$ there is a positive constant $C_g(M)$ such that

$$|g(t, x, \xi) - g(t, x, \xi')| \le C_g(M)|\xi - \xi'|$$

for all ξ, ξ' with $|\xi| \le M, |\xi'| \le M$ and a.e. $(t, x) \in R \times \Gamma$ and there are constants m_1, m_2 with $m_1 \le m_2$ such that $g(t, x, \beta(m_1)) \le 0, g(t, x, \beta(m_2)) \ge 0$ for a.e. $(t, x) \in \Sigma$.

For the sake of simplicity of notations we put

$$X = H^1(\Omega), \quad H = L^2(\Omega) \times L^2(\Gamma),$$

$$A(u, v) = \int_\Omega \nabla u \cdot \nabla v dx \quad \text{for } u, v \in X,$$

$$(u, v)_X = A(u, v) + (\int_\Omega u dx + \int_\Gamma u d\Gamma)(\int_\Omega v dx + \int_\Gamma v d\Gamma) \quad \text{for } u, v \in X,$$

$$(\vec{u}, \vec{v})_H = \int_\Omega uv\, dx + \int_\Gamma u_\Gamma v_\Gamma\, d\Gamma \quad \text{for } \vec{u} = (u, u_\Gamma), \vec{v} = (v, v_\Gamma) \in H,$$

and C_Ω is a positive constant satisfying that

$$|v|_{L^2(\Omega)} \le C_\Omega |v|_X, \quad |v|_{L^2(\Gamma)} \le C_\Omega |v|_X \quad \text{for any } v \in X.$$

Also, we define an operator $E : X \to H$ by putting

$$Ev = (v, v|_\Gamma) \quad \text{for } v \in X.$$

Clearly, X and H are Hilbert spaces with inner product $(\cdot, \cdot)_X$ and $(\cdot, \cdot)_H$, respectively, and the range of E, $R(E)$, is a dense subspace of H and E is linear and compact. We identify H with its dual H^* and denote by X^* the dual space of X. Therefore, denoting by E^* the dual operator of E we have

$$\langle E^*(v, v_\Gamma), \eta \rangle_X = \int_\Omega v\eta\, dx + \int_\Gamma v_\Gamma \eta\, d\Gamma \quad \text{for any } (v, v_\Gamma) \in H \text{ and } \eta \in X,$$

where $\langle \cdot, \cdot \rangle_X$ is a duality pairing between X^* and X.

We now formulate problem SP in variational sense.

Definition 1.1. Let $J = [t_0, t_1]$ be a compact interval, $Q = (t_0, t_1) \times \Omega$, $\Sigma = (t_0, t_1) \times \Gamma$. Then a couple $\{u, V\}$ of functions $u : J \times \Omega \to R$ and $V : J \times \Gamma \to R$ is called a solution of SP on J if $u \in C_w(J; L^2(\Omega)) \cap L^\infty(Q)$, $\beta(u) \in L^2(J; X)$, $V \in W^{1,2}(J; L^2(\Gamma)) \cap L^\infty(\Sigma)$, $\beta(u) = V$ a.e. on Σ and the following variational equation is satisfied:

$$-\int_Q u\eta_t dxdt + \int_\Sigma \rho(V)_t \eta d\Gamma dt + \int_Q \nabla\beta(u) \cdot \nabla\eta dxdt + \int_\Sigma g(t, x, V)\eta d\Gamma dt = 0$$
$$\text{for any } \eta \in Z,$$

where $Z = \{\eta \in C^1(J;X); \eta(t_0) = \eta(t_1) = 0\}$.

Remark 1.1. From the above definition of solution to SP on $J = [t_0, t_1]$ it follows that a couple of functions $\{u, V\}$ is a solution of SP on J if and only if $E^*(u,V) \in W^{1,2}(J; X^*)$, $u \in L^\infty(J \times \Omega)$, $\beta(u) \in L^2(J; Y)$, $V \in W^{1,2}(J; L^2(\Gamma)) \cap L^\infty(J \times \Gamma)$, $\beta(u) = V$ a.e. on $J \times \Gamma$ and for a.e. $t \in J$

$$\langle \frac{d}{dt} E^*(u(t), \rho(V)(t)), \eta \rangle_X + A(\beta(u(t)), \eta) + \int_\Gamma g(t, \cdot, V(t, \cdot))\eta d\Gamma = 0 \quad \text{for any } \eta \in X.$$

Definition 1.2. Let J' be any interval in R. Then a couple $\{u, V\}$ of functions $u : J' \times \Omega \to R$ and $V : J' \times \Gamma \to R$ is called a solution of SP on J' if for every compact subinterval $J = [t_0, t_1]$ of J' the couple $\{u, V\}$ is a solution of SP on J in the sense of Definition 1.1.

Next, we formulate the Cauchy problem and the problem with the periodic condition in time for SP.

Definition 1.3. (i) Let $J' = [t_0, t_1]$ or $[t_0, t_1)$, and let $u_0 \in L^2(\Omega)$, $V_0 \in L^2(\Gamma)$. Then a couple $\{u, V\}$ of functions $u : J' \to L^2(\Omega)$ and $V : J' \to L^2(\Gamma)$ is a solution of the Cauchy problem with initial condition $u(t_0) = u_0$, $V(t_0) = V_0$, denoted by $CSP(u_0, V_0)$ on J', for problem SP on J', if $\{u, V\}$ is a solution of SP on J' with $u(t_0) = u_0$, $V(t_0) = V_0$.

(ii) Let T be a positive number, and let $\{u, V\} : J' \to V$ is a solution of SP on R such that $u(t+T) = u(t)$, $V(t+T) = V(t)$ for all $t \in R$. Then $\{u, V\}$ is called a T-periodic solution of SP on R.

We now recall an existence-uniqueness result for $CSP(u_0, V_0)$.

Theorem 1.1. *(cf. [3], [6, Theorems 1.1 and 1.2]) We suppose that $(\beta 1)$, $(\beta 2)$, (ρ) and $(g1) \sim (g3)$ hold. Let t_0 be any number in R. Then, for any functions $u_0 \in L^\infty(\Omega)$ and $V_0 \in L^\infty(\Gamma)$, there exists one and only one solution $\{u, V\}$ of $CSP(u_0, V_0)$ on $J' = [t_0, \infty)$.*

The main results of this paper are stated in the following theorems. The theorem is concerned with the existence of T-periodic solutions.

Theorem 1.2. *Let T be a positive number, and suppose*

$$(g4) \qquad g(t+T, x, \xi) = g(t, x, \xi) \quad \textit{for all } \xi \in R \textit{ and a.e. } (t,x) \in R \times \Gamma.$$

Then there exists a T-periodic solution $\{u, V\}$ of SP on R.

We denote by $\mathcal{P}_T$ the set of all T-periodic solutions of SP on R; in general $\mathcal{P}_T$ is not a singleton (cf. [7, Example 1.1]). The following two theorems are concerned with the structure of $\mathcal{P}_T$.

Theorem 1.3. *Under the same assumptions as in theorem 1.2 we have*

$$g(\cdot, \cdot, V_1) = g(\cdot, \cdot, V_2) \quad \textit{a.e. on } R \times \Gamma \textit{ for any } \{u_1, V_1\}, \{u_2, V_2\} \in \mathcal{P}_T,$$

and

$$\int_0^T \int_\Gamma g(t, x, V) d\Gamma dt = 0 \quad \textit{for any } \{u, V\} \in \mathcal{P}_T.$$

Theorem 1.4. *Under the same assumptions of theorem 1.2 we have the following statements (a) $\sim$ (c):*

(a) If $\{u_1, V_1\}$, $\{u_2, V_2\} \in \mathcal{P}_T$ and

$$\int_\Omega u_1(0,x)dx + \int_\Gamma \rho(V_1)(0,x)d\Gamma = \int_\Omega u_2(0,x)dx + \int_\Gamma \rho(V_2)(0,x)d\Gamma,$$

then

$$\beta(u_1) = \beta(u_2) \quad a.e. \ on \ R \times \Omega \quad and \quad V_1 = V_2 \quad a.e. \ on \ R \times \Gamma.$$

(b) If $\{u_1, V_1\}$, $\{u_2, V_2\} \in \mathcal{P}_T$ and

$$\int_\Omega u_1(0,x)dx + \int_\Gamma \rho(V_1)(0,x)d\Gamma > \int_\Omega u_2(0,x)dx + \int_\Gamma \rho(V_2)(0,x)d\Gamma,$$

then

$$\beta(u_1) \geq \beta(u_2) \quad a.e. \ on \ R \times \Omega \quad and \quad V_1 \geq V_2 \quad a.e. \ on \ R \times \Gamma.$$

(c) If $\{u_1, V_1\}$, $\{u_2, V_2\} \in \mathcal{P}_T$ and

$$\int_\Omega u_1(0,x)dx + \int_\Gamma \rho(V_1)(0,x)d\Gamma > \int_\Omega u_2(0,x)dx + \int_\Gamma \rho(V_2)(0,x)d\Gamma,$$

then for any $a_0 \in R$ with

$$\int_\Omega u_1(0,x)dx + \int_\Gamma \rho(V_1)(0,x)d\Gamma > a_0 > \int_\Omega u_2(0,x)dx + \int_\Gamma \rho(V_2)(0,x)d\Gamma$$

there exists a T-periodic solution $\{u, V\}$ of SP on R such that

$$a_0 = \int_\Omega u(0,x)dx + \int_\Gamma \rho(V)(0,x)d\Gamma.$$

Finally, as to the asymptotic stability of T-periodic solutions we prove the following.

Theorem 1.5. *Suppose that all conditions of theorem 1.2 are satisfied. Let t_0 be any number in R and let $\{u, V\}$ be any solution of SP on $[t_0, \infty)$. Then there is $\{\hat{u}, \hat{V}\} \in \mathcal{P}_T$ such that*

$$\beta(u(nT + \cdot)) \to \beta(\hat{u}) \quad in \ L^2(0,T; W^{1,2}(\Omega)) \ as \ n \to \infty.$$

2. Known results

In this section we show some known results which will use in the proof of theorems 1.2 $\sim$ 1.5

First, we recall a comparison result for $CSP(u_0, V_0)$.

Theorem 2.1. *(cf. [6, Theorem 1.3]) Suppose that the same assumptions as in Theorem 1.1 hold and a pair $\{\overline{u}_0, \overline{V}_0\} \in L^\infty(\Omega) \times L^\infty(\Gamma)$. Let t_0 be any number in R and $\{u, V\}$ (resp. $\{\overline{u}, \overline{V}\}$) be a solution of $CSP(u_0, V_0)$ (resp. $CSP(\overline{u}_0, \overline{V}_0)$) on $[t_0, \infty)$. Then for any $t \in [t_0, \infty)$,*

$$\begin{aligned} & |[u(t) - \overline{u}(t)]^+|_{L^1(\Omega)} + |[V(t) - \overline{V}(t)]^+|_{L^1(\Gamma)} \\ \leq \ & |[u(t_0) - \overline{u}(t_0)]^+|_{L^1(\Omega)} + |[V(t_0) - \overline{V}(t_0)]^+|_{L^1(\Gamma)}. \end{aligned} \tag{2.1}$$

In particular, if $u(t_0,\cdot) \le \bar{u}(t_0,\cdot)$ a.e. on Ω and $V(t_0,\cdot) \le \bar{V}(t_0,\cdot)$ a.e. on Γ, then

$$u \le \bar{u} \quad \text{a.e. on } R \times \Omega \quad \text{and} \quad V \le \bar{V} \quad \text{a.e. on } R \times \Gamma.$$

Next, we consider the degenerate parabolic equation with linear dynamic boundary condition:

$$\begin{cases} u_t - \Delta\beta(u) = 0 & \text{in } (t_0,\infty) \times \Omega, \\ \dfrac{\partial\beta(u)}{\partial\nu} + \dfrac{\partial\rho(V)}{\partial t} + h = 0 & \text{on } (t_0,\infty) \times \Gamma, \\ \beta(u) = V & \text{on } (t_0,\infty) \times \Gamma, \end{cases} \tag{2.2}$$

where h is a function given in $L^2_{loc}(R; L^2(\Gamma))$ and $-\infty \le t_0 < \infty$; $\beta : R \to R$ and $\rho : R \to R$ are functions satisfying $\{(\beta 1), (\beta 2)\}$ and (ρ), respectively.

Following [5, section2], we say that for any compact interval $J = [t_0, t_1]$, a couple of functions $u : J \to L^2(\Omega)$ and $V : J \to L^2(\Gamma))$ is a solution of (2.2) on J, if $E^*(u,V) \in W^{1,2}(J; X^*)$, $u \in L^\infty(J; L^2(\Omega))$, $\beta(u) \in L^2(J; X)$, $V \in L^\infty(J; L^2(\Gamma))$, $\beta(u) = V$ a.e. on $J \times \Gamma$ and

$$\langle \frac{d}{dt} E^*(u(t), \rho(V)(t)), \eta\rangle_X + A(\beta(u(t)), \eta) + \int_\Gamma h(t,\cdot)\eta d\Gamma = 0 \quad \text{for any } \eta \in X \text{ and a.e. } t \in J.$$

For a general interval $J' \subset R$, solutions of (2.2) on J' are defined in a manner similar to definition 1.2. Also, solutions of Cauchy problems and the problems with T-periodic condition are defined just as definition 1.3.

Next, we mention some results on T-periodic solutions to (2.2) on R.

Theorem 2.2. *We suppose that $h \in L^2_{loc}(R; L^2(\Gamma))$. Let T be a positive number, and assume that*

$$h(t+T,\cdot) = h(t,\cdot) \text{ a.e. on } \Gamma \text{ for any } t \in R,$$

and

$$\int_0^T \int_\Gamma h(t,x) d\Gamma dt = 0.$$

Then the following statements (i) ~ (iv) holds.

(i) For each $a_0 \in R$ there exists a T-periodic solution $\{u, V\}$ of (2.2) on R such that

$$\int_\Omega u(0,x)dx + \int_\Gamma \rho(V)(0,x)d\Gamma = a_0.$$

(ii) Let $\{u, V\}$ be a solution of (2.2) on R. Then $\{u, V\}$ is T-periodic solution of (2.2) if and only if $u \in L^\infty(R; L^2(\Omega))$ and $V \in L^\infty(R; L^2(\Gamma))$.

(iii) Let $\{u_1, V_1\}$, $\{u_2, V_2\}$ be T-periodic solutions of (2.2) on R such that

$$\int_\Omega u_1(0,x)dx + \int_\Gamma \rho(V_1)(0,x)d\Gamma = \int_\Omega u_2(0,x)dx + \int_\Gamma \rho(V_2)(0,x)d\Gamma.$$

Then

$$\beta(u_1) = \beta(u_2) \quad \text{a.e. on } R \times \Omega,$$

and there exist functions $w \in L^2(\Omega)$, $w_\Gamma \in L^2(\Gamma)$ with $\int_\Omega w dx + \int_\Gamma w_\Gamma d\Gamma = 0$ such that

$$\left.\begin{array}{ll} u_1(t,\cdot) - u_2(t,\cdot) = w(\cdot) & \text{a.e. on } \Omega \\ \rho(V_1)(t,\cdot) - \rho(V_2)(t,\cdot) = w_\Gamma(\cdot) & \text{a.e. on } \Gamma \end{array}\right\} \text{ for any } t \in R.$$

(iv) Let $\{u_1, V_1\}$, $\{u_2, V_2\}$ be two T-periodic solutions of (2.2) on R such that

$$\int_\Omega u_1(0,x)dx + \int_\Gamma \rho(V_1)(0,x)d\Gamma > \int_\Omega u_2(0,x)dx + \int_\Gamma \rho(V_2)(0,x)d\Gamma.$$

Then

$$\beta(u_1) \geq \beta(u_2) \quad \text{a.e. on } R \times \Omega.$$

Finally, as to the asymptotic stability of T-periodic solutions we prove the following.

Theorem 2.3. *Suppose that all conditions of theorem 2.2 are satisfied. Let t_0 be any positive number and let $\{u, V\}$ be any solution of (2.2) on $[t_0, \infty)$. Then there exists a T-periodic solution $\{\overline{u}, \overline{V}\}$ of (2.2) on R such that*

$$\int_\Omega u(t,x)dx + \int_\Gamma V(t,x)d\Gamma = \int_\Omega \overline{u}(t,x)dx + \int_\Gamma \overline{V}(t,x)d\Gamma \quad \text{for any } t \geq t_0,$$

$$\begin{array}{ll} u(t) - \overline{u}(t) \to 0 & \text{weakly in } L^2(\Omega) \text{ as } t \to \infty, \\ V(t) - \overline{V}(t) \to 0 & \text{weakly in } L^2(\Gamma) \text{ as } t \to \infty, \end{array}$$

and

$$\beta(u(nT + \cdot)) \to \beta(\overline{u}) \quad \text{in } L^2(0,T;Y) \text{ as } n \to \infty.$$

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相転移問題に対する解の漸近挙動について

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1. Introduction

We consider the following nonlinear system:

$$\frac{\partial \rho(u)}{\partial t} + \frac{\partial w}{\partial t} - \Delta u = f(t,x) \qquad \text{in } Q := (0,+\infty,) \times \Omega, \tag{1.1}$$

$$\nu \frac{\partial w}{\partial t} + \beta(w) + g(w) \ni u \quad \text{in } Q \tag{1.2}$$

with lateral boundary condition:

$$\frac{\partial u}{\partial n} + \alpha_N(x)u = h_N(t,x) \quad \text{on } \Sigma := (0,+\infty) \times \Gamma, \tag{1.3}$$

and initial conditions:

$$u(0,\cdot) = u_0, \quad w(0,\cdot) = w_0 \quad \text{in } \Omega. \tag{1.4}$$

Here Ω is a bounded domain in R^N $(N \geq 1)$ with smooth boundary $\Gamma := \partial\Omega$; ρ is a monotone increasing and bi-Lipschitz continuous function on R; ν is a positive constant; β is a maximal monotone graph in $R \times R$; g is a smooth function defined on R; α_N is a non-negative, bounded and measurable function on Γ such that $\alpha_N > 0$ on a subset of Γ with positive measure; f, h_N, u_0 and w_0 are given data.

For simplicity problem (1.1)-(1.4) is denoted by (CP). This is a simplified model for a class of solid-liquid phase change problems, and in this context u represents a function related to temperature and w a non-conserved order parameter (the state variable characterizing phase). For instance, we have the following examples:

(1) Stefan problem with phase relaxation, in which β is the subdifferential of the indicator function of the interval $[0,1]$ and $g \equiv 0$. This case was discussed as a melting problem with supercooling and superheating effect in [13,5].

(2) Phase-field model with constraint, in which β is the same as in (1), $\rho(u) = u$, $g(w) = w^3 - cw$ with a positive constant c, and a diffusion term $-\kappa \Delta w$ is added to the left side of (1.2). This is a phase-field model with constraint $0 \leq w \leq 1$ and was discussed in [6,9,11]. We may consider system (1.1)-(1.4) as an approximation of this problem with small $\kappa > 0$.

Furthermore we refer to [2,1] for papers dealing with similar problems.

In this paper, we discuss the large-time behavior of the solution $\{u, w\}$. In fact, under the condition that $f(t,x) \to f^\infty(x)$ and $h_N(t,x) \to h_N^\infty(x)$ in an appropriate sense as $t \to +\infty$, it will be shown that as $t \to +\infty$, $u(t,\cdot)$ and $w(t,\cdot)$ converge to a solution $\{u^\infty, w^\infty\}$ of the corresponding steady-state problem

$$\begin{cases} -\Delta u^\infty = f^\infty(x) \quad \text{in } \Omega, \\ \dfrac{\partial u^\infty}{\partial n} + \alpha_N(x) u^\infty = h_N^\infty(x) \quad \text{on } \Gamma, \\ \beta(w^\infty) + g(w^\infty) \ni u^\infty \quad \text{in } \Omega. \end{cases}$$

This paper is much due to [12].

2. Existence and uniqueness result for (CP)

Problem (CP) is discussed under the following assumptions (A1)-(A6):

(A1) $\rho : R \longrightarrow R$ is an increasing and bi-Lipschitz continuous function.

(A2) β is a maximal monotone graph in $R \times R$ such that for some numbers σ_*, σ^* with $-\infty < \sigma_* < \sigma^* < +\infty$

$$\overline{D(\beta)} = [\sigma_*, \sigma^*];$$

note in this case that $R(\beta) = R$, so that there is a non-negative proper l.s.c. convex function $\hat{\beta}$ on R whose subdifferential $\partial\hat{\beta}$ coincides with β in R, and in the context of solid-liquid system we can consider that $w = \sigma_*$ (resp. σ^*) indicates the pure solid (resp. liquid) phase and any intermediate value w indicates a state of mixture.

(A3) $g : R \to R$ is a Lipschitz continuous function with compact support in R; in this case note that there is a non-negative primitive $\hat{g}$ of g.

(A4) $f \in L^2_{loc}(R_+; L^2(\Omega))$.

(A5) $h_N \in W^{1,2}_{loc}(R_+; L^2(\Gamma))$ with $\sup_{t \ge 0} |h_N|_{W^{1,2}(t,t+1;L^2(\Gamma))} < +\infty$.

(A6) $u_0 \in L^2(\Omega)$ and $w_0 \in L^2(\Omega)$ with $\hat{\beta}(w_0) \in L^1(\Omega)$.

We introduce some function spaces and a convex function in order to discuss (CP) in the framework of abstract evolution equations of the form

$$U'(t) + \partial\varphi^t(U(t)) + G(U(t)) \ni \tilde{f}(t).$$

Let $V := H^1(\Omega)$ with norm

$$|z|_V := \{|\nabla z|^2_{L^2(\Omega)} + \int_\Gamma \alpha_N |z|^2 d\Gamma\}^{\frac{1}{2}},$$

and denote by V^* the dual space of V and by $\langle \cdot, \cdot \rangle$ the duality pairing between V^* and V. Then, identifying $L^2(\Omega)$ with its dual space by means of the usual inner product

$$(v, z) := \int_\Omega vz dx,$$

we see that

$$V \subset L^2(\Omega) \subset V^*$$

with compact injections.

Let F be the duality mapping from V onto V^* which is given by the formula

$$\langle Fv, z \rangle = \int_\Omega \nabla v \cdot \nabla z dx + \int_\Gamma \alpha_N vz d\Gamma \quad \text{for any } v, z \in V.$$

It is easy to see that V^* becomes a Hilbert space with inner product $(\cdot, \cdot)_*$ given by

$$(v, z)_* := \langle v, F^{-1}z \rangle \ (= \langle z, F^{-1}v \rangle) \quad \text{for any } v, z \in V^*.$$

Now, consider the product space

$$X := V^* \times L^2(\Omega),$$

which becomes a Hilbert space with inner product $(\cdot, \cdot)_X$ given by

$$([e_1, w_1], [e_2, w_2])_X := (e_1, e_2)_* + \nu(w_1, w_2) \quad \text{for any } [e_i, w_i] \in X \ (i = 1, 2).$$

Next, given the boundary data h_N, choose $h : R_+ \longrightarrow H^1(\Omega)$ such that for each $t \geq 0$

$$\int_\Omega \nabla h(t) \cdot \nabla z dx + \int_\Gamma \alpha_N h(t) z d\Gamma = \int_\Gamma h_N(t) z d\Gamma \quad \text{for all } z \in V;$$

note from (A5) that $\sup_{t\geq 0} |h|_{W^{1,2}(t,t+1;H^1(\Omega))} < +\infty$.

Also, using h and $\hat{\beta}$, for each $t \geq 0$, define a proper l.s.c. convex function φ^t on X by

$$\varphi^t(U) = \begin{cases} \int_\Omega \rho^*(e-w)dx + \int_\Omega \hat{\beta}(w)dx - (h(t), e) \\ \qquad \text{if } U = [e, w] \in L^2(\Omega) \times L^2(\Omega) \text{ with } \hat{\beta}(w) \in L^1(\Omega), \\ +\infty \quad \text{otherwise,} \end{cases}$$

where ρ^* is a non-negative primitive of ρ^{-1}. We denote by $\partial\varphi^t$ the subdifferential of φ^t in X and its characterization is given by the following theorem.

Theorem 2.1. *(cf. [5,9]) Let $t \geq 0$, $[e^*, w^*] \in X$ and $[e, w] \in D(\partial\varphi^t)$. Then $[e^*, w^*] \in \partial\varphi^t([e, w])$ if and only if conditions (a) and (b) below are satisfied:*

(a) $e^* = F(\rho^{-1}(e-w) - h(t))$, *that is,* $\rho^{-1}(e-w) - h(t) \in V$ *and*

$$\langle e^*, z\rangle = \int_\Omega \nabla(\rho^{-1}(e-w) - h(t)) \cdot \nabla z dx + \int_\Gamma \alpha_N(\rho^{-1}(e-w) - h(t)) z d\Gamma$$

$$\text{for all } z \in V;$$

(b) there exists a function $\xi \in L^2(\Omega)$ such that $\xi \in \beta(w)$ a.e. on Ω and

$$\nu w^* = \xi - \rho^{-1}(e-w) \quad \text{in } L^2(\Omega).$$

Moreover, for $U_i^* = [e_i^*, w_i^*] \in \partial\varphi^t(U_i)$ with $U_i = [e_i, w_i] \in D(\partial\varphi^t)$ $(i = 1, 2)$,

$$(U_1^* - U_2^*, U_1 - U_2)_X = |(e_1 - w_1) - (e_2 - w_2)|_{L^2(\Omega)} + (\xi_1 - \xi_2, w_1 - w_2),$$

where $\xi_i \in L^2(\Omega)$ is as any function ξ in (b) for each $i = 1, 2$.

According to Theorem 2.1, (CP) can be reformulated as an evolution equation in X in the following form:

$$\begin{cases} U'(t) + \partial\varphi^t(U(t)) + G(U(t)) \ni \tilde{f}(t), \quad \text{in } X,\ t \geq 0, \\ U(0) = [\rho(u_0) + w_0, w_0], \end{cases}$$

where $U(t) = [\rho(u(t)) + w(t), w(t)], G(U(t)) = [0, \frac{1}{\nu} g(w(t))]$ and $\tilde{f}(t) = [f(t), 0]$.

As to the solvability of (CP) we have:

Theorem 2.2. *(cf. [6,9]) Assume that (A1)-(A6) hold. Then, for any $T > 0$, (CP) admits one and only one solution $\{u, w\}$ on $[0,T]$ such that*

$$\begin{cases} t^{\frac{1}{2}}\rho(u)' \in L^2(0,T;V^*), & t^{\frac{1}{2}}u \in L^2(0,T;H^1(\Omega)), \\ t\rho(u)' \in L^2(0,T;L^2(\Omega)), & tu \in L^\infty(0,T;H^1(\Omega)), \end{cases}$$

$$t^{\frac{1}{2}}w' \in L^2(0,T;L^2(\Omega)), \quad t\hat{\beta}(w) \in L^\infty(0,T;L^1(\Omega)),$$
$$t^{\frac{1}{2}}\xi \in L^2(0,T;L^2(\Omega))$$

where ξ is the function in condition (w3).

3. Large-time behavior of the solution

Further suppose that there are $h_N^\infty \in L^2(\Gamma)$ and $f^\infty \in L^2(\Omega)$ such that

$$h_N - h_N^\infty \in L^2(R_+;L^2(\Gamma)), \quad f - f^\infty \in L^2(R_+;L^2(\Omega)), \tag{3.1}$$

and consider the steady-state problem (3.2)-(3.3):

$$-\triangle u^\infty = f^\infty(x) \text{ in } \Omega, \quad \frac{\partial u^\infty}{\partial n} + \alpha_N(x)u^\infty = h_N^\infty(x) \quad \text{on } \Gamma, \tag{3.2}$$

$$\beta(w^\infty) + g(w^\infty) \ni u^\infty \quad \text{in } \Omega. \tag{3.3}$$

We should note that problem (3.2) does not include w^∞, and it has a unique solution $u^\infty \in H^1(\Omega)$ in the variational sense, i.e.,

$$\int_\Omega \nabla(u^\infty - h^\infty)\cdot\nabla z dx + \int_\Gamma \alpha_N(u^\infty - h^\infty)z d\Gamma = (f^\infty, z) \quad \text{for all } z \in V, \tag{3.4}$$

where $h^\infty \in H^1(\Omega)$ such that

$$\int_\Omega \nabla h^\infty \cdot \nabla z dx + \int_\Gamma \alpha_N h^\infty z d\Gamma = \int_\Gamma h_N^\infty z d\Gamma \quad \text{for all } z \in V.$$

We see from (3.1) that $h - h^\infty \in L^2(R_+;H^1(\Omega))$.

In the sequel we mean by (P$^\infty$) the algebraic relation (3.3) with the solution $u^\infty \in H^1(\Omega)$ of (3.4), and $w^\infty = w^\infty(x)$ is called a solution of (P$^\infty$).

As the following example shows, the steady-state problem (P^∞) has in general infinitely many solutions.

Example 3.1. Consider the case when

$$f^\infty(x) \equiv 0, \quad h_N^\infty(x) \equiv l_0, \quad \alpha_N(x) \equiv 1, \quad \beta = \partial I_{[-1,1]} \text{ and } g(w) = w^3 - w$$

where l_0 is a constant. Then, clearly $u^\infty \equiv l_0$ and we have the following three possibilities:

(i) when $l_0 > \frac{2}{3\sqrt{3}}$ (resp. $l_0 < -\frac{2}{3\sqrt{3}}$), the algebraic relation

$$\beta(r) + g(r) \ni u^\infty (= l_0) \tag{3.5}$$

has exactly one solution $r = 1$ (resp. -1).

(ii) when $l_0 = \frac{2}{3\sqrt{3}}$ (resp. $-\frac{2}{3\sqrt{3}}$), (3.5) has exactly two solutions $r = -\frac{1}{\sqrt{3}}$ (resp. $\frac{1}{\sqrt{3}}$), 1 (resp. -1).

(iii) when $|l_0| < \frac{2}{3\sqrt{3}}$, (3.5) has exactly three solutions $r = \xi_-, \xi_0, \xi_+$ with $-1 \le \xi_- < \xi_0 < \xi_+ \le 1$.

Physically (i) means that if the temperature is kept high (resp. low) enough, then the limit state (as $t \longrightarrow +\infty$) will be of pure liquid (resp. solid). On the other hand, (ii) and (iii) mean that if the temperature is kept near the phase transition temperature, then the limit state possibly includes a mushy region. In particular, in the case of (iii), all step functions w^∞ with range in $\{\xi_-, \xi_0, \xi_+\}$ are solutions of (P^∞) and hence (P^∞) has in general infinitely many solutions.

Our main result is stated in the following theorem.

Theorem 3.1. *Suppose that conditions (A1)-(A6) and (3.1) hold, and let $\{u, w\}$ be the solution to (CP) on R_+. Further, suppose that for each $p \in R$ the (algebraic) inclusion*

$$\beta(r) + g(r) \ni p$$

has a finite number of solutions r in $\overline{D(\beta)}$. Then,

$$u(t) \longrightarrow u^\infty \text{ weakly in } H^1(\Omega) \text{ as } t \longrightarrow +\infty, \tag{3.6}$$

where u^∞ *is the unique solution of (3.4), and there exists a function* $w^\infty \in L^\infty(\Omega)$ *such that*

$$\beta(w^\infty(x)) + g(w^\infty(x)) \ni u^\infty(x) \quad \text{for a.e. } x \in \Omega$$

and

$$w(t,x) \longrightarrow w^\infty(x) \text{ for a.e. } x \in \Omega \text{ as } t \longrightarrow +\infty.$$

We prove the theorem by the following four lemmas.

Lemma 3.1. *Under the same assumptions of Theorem 3.1, for the solution* $\{u, w\}$ *to (CP) on* R_+*, we have*

$$u - u_\infty \in L^2(R_+; H^1(\Omega)),\ w' \in L^2(R_+; L^2(\Omega)) \text{ and } \hat{\beta}(w) \in L^\infty(R_+; L^1(\Omega)), \tag{3.7}$$

$$u \in L^\infty([1, +\infty); H^1(\Omega)). \tag{3.8}$$

Lemma 3.2. *Under the same assumptions of Theorem 3.1, put*

$$U^t(x) := \int_t^{t+1} |w'(\tau, x)|^2 d\tau \quad \text{for } x \in \Omega.$$

Then, $U^t(x) \longrightarrow 0$ *as* $t \longrightarrow +\infty$ *for a.e.* $x \in \Omega$.

Lemma 3.3. *Under the same assumptions of Theorem 3.1, (3.6) holds.*
Lemma 3.4. *Under the same assumptions of Theorem 3.1, put*

$$V(x) := \{r \in \overline{D(\beta)}; w(t_n, x) \to r \text{ for some } t_n \text{ with } t_n \to +\infty\} \quad \text{for } x \in \Omega.$$

Then,

(1) $V(x) \neq \emptyset$ *for a.e.* $x \in \Omega$*;*

(2) $\beta(r) + g(r) \ni u^\infty(x)$ *for all* $r \in V(x)$ *and a.e.* $x \in \Omega$*;*

(3) $V(x)$ *is a singleton for a.e.* $x \in \Omega$*.*

In particular, (2) and (3) of Lemma 3.4 imply that $w(t,x)$ converges to a solution $w^\infty(x)$ for a.e. $x \in \Omega$ as $t \longrightarrow +\infty$ and the limit w^∞ is a solution of (P^∞). Thus we complete the proof of Theorem 3.1.

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ON SOME ESTIMATE OF CAUCHY PROBLEM FOR DEGENERATE ELLIPTIC EQUATIONS OF MONGE-AMPÈRE TYPE WITH TWO VARIABLES

Takaaki Yamashiro

1 Results

Let D be a bounded domain in the (x, y)-plane with its boundary ∂D. Let Γ be a connected open subset of ∂D. We assume that $D \subset \{y > 0\}$, $\Gamma \ni O$(the origin) and Γ is of class C^1.

We write for $\rho > 0$, $D_\rho = D \cap \{y < \rho\}$, $\Gamma_\rho = \Gamma \cap \{y < \rho\}$, $l_\rho = D \cap \{y = \rho\}$. We define the following definitions :

(H.1) There is a real number a with $0 < a < 1$ such that each l_ρ is an open segment and $|l_\rho| \leq |l_{\rho'}| \leq 1/2$ for any ρ, ρ' with $0 < \rho < \rho' \leq a$.

If (H.1) is satisfied, let us say often that (H.1) holds for D_a.

(H.2) Under the hypothesis of (H.1), there is a number $c > 0$ and a function $\varphi(x) \in C^2(\{|x| \leq c\})$ such that $\varphi(0) = 0$, $\varphi(\pm c) \geq a$, $\{(x, \varphi(x)); |x| \leq c\} \subset \Gamma$ and $\varphi''(x) > 0$ in $\{|x| \leq c\}$.

In (1.1) we assume that the lower order term g has the form

$$g(x, y, z) \leq Kz^2$$

for some positive constant K. So the equation (1.1) becomes

$$(\partial_x \partial_y u)^2 - \partial_x^2 u \cdot \partial_y^2 u \leq K u^2. \tag{1.1}$$

We denote the norms of $L^\infty(D_\rho)$ and $L^\infty(\Gamma_\rho)$ by $\| \quad \|_\rho$ and $\langle \quad \rangle_\rho$, respectively.

Our aim is to prove

Theorem 1 *Suppose that* (H.1) *is satisfied. Suppose that u belongs to $C^2(\overline{D_a})$ and it is a solution of* (2.1) *in D_a. Let*

$$\varepsilon = \langle u \rangle_a + \langle \partial_x u \rangle_a + \langle \partial_y u \rangle_a + \langle \partial_x \partial_y u \rangle_a + \langle \partial_y^2 u \rangle_a,$$

And let

$$\varepsilon \cdot \max(e^a, e^{\sqrt{2K}a}) \leq 1.$$

Then it holds that

$$\|u\|_{\frac{a}{2}} + \|\partial_x u\|_{\frac{a}{2}} \le Ca^{-2}\varepsilon^{\frac{1}{6}},$$

where C is a positive constant independent of a, K, ε, Γ and D.

Remark 1 The inequality (2.1) is invariant under the orthognal transformation of coordinates. So we can generalize the domain by the rotation of D arround the origin.

Remark 2 The convexity of D at O can not be removed, becauce any function independent of y is always the solution of (2.1).

Next we assume (H.2). Let x_0 be a real number such that $0 < |x_0| < c/2$ and $|\varphi'(x_0)| < 1/2$. Around the point $(x_0, \varphi(x_0))$ we take the orthognal transformation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - \varphi(x_0) \end{pmatrix},$$

where $\sin\theta = \varphi'(x_0)/\sqrt{1+\varphi'(x_0)^2}$. That is, $\xi(\eta)$-axis is the tangent(normal) line of Γ at $(x_0, \varphi(x_0))$, respectivelyWe define for $\rho > 0$, $E_\rho = D \cap \{(\xi,\eta); 0 < \eta < \rho\}$. We look at D as a domain in the new plane with (ξ,η)-coordinates. Under the assumption (H.2), the following is easily verified: There are x_0, a, $\tilde{a}$ such that $0 < |x_0| < c/2$, $|\varphi'(x_0)| < 1/2$, $0 < \tilde{a} < a/2$, $D_{\tilde{a}} \subset D_{\frac{a}{2}} \cap E_{\frac{a}{2}}$, and (H.1) holds for both D_a and E_a. We denote $\beta = \sin\theta$.

Under these assumptions we have

Theorem 2 *Let u be the function in Theorem 1. Let*

$$\tilde{\varepsilon} = \langle u\rangle_\Gamma + \langle\partial_x u\rangle_\Gamma + \langle\partial_y u\rangle_\Gamma + \langle\partial_x\partial_y u\rangle_\Gamma + \langle\partial_y^2 u\rangle_\Gamma,$$

And let

$$\tilde{\varepsilon}\cdot\max(e^a, e^{\sqrt{2K}a}) \le 1.$$

Then it holds that

$$\|u\|_{\tilde{a}} + \|\partial_x u\|_{\tilde{a}} + |\beta|\|\partial_y u\|_{\tilde{a}} \le Ca^{-2}(\tilde{\varepsilon} + |\beta|\langle\partial_x^2 u\rangle_\Gamma)^{\frac{1}{6}},$$

where C is a positive constant independent of a, $\tilde{a}$, K, ε, Γ, D, x_0 and β.

2 Lemma

We assume (H.1) and a is the real number in (H.1). This lemma is known to all (see e.g., Lemma 3 in [1]).

Lemma 1 *Let $p \ge 1$ and f belong to $C^1(\overline{D}_a)$. Then it holds that for ρ with $0 < \rho < a$*

$$\iint_{D_\rho} |f|^p dxdy \le 2^p \int_{\Gamma_\rho} |f|^p d\sigma + (2|l_\rho|)^p \iint_{D_\rho} |\partial_x f|^p dxdy.$$

3 Proof of Theorem 1

We give the proof of Theorem 1 in this section. Let $0 < \rho \leq a$ and u be the function in Theorem 1. We denote by $(\, , \,)_\rho$ the inner product of $L^2(D_\rho)$.

Let us set $v(x,y) = e^{\lambda y}u(x,y)$ for $\lambda \leq -1$. It is easily verified from (2.1) that

$$(\partial_x\partial_y v)^2 - \partial_x^2 v \cdot \partial_y^2 v - 2\lambda\partial_x v \cdot \partial_x\partial_y v + \lambda^2(\partial_x v)^2 + 2\lambda\partial_y v \cdot \partial_x^2 v - \lambda^2 v\partial_x^2 v \leq K v^2.$$

From this it follows that for $k \geq 0$

$$\begin{aligned}(3.1) \qquad & ((\partial_x\partial_y v)^2, |\partial_x v|^k)_\rho - (\partial_x^2 v \cdot \partial_y^2 v, |\partial_x v|^k)_\rho - 2\lambda(\partial_x v \cdot \partial_x\partial_y v, |\partial_x v|^k)_\rho \\ & +\lambda^2((\partial_x v)^2, |\partial_x v|^k)_\rho + 2\lambda(\partial_y v \cdot \partial_x^2 v, |\partial_x v|^k)_\rho - \lambda^2(v\partial_x^2 v, |\partial_x v|^k)_\rho \leq K(v^2, |\partial_x v|^k)_\rho.\end{aligned}$$

After here let $\mathbf{n}$ be the outer normal of ∂D_ρ. And let $(x, \mathbf{n})$ $((y, \mathbf{n}))$ be the angle between x-axis (y-axis) and $\mathbf{n}$, respectively. By integration by parts we see that

$$\begin{aligned}-(\partial_x^2 v \cdot \partial_y^2 v, |\partial_x v|^k)_\rho &= -\frac{1}{1+k}(\partial_x(|\partial_x v|^k \partial_x v), \partial_y^2 v)_\rho \\ &= -\frac{1}{1+k}\int_{\partial D_\rho} |\partial_x v|^k \partial_x v \cdot \partial_y^2 v \cos(x, \mathbf{n}) d\sigma + \frac{1}{1+k}(|\partial_x v|^k \partial_x v, \partial_x \partial_y^2 v)_\rho,\end{aligned}$$

and

$$(|\partial_x v|^k \partial_x v, \partial_x\partial_y^2 v)_\rho = \int_{\partial D_\rho} |\partial_x v|^k \partial_x v \cdot \partial_x \partial_y v \cos(y, \mathbf{n}) d\sigma - (1+k)(|\partial_x v|^k, (\partial_x\partial_y v)^2)_\rho.$$

Here the third derivatives of v appear. But it is not necessary to assume three times differentiability of v, if we take an approximating sequence of v. Thus we have

$$\begin{aligned}-(\partial_x^2 v \cdot \partial_y^2 v, |\partial_x v|^k)_\rho &= -\frac{1}{1+k}\int_{\partial D_\rho} |\partial_x v|^k \partial_x v \cdot \partial_y^2 v \cos(x, \mathbf{n}) d\sigma \\ &+\frac{1}{1+k}\int_{\partial D_\rho} |\partial_x v|^k \partial_x v \cdot \partial_x\partial_y v \cos(y, \mathbf{n}) d\sigma - (|\partial_x v|^k, (\partial_x\partial_y v)^2)_\rho.\end{aligned}$$

Further we have the following equalities:

$$\begin{aligned}(\partial_x v \cdot \partial_x\partial_y v, |\partial_x v|^k)_\rho &= \frac{1}{2+k}(1, \partial_y(|\partial_x v|^{2+k}))_\rho \\ &= \frac{1}{2+k}\int_{\partial D_\rho} |\partial_x v|^{2+k} \cos(y, \mathbf{n}) d\sigma,\end{aligned}$$

$$\begin{aligned}(\partial_y v \cdot \partial_x^2 v, |\partial_x v|^k)_\rho &= \frac{1}{1+k}(\partial_y v, \partial_x(|\partial_x v|^k \partial_x v))_\rho \\ &= \frac{1}{1+k}\int_{\partial D_\rho} |\partial_x v|^k \partial_x v \cdot \partial_y v \cos(x, \mathbf{n}) d\sigma - \frac{1}{1+k}(\partial_x\partial_y v, |\partial_x v|^k \partial_x v)_\rho \\ &= \frac{1}{1+k}\int_{\partial D_\rho} |\partial_x v|^k \partial_x v \cdot \partial_y v \cos(x, \mathbf{n}) d\sigma \\ &\qquad -\frac{1}{(1+k)(2+k)}\int_{\partial D_\rho} |\partial_x v|^{2+k} \cos(y, \mathbf{n}) d\sigma,\end{aligned}$$

$$
\begin{aligned}
(v\partial_x^2 v, |\partial_x v|^k)_\rho &= \frac{1}{1+k}(v, \partial_x(|\partial_x v|^k \partial_x v))_\rho \\
&= \frac{1}{1+k}\int_{\partial D_\rho} |\partial_x v|^k \partial_x v \cdot v \cos(x, \mathrm{n}) d\sigma - \frac{1}{1+k}(1, |\partial_x v|^{2+k})_\rho.
\end{aligned}
$$

Combining the above equalities with (3.1), we obtain

$$
\begin{aligned}
(3.2)\quad \frac{2+k}{1+k}\lambda^2(1, |\partial_x v|^{2+k})_\rho &\leq \frac{1}{1+k}\Big[\int_{\partial D_\rho} |\partial_x v|^k \partial_x v \cdot \partial_y^2 v \cos(x, \mathrm{n}) d\sigma \\
&\qquad - \int_{\partial D_\rho} |\partial_x v|^k \partial_x v \cdot \partial_x \partial_y v \cos(y, \mathrm{n}) d\sigma\Big] \\
&+ \frac{2}{2+k}\lambda \int_{\partial D_\rho} |\partial_x v|^{2+k} \cos(y, \mathrm{n}) d\sigma \\
&- \frac{2}{1+k}\lambda \int_{\partial D_\rho} |\partial_x v|^k \partial_x v \cdot \partial_y v \cos(x, \mathrm{n}) d\sigma \\
&+ \frac{2}{(1+k)(2+k)}\lambda \int_{\partial D_\rho} |\partial_x v|^{2+k} \cos(y, \mathrm{n}) d\sigma \\
&+ \frac{1}{1+k}\lambda^2 \int_{\partial D_\rho} |\partial_x v|^k \partial_x v \cdot v \cos(x, \mathrm{n}) d\sigma \\
&+ K(v^2, |\partial_x v|^k)_\rho.
\end{aligned}
$$

From now on let k be sufficiently large and let us take λ with $2K \leq \lambda^2$. Here we have the following equalities:

$$
\begin{aligned}
&\int_{\partial D_\rho} |\partial_x v|^k \partial_x v \cdot \partial_x \partial_y v \cos(x, \mathrm{n}) d\sigma \\
&= \int_{\Gamma_\rho} |\partial_x v|^k \cdot \partial_x v \cdot \partial_x \partial_y v d\sigma - \frac{1}{2+k}\int_{l_\rho} \partial_y(|\partial_x v|^{2+k}) d\sigma,
\end{aligned}
$$

$$
\begin{aligned}
\int_0^a \int_{l_\rho} \partial_y(|\partial_x v|^{2+k}) d\sigma dx &= (1, \partial_y(|\partial_x v|^{2+k}))_a = \int_{\partial D_a} |\partial_x v|^{2+k} \cos(y, \mathrm{n}) d\sigma \\
&= \int_{\Gamma_a} |\partial_x v|^{2+k} \cos(y, \mathrm{n}) d\sigma + \int_{l_a} |\partial_x v|^{2+k} d\sigma.
\end{aligned}
$$

Hence, integrating the both sides of (3.2) from zero to a with respect to ρ, we obtain

$$
\begin{aligned}
(3.3)\quad \frac{2+k}{1+k}\lambda^2 \int_0^a (1, |\partial_x v|^{2+k})_\rho d\rho &\leq \frac{1}{1+k}\Big[\int_{\Gamma_a} |\partial_x v|^{1+k} |\partial_y^2 v| d\sigma \\
&+ \int_{\Gamma_a} |\partial_x v|^{1+k} |\partial_x \partial_y v| d\sigma + \frac{1}{2+k}\int_{\Gamma_a} |\partial_x v|^{2+k} d\sigma\Big] \\
&+ \frac{2}{2+k}|\lambda| \int_{\Gamma_a} |\partial_x v|^{2+k} d\sigma \\
&+ \frac{2}{1+k}|\lambda| \int_{\Gamma_a} |\partial_x v|^{1+k} |\partial_y v| d\sigma \\
&+ \frac{2}{(1+k)(2+k)}|\lambda| \int_{\Gamma_a} |\partial_x v|^{2+k} d\sigma
\end{aligned}
$$

$$+ \quad \frac{1}{1+k}\lambda^2 \int_{\Gamma_a} |\partial_x v|^{1+k}|v| d\sigma$$
$$+ \quad K\int_0^a (v^2, |\partial_x v|^k)_\rho d\rho.$$

By Cauchy-Young's inequality, we have for any function f, g

$$\int_{\Gamma_a} |f|^{1+k}|g| d\sigma \le \frac{1+k}{2+k}\int_{\Gamma_a} |f|^{2+k} d\sigma + \frac{1}{2+k}\int_{\Gamma_a} |g|^{2+k} d\sigma,$$

$$\int_{\Gamma_a} |f|^k g^2 d\sigma \le \frac{k}{2+k}\int_{\Gamma_a} |f|^{2+k} d\sigma + \frac{2}{2+k}\int_{\Gamma_a} |g|^{2+k} d\sigma.$$

Thus we obtain from (3.3)

$$\begin{aligned}
&(\frac{2+k}{1+k}\lambda^2 - \frac{k}{2+k}K)\int_0^a (1, |\partial_x v|^{2+k})_\rho d\rho \\
\le\ & \frac{2}{2+k}\int_{\Gamma_a} |\partial_x v|^{2+k} d\sigma \\
+\ & \frac{1}{(1+k)(2+k)}[\int_{\Gamma_a} |\partial_x v|^{2+k} d\sigma + \int_{\Gamma_a} |\partial_y^2 v|^{2+k} d\sigma + \int_{\Gamma_a} |\partial_x\partial_y v|^{2+k} d\sigma] \\
+\ & \frac{4}{2+k}|\lambda|\int_{\Gamma_a} |\partial_x v|^{2+k} d\sigma \\
+\ & \frac{2}{(1+k)(2+k)}|\lambda|[\int_{\Gamma_a} |\partial_x v|^{2+k} d\sigma + \int_{\Gamma_a} |\partial_y v|^{2+k} d\sigma] \\
+\ & \frac{1}{2+k}\lambda^2\int_{\Gamma_a} |\partial_x v|^{2+k} d\sigma \\
+\ & \frac{1}{(1+k)(2+k)}\lambda^2\int_{\Gamma_a} |v|^{2+k} d\sigma \\
+\ & \frac{2K}{2+k}\int_0^a (1, |v|^{2+k})_\rho d\rho.
\end{aligned}$$

Since $\lambda^2/2 < (2+k)\lambda^2/(1+k) - kK/(2+k)$ and k is large, this becomes

$$\begin{aligned}
\int_0^a (1, |\partial_x v|^{2+k})_\rho d\rho \ \le\ & \int_{\Gamma_a} |\partial_x v|^{2+k} d\sigma + \int_{\Gamma_a} |\partial_y^2 v|^{2+k} d\sigma \\
+\ & \int_{\Gamma_a} |\partial_x\partial_y v|^{2+k} d\sigma \\
+\ & \int_{\Gamma_a} |\partial_y v|^{2+k} d\sigma + \int_{\Gamma_a} |v|^{2+k} d\sigma \\
+\ & \frac{4K}{\lambda^2(2+k)}\int_0^a (1, |v|^{2+k})_\rho d\rho.
\end{aligned} \tag{3.4}$$

From Lemma

$$(1, |v|^{2+k})_\rho \le 2^{2+k}\int_{\Gamma_\rho} |v|^{2+k} d\sigma + (1, |\partial_x v|^{2+k})_\rho. \tag{3.5}$$

Hence we have from (3.4)

$$\int_0^a (1, |\partial_x v|^{2+k})_\rho d\rho \ \le\ 2[\int_{\Gamma_a} |\partial_x v|^{2+k} d\sigma + \int_{\Gamma_a} |\partial_y^2 v|^{2+k} d\sigma \tag{3.6}$$

$$
\begin{aligned}
&+ \int_{\Gamma_a} |\partial_x \partial_y v|^{2+k} d\sigma \\
&+ \int_{\Gamma_a} |\partial_y v|^{2+k} d\sigma + \int_{\Gamma_a} |v|^{2+k} d\sigma \\
&+ \frac{2K}{\lambda^2(2+k)} 2^{2+k} \int_{\Gamma_a} |v|^{2+k} d\sigma].
\end{aligned}
$$

Now in general it holds that for any $f \in C^0(\overline{D_a})$, $f \geq 0$ in D_a

$$
\int_0^a (1,f)_\rho d\rho = \int_0^a \rho'(1,f)_\rho d\rho = a(1,f)_a - \int_0^a \rho \partial_\rho (1,f)_\rho d\rho = (a-y,f)_a \geq \frac{a}{2}(1,f)_{\frac{a}{2}}.
$$

From the above (3.6) becomes

$$
\begin{aligned}
\frac{a}{2}(1, |\partial_x v|^{2+k})_{\frac{a}{2}} \leq\ & 3[\int_{\Gamma_a} |\partial_x v|^{2+k} d\sigma + \int_{\Gamma_a} |\partial_y v|^{2+k} d\sigma + \int_{\Gamma_a} |v|^{2+k} d\sigma \\
&+ \int_{\Gamma_a} |\partial_y^2 v|^{2+k} d\sigma + \int_{\Gamma_a} |\partial_x \partial_y v|^{2+k} d\sigma + 2^{2+k} \int_{\Gamma_a} |v|^{2+k} d\sigma].
\end{aligned}
$$

Hence

$$
\begin{aligned}
(3.7)\quad (\int_{D_{\frac{a}{2}}} |\partial_x v|^{2+k} dxdy)^{\frac{1}{2+k}} \leq\ & (6/a)^{\frac{1}{2+k}} [(\int_{\Gamma_a} |\partial_x v|^{2+k} d\sigma)^{\frac{1}{2+k}} + (\int_{\Gamma_a} |\partial_y v|^{2+k} d\sigma)^{\frac{1}{2+k}} \\
&+ (\int_{\Gamma_a} |v|^{2+k} d\sigma)^{\frac{1}{2+k}} + (\int_{\Gamma_a} |\partial_y^2 v|^{2+k} d\sigma)^{\frac{1}{2+k}} \\
&+ (\int_{\Gamma_a} |\partial_x \partial_y v|^{2+k} d\sigma)^{\frac{1}{2+k}} + 2(\int_{\Gamma_a} |v|^{2+k} d\sigma)^{\frac{1}{2+k}}],
\end{aligned}
$$

where we have used the inequality $(\sum_i a_i)^{\frac{1}{p}} \leq \sum_i a_i^{\frac{1}{p}}$ for $p \geq 1$ and $a_i \geq 0$. Letting $k \to \infty$ in (3.6), we obtain

$$
\|\partial_x v\|_{\frac{a}{2}} \leq \langle \partial_x v \rangle_a + \langle \partial_y v \rangle_a + 3\langle v \rangle_a + \langle \partial_y^2 v \rangle_a + \langle \partial_x \partial_y v \rangle_a
$$

And from (3.5)

$$
\|v\|_{\frac{a}{2}} \leq 2\langle v \rangle_{\frac{a}{2}} + \|\partial_x v\|_{\frac{a}{2}}.
$$

Combining these two inequalities, we conclude that

$$
(3.8)\qquad \|v\|_{\frac{a}{2}} + \|\partial_x v\|_{\frac{a}{2}} \leq C[\langle v \rangle_a + \langle \partial_x v \rangle_a + \langle \partial_y v \rangle_a + \langle \partial_x \partial_y v \rangle_a + \langle \partial_y^2 v \rangle_a]
$$

where C is independent of a, K, ε, λ, Γ and D.

Here we use the following inequalities:

$$
\begin{aligned}
&\langle v \rangle_a \leq \langle u \rangle_a, \qquad \langle \partial_x v \rangle_a \leq \langle \partial_x u \rangle_a, \\
&\langle \partial_y v \rangle_a \leq \langle \partial_y u \rangle_a + |\lambda| \langle u \rangle_a, \\
&\langle \partial_x \partial_y v \rangle_a \leq \langle \partial_x \partial_y u \rangle_a + |\lambda| \langle \partial_x u \rangle_a, \\
&\langle \partial_y^2 v \rangle_a \leq \langle \partial_y^2 u \rangle_a + 2|\lambda| \langle \partial_y u \rangle_a + \lambda^2 \langle u \rangle_a,
\end{aligned}
$$

Then (3.8) becomes

$$e^{\frac{\lambda a}{2}}(\|u\|_{\frac{a}{2}} + \|\partial_x u\|_{\frac{a}{2}}) \leq C\lambda^2[\langle u\rangle'_a + \langle\partial_x u\rangle'_a + \langle\partial_y u\rangle'_a + \langle\partial_x\partial_y u\rangle'_a + \langle\partial_y^2 u\rangle'_a].$$

From the definition of ε we can write

$$\|u\|_{\frac{a}{2}} + \|\partial_x u\|_{\frac{a}{2}} \leq C\lambda^2 e^{\frac{|\lambda|a}{2}}\varepsilon.$$

We set

$$\lambda = -\frac{1}{a}\log(\frac{1}{\varepsilon}).$$

Then $\lambda \leq -1$ and $2K \leq \lambda^2$ from our assumption on ε. Since $\frac{1}{2}(a|\lambda|/3)^2 \leq e^{\frac{|\lambda|a}{3}}$, we have $\lambda^2 \leq Ca^{-2}e^{\frac{|\lambda|a}{3}}$. Therefore

$$\|u\|_{\frac{a}{2}} + \|\partial_x u\|_{\frac{a}{2}} \leq Ca^{-2}e^{\frac{5|\lambda|a}{6}}\varepsilon = Ca^{-2}\varepsilon^{\frac{1}{6}}.$$

This completes the proof of Theorem 1.

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HARDY 空間の偏微分方程式への応用
(EVANS-MÜLLER の論文より)

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つぎの Poisson 方程式

$$-\Delta\psi = \omega \qquad in\ \mathbb{R}^2 \tag{1}$$

の 1 階微分の評価を考える。その為に、次の関数空間を用意する。

定義. $\phi \in C^\infty(\mathbb{R}^n)$ は、

$$\mathrm{supp}\phi \subset B(0,1), \int_{\mathbb{R}^n} \phi = 1 \tag{2}$$

を満たすとする。このとき、$f \in L^1_{loc}(\mathbb{R}^n)$ に対し、

$$f^{**}(x) := \sup_{0<r<1} \left| \frac{1}{r^n} \int_{\mathbb{R}^n} f(y)\phi\left(\frac{x-y}{r}\right) dy \right| \tag{3}$$

$$\mathcal{H}^1_{loc}(\mathbb{R}^n) := \{f \in L^1_{loc}(\mathbb{R}^n) | f^{**} \in L^1_{loc}(\mathbb{R}^n)\} \tag{4}$$

とおく。$\mathcal{H}^1_{loc}(\mathbb{R}^n)$ を、局所 **Hardy** 空間という。

ここでは、Evans-Muller[1] による、次の定理と、その証明について紹介する。

定理. $\phi \in H^1_{loc}(\mathbb{R}^2)$ は、*(1)* の弱い解とする。ただし、$\omega \in L^1 loc(\mathbb{R}^2), \omega \geqq 0$ とする。このとき、$\phi_{x_1}\phi_{x_2}, (\phi_{x_1})^2 - (\phi_{x_2})^2 \in \mathcal{H}^1_{loc}(\mathbb{R}^2)$ である。

証明. $R > 8$ を取り、固定する。

$$v(x) = -\frac{1}{2\pi} \int_{B(0,R)} \omega(y) \log(|x-y|) dy$$

とおく。すると、$r := \phi - v$は、B(0,R) で調和、さらに、

$$v_{x_i}(x) = -\frac{1}{2\pi} \int_{B(0,R)} \omega(y) \frac{x_i - y_i}{|x-y|^2} dy \tag{5}$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

である。
$\psi:\mathbb{R}^2\longrightarrow\mathbb{R}$ を、(2) を満たし、更に$\psi\geqq 0$ となるようにとる。$\mathbb{R}^2$上の点 p をとり、固定する。$0<r<1$ に対し、

(6)
$$\Upsilon_r(p):=\frac{1}{r^2}\int_{B(0,R)} v_{x_1}(x)v_{x_2}(x)\psi\left(\frac{x-p}{r}\right)dx$$
$$=\frac{1}{4\pi r^2}\int_{B(0,R)}\omega(y)\omega(z)dydz\int_{\mathbb{R}^2}\frac{x_1-y_1}{|x-y|^2}\frac{x_2-z_2}{|x-z|^2}\psi\left(\frac{x-p}{r}\right)dx$$

とおく。

(7)
$$K_r(p;y,z):=\int_{\mathbb{R}^2}\frac{x_1-y_1}{|x-y|^2}\frac{x_2-z_2}{|x-z|^2}\psi\left(\frac{x-p}{r}\right)dx\quad(y\neq z)$$

と書くと、

(8)
$$|K_r(p;y,z)|\leqq C_\psi\left(\frac{r}{r+|y-p|}\right)\left(\frac{r}{r+|z-p|}\right)$$

なる評価を得る事が出来る。
(8) の評価を (6) に代入すると、

(9)
$$|\Upsilon_r(p)|\leqq C\left(\int_{B(0,R)}\frac{\omega(y)}{r+|y-p|}dy\right)^2$$

となる。$\omega\geqq 0$ に注意すると、

(10)
$$(v_{x_1}v_{x_2})^{**}(p)=\sup_{0<r<1}|\Upsilon_r(p)|\leqq C\left(\int_{B(0,R)}\frac{\omega(y)}{|y-p|}dy\right)^2$$

を得る。

(11)
$$\sigma(p):=\int_{B(0,R)}\frac{\omega(y)}{|y-p|}dy$$

を計算してゆくことで、$\sigma(p)\in L^2_{loc}(\mathbb{R}^2)$ となる事がわかる。(10) と組み合わせることで、次の評価を得る:

(12)
$$\|(v_{x_1}v_{x_2})^{**}\|_{L^1(B(0,R/2))}\leqq C_R\|\nabla\phi\|^2_{L^1(B(0,2R))}$$

一方、τを含む項については、

(13)
$$\|(\tau_{x_1}\tau_{x_2})^{**}\|_{L^1(B(0,R/2))}\leqq C_R\left(\|\nabla\phi\|^2_{L^2(B(0,R))}+\|\omega\|^2_{L^1(B(0,R))}\right)$$

(14)
$$\|(\tau_{x_1}v_{x_2})^{**}\|_{L^1(B(0,R/2))}\leqq C_R\left(\|\nabla\phi\|^2_{L^2(B(0,R))}+\|\omega\|^2_{L^1(B(0,R))}\right)$$

$$\|(v_{x_1}\tau_{x_2})^{**}\|_{L^1(B(0,R/2))} \leqq C_R\big(\|\nabla\phi\|^2_{L^2(B(0,R))} + \|\omega\|^2_{L^1(B(0,R))}\big) \tag{15}$$

が成り立つが、実は、

$$\|\omega\|^2_{L^1(B(0,R))} \leqq \|\nabla\phi\|^2_{L^2(B(0,2R))} \tag{16}$$

となっている。(12)~(16) を組み合わせることで、

$$\|(\phi_{x_1}\phi_{x_2})^{**}\|_{L^1(B(0,R/2))} \leqq C_R\|\nabla\phi\|^2_{L^2(B(0,2R))} \tag{17}$$

従って、$\phi_{x_1}\phi_{x_2} \in \mathcal{H}^1_{loc}(\mathbb{R}^2)$ がいえる。
+ もう一方の式については、$x_1 \longrightarrow (x_1+x_2)/\sqrt{2}$, $x_2 \longrightarrow (x_1-x_2)/\sqrt{2}$なる変数変換を行なうことで得ることが出来る。 □

注意. 評価 (8) を得るところがこの証明のメインである。特異積分の形をした部分の処理を行なうのに工夫がなされている。

References

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Semigroups of differentiable transformations : The semilinear case

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§1 Introduction

バナッハ空間 $(X, \|\cdot\|)$ における半線形発展方程式

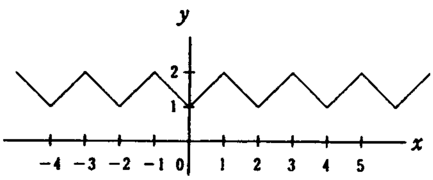
(DE)　$u'(t) = (A+B)u(t),\ t > 0\ ;\ u(0) = x$

の弱解を与える非線形作用素の半群を$\{S(t) : t \geqslant 0\}$とする。ここでAはX上のC_0－半群$\{T(t) : t \geqslant 0\}$の生成作用素、BはXの凸部分集合Cで定義された非線形作用素であるとする。作用素Bが微分可能である場合に、非線形半群$\{S(t)\}$が持つ性質について論じたい。

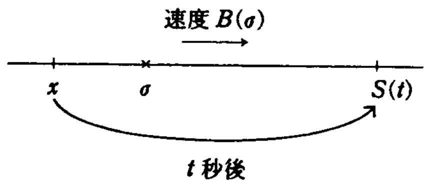
§2

すでに多くの研究者が様々な前提のもとに示している通り、Bが微分可能な場合、各$t \geqslant 0$に対し、作用素$S(t)$はフレッシェ又はガトーの意味で微分可能である。一方各$t \geqslant 0$に対し、作用素$S(t)$が微分可能であってもBは微分可能とはいえない。次の様な単純な例を挙げておく。

<u>反例</u>　$X = C = R^1$　$A = 0$。従って各$t \geqslant 0$に対し、$T(t)$は恒等写像。更に実数値関数$y = B(x)$を (2.1) 図の様に定める。



(2.1)



(2.2)

これは次の性質をを持つ

- 任意の数Xに対し、$B(x) > 0$
- 任意の数Pに対し、$\int_p^\infty \frac{d\sigma}{B(\sigma)} = \infty$
- 関数$B(x)$はリップシッツ連続
- あるxにおいて関数$B(x)$は微分不可能

R^1 上の任意の点 σ において、速度 $B(\sigma)$ で動く様な動点を考える。(2.2) 図

この動点の時刻 0 における座標が x である場合の、時刻 t における動点の座標を $S(t)x$ と定める事にする。これは式で表すと次の様になる：

$$\int_x^{S(t)x} \frac{1}{B(\sigma)} d\sigma = t$$

$\{ S(t) : t \geqslant 0 \}$ は $X = R^1$ 上の非線形半群になり、更に

$$\begin{cases} \cdot \dfrac{\partial}{\partial t} S(t)x = B(S(t)x) = (A+B)(S(t)x) \\ \\ \cdot \dfrac{\partial}{\partial x} S(t)x = \dfrac{B(S(t)x)}{B(x)} \end{cases}$$

そして両方共、2 変数 (t, x) に関し連続。ところが仮定により関数 $B(x)$ は微分可能でない。 ∎

この様に各 $S(t)$ の微分可能性だけでは、B の微分可能性を特徴付けた事にならない。

そこで $\{ S(t) : t \geqslant 0 \}$ に関し何か別の性質も挙げる事により、B の微分可能性の特徴付けに成る様にしたい。

§3

以下次の事を仮定する。

仮定［0］：

- C は Banach 空間 X の convex set
- $\{ S(t) : t \geqslant 0 \}$ は C を定義域とし C の値をとる写像のクラスで

  (s, 1)　$S(0)z = z$　$S(t+s)z = S(t)S(s)z$　for　$s, t \geqslant 0,\ z \in C$

  (s, 2)　任意の $x \in C$ に対し $t \longrightarrow S(t)x$ は $[0, \infty)$ で連続
- $\{ T(t) : t \geqslant 0 \}$ は X 上の C_0－半群、その生成素は A
- μ は C を定義域とし、$[0, \infty)$ に値をとる下半連続汎関数。

  各 $\alpha \geqslant 0$ に対し $C_\alpha = \{ x \in c : \mu(x) \leqslant \alpha \}$ とおく。各 C_α は凸閉集合。

• g は $[0, \infty)$ から $[0, \infty)$ への連続関数で、任意の $\alpha \geqq 0$ に対し初期値問題

$$w'(t) = g(w(t)),\ t > 0\ ;\ w(0) = \alpha$$

は $[0, \infty)$ で最大解 $m(t;\alpha)$ を持つ

• 任意の $t \geqq 0$, $\alpha \geqq 0$, $z \in C_\alpha$ に対し

$$\mu(S(t)z) \leqq m(t;\mu(z))$$

以上は文献［3］で使われている仮定。更に次の仮定［I］、［II］、［III］を置く。

［I］定義域を C とし X に値をとる作用素 B が存在し、任意の $\alpha \geqq 0$ に対し B は C_α 上連続。

［II］任意の $z \in C$, $t \geqq 0$ に対し

$$S(t)Z = T(t)Z + \int_0^t T(t-s)B\{S(s)z\}ds \quad \text{が成立}$$

集合 $\{x - y : x, y \in C_\alpha\}$ を含む最小の閉部分空間を $\widetilde{C_\alpha}$ とする。

［III］各 $\alpha \geqq 0$, $z \in C_\alpha$ に対し $\widetilde{C_\alpha}$ から X への有界線形作用素 $B'_\alpha(z)$ が存在し、任意の $z + \Delta z \in C_\alpha$ に対し

① $\lim_{h \downarrow 0} \dfrac{1}{h}\{B(z+h\Delta z) - B(z)\} = \{B'_\alpha(z)\}\Delta z$

② 任意の $\alpha \geqq 0$, $w \in \widetilde{C_\alpha}$ に対し $z \longmapsto B'_\alpha(z)w$ は C_α 上連続

<u>**命題 1**</u>　［0］、［I］、［II］、［III］を仮定とすると、次の［A］、［B］、［C］、［D］が成立

［A］各 $\alpha \geqq 0$ に対し、$(t, z) \longmapsto S(t)z$ は $[0, \infty) \times C_\alpha$ 上連続

［B］任意の $t \geqq 0, \alpha \geqq 0, z \in C_\alpha$ に対し $\widetilde{C_\alpha}$ から X への有界線形作用素 $S'_\alpha(t, z)$ が存在し、以下①、②が成立

① $\lim_{h \downarrow 0} \dfrac{S(t)(z + h\Delta z) - S(t)z}{h} = S'_\alpha(t, z)\Delta z \qquad \text{for all} \quad z + \Delta z \in C_\alpha$

② 任意の $\alpha \geqslant 0$, $w \in \widetilde{C_\alpha}$ に対し $(t,z) \longmapsto \{S'_\alpha(t,z)\}w$ は $(t,z) \in [0,\infty) \times C_\alpha$ 上連続

[C] ① 任意の$\alpha \geqslant 0$, $z \in C_\alpha$, $w \in \widetilde{C_\alpha}$に対し

$$lim_{t \downarrow 0} \frac{1}{t}\{S'_\alpha(t,z)w - T(t)w\} = B'_\alpha(z)w$$

② 任意の$\alpha \geqslant 0$, $w \in \widetilde{C_\alpha}$に対し

$z \longmapsto B'_\alpha(z)w$ は C_α で連続

[D] $lim_{t \downarrow 0} \frac{1}{t}\{S(t)z - T(t)z\} = B(z) \quad \text{for} \quad z \in C$

㊟ §2の反例では$\mu \equiv 0$とする事（従って任意の$\alpha \geqslant 0$について、$C_\alpha = C = X$）により［0］、［I］、［II］と［A］、［B］、［D］が成立。［III］と［C］が成り立たない。

㊟ 仮定［0］と［I］のもとで、［II］と［D］は同値。これに関連したより詳しい特徴付けは文献［3］の定理3.1に記載されている。

命題1の証明のあらすじ　特に［B］①について述べたい。いくつかのレンマを通して証明する。

レンマ1　任意の $\alpha \geqslant 0$ と $z_0 \in C_\alpha$ に対し、ある $\varepsilon > 0$ が存在して以下が成立。

$$Sup\{\|B'_\alpha(z)\| : \|z - z_0\| \leqslant \varepsilon, \ z \in C_\alpha\} < \infty$$

ここに $\|B'_\alpha(z)\|$ は作用素ノルムを表す。

証　明　仮定［III］②に注目。一様有界性定理を使う。■

レンマ2　任意の $\alpha \geqslant 0$ と、$z \in C_\alpha$, $t_0 > 0$ に対し、ある $\varepsilon > 0$ と $M > 0$ が存在して、

$$\|S(t)(z+\Delta z) - S(t)z\| \leqslant M \cdot \|\Delta z\|$$

$$\text{if} \quad z + \Delta z \in C_\alpha, \ \|\Delta z\| < \varepsilon, \ 0 \leq t \leq t_0$$

証　明　[II] に注目して

$$S(t)(z+\Delta z) - S(t)z = T(t)\Delta z + \int_0^t T(t-s)\{BS(s)(z+\Delta z) - BS(s)z\}ds$$

そしてレンマ1を用いる。 ∎

レンマ3　$z,\ z+\Delta z \in C_\alpha,\ t_0 > 0$ を任意に固定すると

$$lim_{h\downarrow 0}\frac{1}{h}\{S(t)(z+h\Delta z) - S(t)z\}$$

は $0 \leqslant t \leqslant t_0$ において一様収束する。

証　明　$1 \geqslant h_1 > h_2 > h_3 > \cdots\cdots > 0$　$\lim_{i\to\infty} h_i = 0$ を満たす任意の数列$\{h_i\}$に対し、

$$P_i(t) = \frac{1}{h_i}\{S(t)(z+h_i\Delta z) - S(t)z\}$$

と定める。$\lim_{i\to\infty} P_i(t)$ が $t \in [0, t_0]$ で一様収束する事を示せばよい。レンマ2より関数列$\{P_i\}_i$ は $[0, t_0]$ で一様有界。そこで $0 \leqslant t \leqslant t_0,\ n = 1, 2, 3, \cdots\cdots\cdots$ に対し

$$q_n(t) = Sup\{\|P_i(s) - P_j(s)\| : 0 \leqslant s \leqslant t,\ i, j \geqslant n\}$$

$$q(t) = \inf_n q_n(t) = \lim_{n\to\infty} q_n(t)$$

と定める。[III] ①に注目してレンマ1とレンマ2を用いて $q(t_0) = 0$ が示される。すると $\lim_{i\to\infty} P_i(t)$ は $0 \leqslant t \leqslant t_0$ で一様収束し、故にレンマ3は成り立つ。 ∎

[B] の①の証明　$\alpha \geqslant 0,\ z \in C_\alpha,\ t_0 > 0$ を任意に採る。レンマ3に注目して

$$P(t, \Delta z) = \lim_{h\downarrow 0}\frac{1}{h}\{s(t)(z+h\Delta z) - S(t)z\}$$

と定める。ただし $z+\Delta z \in C_\alpha,\ 0 \leqslant t \leqslant t_0$ とする。$\widetilde{C_\alpha}$ の定義から任意の $w \in \widetilde{C_\alpha}$ に対し、C_α の点列$\{z+\Delta z_i\}_i$ と自然数の列 $n(1),\ n(2),\ n(3)\cdots$ および実数 $a_{ij}(i=1, 2, \cdots\cdots.\ j=1, 2, \cdots\cdots n(i))$ が存在し

$$w = \lim_{i\to\infty}\sum_{j=1}^{n(i)} a_{ij}\Delta z_j$$

が成り立つ。これに対し $\lim_{i\to\infty}\sum_{j=1}^{n(i)} a_{ij}P(t, \Delta z_j)$ は $0 \leqslant t \leqslant t_0$ で一様収束するので

この値を $P(t,w)$ と定める。この値は $\{z+\Delta z_i\}_i$, $\{n(i)\}_i$, $\{a_{ij}\}_{ij}$ の選び方に依存しない。更に t を固定すると、$w \longrightarrow P(t,w)$ は $\widetilde{C_\alpha}$ から X への有界線形作用素である。これを $S'_\alpha(t,z)$ と定めると、これは [B] ①を満足する。 ∎

次に命題1の逆に当る命題を述べる。

<u>**命 題 2**</u> [0]、[A]、[B]が成り立ち、更に任意の $\alpha \geqslant 0$ と $z \in C_\alpha$ に対し、$\widetilde{C_\alpha}$ から X への有界線形作用素 $B'_\alpha(z)$ が存在して [C]が成り立ち、更に次の [D]'が成り立つと仮定する。

[D]' ある $z_0 \in C$ について $\lim_{t\downarrow 0} \dfrac{S(t)z_0 - T(t)z_0}{t}$ が存在する。

このとき定義域を C とし、X に値をとる作素用 B が存在して [Ⅰ]、[Ⅱ]、[Ⅲ] が成立する。

㊟ 命題2の [D]' 以外の仮定は満たされるが [D]' は満たされない例を挙げる。

このとき仮定 [Ⅰ] [Ⅱ] を満足する作用素 B は存在しない。

<u>**例**</u> X は Banach 空間、$X = C$、$\mu \equiv 0$、A は X 上の C_0半群 $\{T(t) : t \geqslant 0\}$ の生成作用素とする。更に $z_1 \in X$, $z_1 \notin D(A)$ と仮定。更に $S(t)$ を次の様に定める。

$$S(t)z = T(t)z + T(t)z_1 - z_1 \quad (t \geqslant 0,\ z \in X)$$

Referenccs

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Integrated semigroups and the application to the degenerate Cauchy problem

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§1. INTRODUCTION

In this talk, we discuss the inhomogeneous abstract Cauchy problem associated α-times integrated semigroups which are not necessarily non-degenerate. Moreover, we consider the degenerate Cauchy problem by using the result on the inhomogeneous abstract Cauchy problem.

Let X be a Banach space. Let $\mathcal{A}$ be a closed multivalued linear operator in X, $x \in X$ and $f \in C([0,T];X)$, where $0 < T < \infty$. We consider the following inhomogeneous Cauchy problem (for $\mathcal{A}$):

(CP;$\mathcal{A},x,f$) $\qquad (d/dt)u(t) \in \mathcal{A}u(t) + f(t) \quad \text{for } 0 \leq t \leq T, \quad \text{and} \quad u(0) = x.$

where by a *classical solution* u to (CP;$\mathcal{A},x,f$) we mean that $u \in C^1([0,T];X)$ and $u(t)$ satisfies the above equation (CP;$\mathcal{A},x,f$).

Next we give the definition of an α-times integrated semigroup. We denote by $B(X)$ the set of all bounded linear operators from X into itself. Let α be a positive number. A one-parameter family $\{U(t) : t \geq 0\}$ in $B(X)$ is called an *α-times integrated semigroup* on X, if

(1.1) $U(\cdot)x : [0,\infty) \to X$ is continuous for every $x \in X$,

(1.2) $$U(t)U(s)x = \frac{1}{\Gamma(\alpha)}\left(\int_t^{t+s}(t+s-r)^{\alpha-1}U(r)x\,dr - \int_0^s(t+s-r)^{\alpha-1}U(r)x\,dr\right) \quad \text{for every } x \in X \text{ and } t,s \geq 0,$$

(1.3) $U(0) = 0$ (the zero operator).

For convenience we call a (C_0)-semigroup on X also 0-times integrated semigroup on X.

If an α-times integrated semigroup $\{U(t) : t \geq 0\}$ on X, where $\alpha \geq 0$, satisfies the non-degenerate condition:

(1.4) $U(t)x = 0$ for all $t > 0$ implies $x = 0$,

then it is called a *non-degenerate* α-times integrated semigroup on X, or simply it is called *non-degenerate*. Clearly 0-times integrated semigroups are non-degenerate.

Definition 1.1. Let $\{U(t) : t \geq 0\}$ be an α-times integrated semigroup on X, where $\alpha \geq 0$. The *generator* $\mathcal{A}$ of $\{U(t) : t \geq 0\}$ is a multivalued operator defined as follows:

$x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$ if and only if

$$U(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x = \int_0^t U(s)y\,ds \quad \text{for all } t \geq 0. \tag{1.5}$$

Remark 1.1. Let $\mathcal{A}$ be the generator of an α-times integrated semigroup $\{U(t) : t \geq 0\}$ on X. Then $\{U(t) : t \geq 0\}$ is non-degenerate if and only if $\mathcal{A}$ is singlevalued.

We refer to [9] for further information on α-times integrated semigroups which are not necessarily non-degenerate.

§2. Inhomogeneous abstract Cauchy problems

The following result is a generalization of [1, Theorem 5.2].

Theorem 2.1. *Let $n \geq 0$ be an integer. Let $\mathcal{A}$ be the generator of an n-times integrated semigroup $\{U(t) : t \geq 0\}$ on X, $x \in X$ and $f \in C([0,T]; X)$, and set*

$$w(t) = U(t)x + \int_0^t U(t-s)f(s)\,ds \quad \text{for } 0 \leq t \leq T. \tag{2.1}$$

Then (CP;$\mathcal{A}, x, f$) *has a classical solution if and only if $w \in C^{n+1}([0,T]; X)$ and $w^{(n)}(0) = x$. In this case, the classical solution u of* (CP;$\mathcal{A}, x, f$) *is given by $u = w^{(n)}$.*

The following result which extends [7, Corollary 4.5] is a direct consequence of Theorem 2.1.

Proposition 2.2. *Let $n \geq 0$ be an integer and $\mathcal{A}$ be a multivalued linear operator in X. Let $f : [0,T] \to X$ be a function. Suppose that*

(a_1) *$\mathcal{A}$ is the generator of an n-times integrated semigroup $\{U(t) : t \geq 0\}$ on X,* (a$_2$) $f \in C^{n+1}([0,T];X)$,

(a$_3$) $x \in D(\mathcal{A})$, $x_1 \in \{\mathcal{A}x + f(0)\} \bigcap D(\mathcal{A})$, $x_2 \in \{\mathcal{A}x_1 + f'(0)\} \bigcap D(\mathcal{A})$, $\cdots$,

$x_{k+1} \in \{\mathcal{A}x_k + f^{(k)}(0)\} \bigcap D(\mathcal{A})$, $\cdots$, and $x_n \in \{\mathcal{A}x_{n-1} + f^{(n-1)}(0)\} \bigcap D(\mathcal{A})$.

Then (CP;$\mathcal{A}, x, f$) *has a unique classical solution* u. *Moreover we have* $u'(0) = x_1$.

We give the definition and a generation theorem of an exponentially bounded α-times integrated semigroup. In §3, we consider the degenerate Cauchy problem by using Proposition 2.2 and the generation theorem.

Definition 2.1. If an α-times integrated semigroup $\{U(t) : t \geq 0\}$ on X, where $\alpha \geq 0$, satisfies the exponential growth condition:

(2.2) there are constants $a \geq 0$ and $M \geq 0$ such that $\|U(t)\| \leq Me^{at}$ for $t \geq 0$,

then it is called an exponentially bounded α-times integrated semigroup on X.

The following result is obtained by the same argument as [2, Proposition 3.1] and [5, Lemma 2.2].

Theorem 2.3. *Let* $\mathcal{A}$ *be a multivalued linear operator in* X. *Suppose that there are constants* $a \geq 0$, $M \geq 0$ *and* $\beta \geq -1$ *such that* $\{\lambda \in \mathbb{C} : Re\lambda \geq a\} \subset \rho(\mathcal{A})$ *and*

$$\|R(\lambda : \mathcal{A})\| \leq M|\lambda|^{\beta} \quad \text{for } Re\ \lambda \geq a. \tag{2.3}$$

Then, $\mathcal{A}$ *is the generator of an exponentially bounded* α*-times integrated semigroup on* X *for* $\alpha > \beta + 1$.

Next we introduce the notion of locally Lipschitz continuous α-times integrated semigroups.

Definition 2.2. If an α-times integrated semigroup $\{U(t) : t \geq 0\}$ on X, where $\alpha \geq 0$, satisfies the following condition:

$$\begin{aligned} &\text{for all } T > 0 \text{ there exists } M_T > 0 \text{ such that} \\ &\|U(t) - U(s)\| \leq M_T|t - s| \text{ for all } t, s \in [0,T], \end{aligned} \tag{2.4}$$

then it is called a locally Lipschitz continuous α-times integrated semigroup on X.

Lemma 2.4. *Let $\alpha \geq 0$ and $\mathcal{A}$ the generator of a locally Lipschitz continuous $(\alpha+1)$-times integrated semigroup on X. Then $\mathcal{A}$ is the generator of a β-times integrated semigroup on X for $\beta > \alpha$.*

The following result is an extension of [4, Theorem 4.6].

Theorem 2.5. *Let $\alpha \geq 0$ and n the smallest integer such that $n > \alpha$. Let $\mathcal{A}$ be a multivalued linear operator in X, $f : [0, T] \to X$ and $x \in X$. Suppose that*

(b_1) *$\mathcal{A}$ is the generator of a locally Lipschitz continuous $(\alpha+1)$-times integrated semigroup on X,*

(b_2) *$f \in C^n([0,T]; X)$ and $f^{(n)}(t) = f^{(n)}(0) + \int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)} g(s)ds$ for $0 \leq t \leq T$, where $g \in L^1(0,T;X)$,*

(b_3) *$x \in D(\mathcal{A})$, $x_1 \in \{\mathcal{A}x + f(0)\} \bigcap D(\mathcal{A})$, $x_2 \in \{\mathcal{A}x_1 + f'(0)\} \bigcap D(\mathcal{A})$, $\cdots$,*
$x_{k+1} \in \{\mathcal{A}x_k + f^{(k)}(0)\} \bigcap D(\mathcal{A})$, $\cdots$, and $x_n \in \{\mathcal{A}x_{n-1} + f^{(n-1)}(0)\} \bigcap D(\mathcal{A})$.

Then (CP;$\mathcal{A}, x, f$) *has a unique classical solution u. Moreover we have $u'(0) = x_1$.*

By [9, Theorem 5.2], we have the following.

Corollary 2.6. *Let $\alpha \geq 0$ and n the smallest integer such that $n > \alpha$. Let $\mathcal{A}$ be a multivalued linear operator in X, $f(\cdot) : [0,T] \to X$ and $x \in X$. Suppose that*

(b_1') *there are constants $a \geq 0$ and $M \geq 0$ such that $(a, \infty) \subset \rho(\mathcal{A})$ and*

$$\left\| \frac{(\lambda - a)^{k+1}}{k!} \left[\frac{R(\lambda : \mathcal{A})}{\lambda^\alpha} \right]^{(k)} \right\| \leq M \quad \text{for } \lambda > a \text{ and } k \in \mathbb{N} \cup \{0\}. \tag{2.5}$$

Moreover, if we assume (b_2) and (b_3), then (CP;$\mathcal{A}, x, f$) *has a unique classical solution u. Moreover we have $u'(0) = x_1$.*

Remarks 2.1. (i) In the case that α is nonnegative integer, the weaker conditions

(b_2') $f \in C^{n-1}([0,T]; X)$ and $f^{(n-1)}(t) = f^{(n-1)}(0) + \int_0^t g(s)ds$ for $0 \leq t \leq T$, where $g \in L^1(0,T;X)$,

(b_3') $x \in D(\mathcal{A})$, $x_1 \in \{\mathcal{A}x + f(0)\} \bigcap D(\mathcal{A})$, $x_2 \in \{\mathcal{A}x_1 + f'(0)\} \bigcap D(\mathcal{A})$, $\cdots$,
$x_{k+1} \in \{\mathcal{A}x_k + f^{(k)}(0)\} \bigcap D(\mathcal{A})$, $\cdots$, and $x_n \in \{\mathcal{A}x_{n-1} + f^{(n-1)}(0)\} \bigcap \overline{D(\mathcal{A})}$.

instead of (b_2) and (b_3) are sufficient to yield the same consequence as in Theorem 2.5 and Corollary 2.6.

(ii) In the case that $\alpha = 0$ and X is reflexive, if we assume the conditions (b_1'), (b_2') and "(b_3'') $x \in D(\mathcal{A})$" in Corollary 2.6, (CP;$\mathcal{A}, x, f$) has a unique classical solution (see [10]).

§3. Degenerate Cauchy problems

Firstly, we consider the following degenerate Cauchy problem:

$$\text{(DE-1;}A,B,x,f) \qquad \begin{cases} (d/dt)Bv(t) = Av(t) + f(t) & \text{for } 0 \le t \le T, \\ Bv(0) = x, \end{cases}$$

where A and B are closed linear operators in X, $f \in C([0,T];X)$ and $x \in X$. By a *solution* v to (DE-1;A, B, x, f) we mean that $v(t) \in D(A) \cap D(B)$ for $0 \le t \le T$, $Bv(t)$ is continuously differentiable in $t \in [0,T]$, $Av(t)$ is continuous in $t \in [0,T]$ and $v(t)$ satisfies (DE-1;A, B, x, f).

Proposition 3.1. *(See [10].) Let A and B be closed linear operators in X, $f \in C([0,T];X)$ and $x \in X$. Then the following statements are equivalent:*

(i) (DE-1;A, B, x, f) has a solution v,

(ii) (CP;AB^{-1}, x, f) has a classical solution u.

In the case, we have $u(t) = Bv(t)$ for $0 \le t \le T$.

Next, we consider the following degenerate Cauchy problem:

$$\text{(DE-2;}A,B,x,f) \qquad \begin{cases} B(d/dt)Bv(t) = Av(t) + f(t) & \text{for } 0 \le t \le T, \\ Bv(0) = x, \end{cases}$$

where $B \in B(X)$, A is a closed linear operator in X, $f \in C([0,T];X)$ and $x \in X$. By a *solution* v to (DE-2;A, B, x, f) we mean that $v(t) \in D(A)$ for $0 \le t \le T$, $Bv(t)$ is continuously differentiable in $t \in [0,T]$, $Av(t)$ is continuous in $t \in [0,T]$ and $v(t)$ satisfies (DE-2;A, B, x, f).

Proposition 3.2. *Let A be a closed linear operator in X, $B \in B(X)$, $f \in C([0,T];X)$ and $x \in X$. Then the following statements are equivalent:*

(i) (DE-2;A, B, x, Bf) has a solution v,

(ii) (CP;$B^{-1}AB^{-1}, x, f$) has a classical solution u.

In the case, we have $u(t) = Bv(t)$ for $0 \le t \le T$.

Example. (See also [10].) We consider the following initial value problem in $L^p(\mathbb{R})$ $(1 \le p \le \infty)$:

$$
(3.1) \qquad \begin{cases} \dfrac{\partial}{\partial t} m_E(x)v(t,x) = -\dfrac{\partial}{\partial x}v(t,x) + f(t,x) & \text{for } 0 \le t \le T \text{ and } x \in \mathbb{R}, \\ m_E(x)v(0,x) = v_0(x) & \text{for } x \in \mathbb{R}, \end{cases}
$$

where $m_E(x)$ is a characteristic function of a measurable set $E \subset \mathbb{R}$, that is, $m_E(x) = 1$ for $x \in E$, $m_E(x) = 0$ for $x \notin E$, $f(t,x)$ is a given function, $v_0(x)$ is an initial function, and $v(t,x)$ is an unknown function. We define linear operators A and B by $(Av)(x) := -(d/dx)v(x)$ for $v \in D(A) := W^{1,p}(\mathbb{R})$ and $(Bv)(x) := m_E(x)v(x)$ for $v \in D(B) := L^p(\mathbb{R})$ respectively. Clearly B is a bounded linear operator on $L^p(\mathbb{R})$ satisfying $B^2 = B$ and $\|B\| \le 1$. Therefore (3.1) can be written as (DE-1;A, B, v_0, f) or (DE-2;A, B, v_0, f). We shall investigate the initial value problem (3.1) for some special characteristic functions $m_E(x)$.

(I) In the case of $E = (-\infty, a) \cup (b, \infty)$, we consider the problem (3.1) in $L^p(\mathbb{R})$ $(1 \le p \le \infty)$ and treat (3.1) as (DE-1;A, B, v_0, f). Now, for any $f \in L^p(\mathbb{R})$ and $Re\ \lambda \ge 1$, we find a solution $v \in W^{1,p}(\mathbb{R})$ to the follwing problem:

$$
(3.2) \qquad \lambda m_E(x)v(x) + v'(x) = f(x) \quad \text{for } x \in \mathbb{R}.
$$

By the same argument as [10, Example 5.1(1)], (3.2) consists of three problems: $\lambda v_1(x) + v_1'(x) = f(x)$ in $x < a$, under $\lim_{x \to -\infty} v_1(x) = 0$ if $1 \le p < \infty$ ($\sup_{x<a} |v_1(x)| < \infty$ if $p = \infty$); $v_2'(x) = f(x)$ in $a < x < b$ under $v_2(a) = v_1(a)$; and $\lambda v_3(x) + v_3'(x) = f(x)$ in $x > b$ under $v_3(b) = v_2(b)$. By simple computation, we see that (3.2) has a unique solution $v \in W^{1,p}(\mathbb{R})$, that $\|v\|_p \le M_p \|f\|_p$ for $f \in L^p(\mathbb{R})$ and $Re\ \lambda \ge 1$, where $M_p > 0$ is a constant depending on only $p \in [1, \infty]$ and $\|\cdot\|_p$ is the norm of $L^p(\mathbb{R})$. In fact, the above argument shows that $\lambda B - A$ is bijective, $(\lambda B - A)^{-1} \in B(L^p(\mathbb{R}))$ and $\|(\lambda - AB^{-1})^{-1}\| = \|B(\lambda B - A)^{-1}\| \le \|B\| M_p \le M_p$ for $Re\ \lambda \ge 1$. By Theorem 2.3. AB^{-1} is the generator of α-times integrated semigroup on $L^p(\mathbb{R})$ for $\alpha > 1$. Hence we can use Proposition 2.2.

(II) In the case of $E = (a, \infty)$, we consider the problem (3.1) in $L^p(\mathbb{R})$ $(1 \le p < \infty)$ and treat (3.1) as (DE-2;A, B, v_0, f), where $f(t,x) = m_E(x)g(t,x)$ for $0 \le t \le T$ and $x \in \mathbb{R}$ ($g \in C([0,T]; L^p(\mathbb{R}))$). Now, for any $f \in L^p(\mathbb{R})$ and $\lambda > 0$, we find a solution $v \in W^{1,p}(\mathbb{R})$ to the following:

$$
(3.3) \qquad \lambda m_E(x)v(x) + v'(x) = m_E(x)f(x) \quad \text{for } x \in \mathbb{R}.
$$

The problem (3.3) is essentially the following: $\lambda v_1(x) + v_1'(x) = f(x)$ in $x > a$, under $v_1(a) = 0$. We see that (3.3) has a unique solution $v \in W^{1,p}(\mathbb{R})$. Then we get that $\|v\|_p \leq \frac{1}{\lambda}\|f\|_p$ for $f \in L^p(\mathbb{R})$ and $\lambda > 0$. In fact, the above argument shows that $R(\lambda B^2 - A) \supset R(B)$, $(\lambda B^2 - A)^{-1}B \in B(L^p(\mathbb{R}))$ and $\|(\lambda - B^{-1}AB^{-1})^{-1}\| = \|B(\lambda B^2 - A)^{-1}B\| \leq \frac{\|B\|}{\lambda} \leq \frac{1}{\lambda}$ for $\lambda > 0$. Hence we can use Corollary 2.6. In particular, if $1 < p < \infty$, by (ii) of Remarks 2.1 (DE-2;A, B, v_0, Bg) has a unique solution v for $v_0 \in D(B^{-1}AB^{-1})$, for example, $v_0(x) = m_E(x)w(x)$ for some $w \in L^p(\mathbb{R})$ and $v_0 \in W^{1,p}(\mathbb{R})$.

(III) In the case of $E = (-\infty, a)$, we consider the problem (3.1) in $L^\infty(\mathbb{R})$ and treat (3.1) as (DE-2;A, B, v_0, f), where $f(t,x) = m_E(x)g(t,x)$ for $0 \leq t \leq T$ and $x \in \mathbb{R}$ ($g \in C([0,T]; L^p(\mathbb{R}))$). Now, for any $f \in L^\infty(\mathbb{R})$ and $\lambda > 0$, we find a solution $v \in W^{1,p}(\mathbb{R})$ to the following:

$$\lambda m_E(x)v(x) + v'(x) = m_E(x)f(x) \quad \text{for } x \in \mathbb{R}. \tag{3.4}$$

The problem (3.4) consists of two problems: $\lambda v_1(x) + v_1'(x) = f(x)$ in $x < a$, under $\sup_{x<a} |v_1(x)| < \infty$; and $v_2'(x) = 0$ in $x > a$ under $v_2(a) = v_1(a)$. We see that (3.4) has a unique solution $v \in W^{1,\infty}(\mathbb{R})$. Then the estimate $\|v\|_\infty \leq \frac{1}{\lambda}\|f\|_\infty$ holds for $f \in L^\infty(\mathbb{R})$ and $\lambda > 0$. Hence we can use Corollary 2.6 the same as the case (II).

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散逸関数をともなう熱対流方程式について

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下から一様に熱せられた水平流体層に発生する対流は、流れの不安定性や乱流の発生などの研究対象として古くから研究されており、初期の研究者の名をとって Rayleigh-Bénard 対流と呼ばれている。

上下面の温度差を大きくしていくと、対流運動は静止状態から定常運動、周期運動そして準周期運動へと運動形態を変えていき、最終的には乱流へと変化していく。このような Rayleigh-Bénard 対流に見られる乱流への遷移過程は、Ruelle-Takens による乱流発生のシナリオを支持するものとなっている。Ruelle-Takens [6] は、“ストレンジ・アトラクター”による乱流発生の説明づけを提案した。系を制御するパラメータを変化させたとき、有限回の分岐の後に“ストレンジ・アトラクター”という複雑な構造をもつアトラクターが現れ、それが乱流の発生を数学的に説明づけるとしたのである。Ruelle-Takens のこの研究に刺激され、Navier-Stokes 方程式を中心とする流体力学方程式のアトラクターの研究が活発になった。ここでは、2 次元 Rayleigh-Bénard 対流を記述する方程式系の解の存在と一意性の問題、およびそのアトラクターについて考える。

2 次元 Rayleigh-Bénard 対流を記述する方程式系の無次元形は、水平方向を x_1-方向、鉛直方向を x_2-方向にとると、次のようになる ([1],[2]) :

$(1)_\eta$

$$\frac{\partial u}{\partial t} - \nu\Delta u + u\cdot\nabla u + \nabla p = e_n f(\theta),$$

$$\nabla\cdot u = 0, \qquad t>0,\ x\in \mathrm{R}\times(0,1)$$

$$\frac{\partial \theta}{\partial t} - \kappa\Delta\theta + u\cdot\nabla\theta - e_n\cdot u = \eta D(u):D(u).$$

ここで、 $e_2=(0,1)$; $u=(u^1,u^2)$ は速度場 ; p は圧力 ; θ は温度と静止状態の温度分布 $1-x_2$ との差を表す。$D(u):D(u)$ は散逸関数と呼ばれるもので

$$D(u):D(u) = \frac{\nu}{2}\sum_{i,k=1}^{2}\left(\frac{\partial u^i}{\partial x_k}+\frac{\partial u^k}{\partial x_i}\right)^2$$

とかけ、f は R 上の滑らかな関数である。$\nu=(\frac{Ra}{Pr})^{1/2}$, $\kappa=(\frac{1}{RaPr})^{1/2}$ であり、Ra 、Pr はそれぞれ、Rayleigh 数、Prandtl 数 と呼ばれる無次元パラメータである。$\eta>0$ は無次元パラメータで、通常の物質では $\eta\ll 1$

である。$(1)_\eta$ で $\eta=0$ とし、$f(\theta)=\theta$とすると、いわゆる Boussinesq 方程式が得られる。

u, θ に対する $x_2=0,1$ における境界条件は

$$(2) \qquad u=0 \ ; \ \theta=0 \ \text{ at } \ x_2=0,1$$

である。また、ここでは, x_1-方向について次の周期境界条件

$$(3) \qquad \left.\frac{\partial^j u}{\partial x_1^j}\right|_{x_1=0}=\left.\frac{\partial^j u}{\partial x_1^j}\right|_{x_1=\alpha}, \quad \left.\frac{\partial^j \theta}{\partial x_1^j}\right|_{x_1=0}=\left.\frac{\partial^j \theta}{\partial x_1^j}\right|_{x_1=\alpha}, \quad j=0,1$$

を課すことにする。これらに初期条件

$$(4) \qquad u|_{t=0}=u_0 \ ; \ \theta|_{t=0}=\theta_0$$

を加えると、$(1)_\eta$ に対する初期値境界値問題が設定される。

Foias-Manley-Temam [3] は、2 次元 Boussinesq 方程式の初期値境界値問題を上の境界条件と初期条件の下で考え、大域アトラクターの存在を示した。大域アトラクターはある Hilbert 空間のなかのコンパクト集合で、すべての解を $t\to\infty$ のとき引き寄せるものであり、適当な Ra や Pr に対して、乱流を記述するものである。また、彼らはこのアトラクターの Hausdorff 次元が有限であること示し、その次元の評価を

$$(6) \qquad c|\Omega|(1+Pr)(1+Gr+Ra), \quad c=c(\alpha), \quad \Omega=(0,\alpha)\times(0,1)$$

の形で与えた。ここで、$Gr=\frac{Ra}{Pr}$ である。大域アトラクターの Hausdorff 次元の有限性は、乱流の長時間経過後の振る舞いが有限自由度系のように見えるという経験的事実を数学的に説明づけるものである。

Rayleigh-Bénard 対流に対する数学的研究では、$(1)_\eta$ で $\eta=0$ と理想化した方程式がモデル方程式として用いられてきたが、ここでは散逸関数を考慮に入れた方程式系 $(1)_\eta$ に対する初期値境界値問題を考え、以下のような解の存在と一意性、および大域アトラクターに関する結果を得た。

関数 f については次の (5) または (6) を仮定する。

$$(5) \qquad f(\theta)=\theta,$$

$$(6) \qquad |f|_\infty \equiv \sup_{\theta\in R}|f(\theta)|<\infty, \quad |f'|_\infty<\infty \quad \text{and} \quad |f''|_\infty<\infty.$$

ここで、f' は f の導関数を表す。

解の存在については次のことが成立する。

定理1. (i) f は (5) または (6) を満たすとする。ただし、f が (5) を満たすときは $\eta < 1$ とする。このとき、任意の初期値 $\{u_0, \theta_0\} \in L^2_\sigma \times L^2(\Omega)$ に対して、次を満たす時間大域的弱解 $\{u, \theta\}$ が存在する：任意の $T > 0$ に対して

$$u \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;V),$$
$$\theta \in L^{4/3}(0,T;L^2(\Omega)) \cap L^2(0,T;L^{4/3}(\Omega)).$$

ここで

$$L^2_\sigma = \{u \in L^2(\Omega)^2 \ ; \ \nabla \cdot u = 0,\ u^2|_{x_2=0,1} = 0,\ u^1|_{x_1=0} = u^1|_{x_1=\alpha}\}$$
$$V = \{u \in (H^1_{0,per})^2 \ ; \ \nabla \cdot u = 0\}.$$

(ii) f は (6) を満たし、$\{u_0, \theta_0\} \in V \times L^2(\Omega)$ とする。このとき、(i) の弱解 $\{u, \theta\}$ は次を満たす：任意の $T > 0$ に対して

$$u \in C([0,T];V) \cap L^2(0,T;(H^2_{per})^2),$$
$$\theta \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1_{0,per}).$$

さらに、この弱解 $\{u, \theta\}$ は次の関数空間

$$(C([0,T];V) \cap L^2(0,T;(H^2_{per})^2)) \times (C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1_{0,per}))$$

の中で一意である。

(iii) f は (6) を満たし、$\{u_{0,i}, \theta_{0,i}\} \in V \times L^2(\Omega)$, $i = 1,2$ とする。また、$\{u_i(t), \theta_i(t)\}$, $i = 1,2$ をそれぞれ $\{u_{0,i}, \theta_{0,i}\}$, $i = 1,2$ を初期値とする弱解とする。このとき、任意の $T > 0$ 対して、ある正数 $C = C(T)$ が存在して

$$\begin{aligned} &\|\nabla u_2(t) - \nabla u_1(t)\|_2^2 + \|\theta_2(t) - \theta_1(t)\|_2^2 \\ \le\ & C(\|\nabla u_{0,2} - \nabla u_{0,1}\|_2^2 + \|\theta_{0,2} - \theta_{0,1}\|_2^2) \end{aligned}$$

が成り立つ。

定理1で構成した $(1)_\eta$ の弱解と $\eta = 0$ とした $(1)_0$ の弱解との間には次のような関係が成立する。

定理２. (i) f は (5) または (6) を満たすとする。$\{u_\eta, \theta_\eta\}$ を 定理 1 で構成した $\{u_0, \theta_0\} \in L^2_\sigma \times L^2(\Omega)$ を初期値とする $(1)_\eta$ の弱解とする。また、$\{u, \theta\}$ を同じ初期値をもつ $(1)_0$ の（一意な）弱解とする。このとき任意の $T > 0$ に対して、$\eta \to 0$ のとき

$$u_\eta \to u \quad \text{strongly in} \quad C([0,T]; L^2_\sigma) \cap L^2(0,T;V),$$

$$\theta_\eta \to \theta \quad \text{strongly in} \quad L^{4/3}(0,T;L^2(\Omega))$$

が成り立つ。

(ii) さらに、f は (6) を満たし、$\{u_0, \theta_0\} \in V \times L^2(\Omega)$ とすると、任意の $T > 0$ に対して、$\eta \to 0$ のとき

$$u_\eta \to u \quad \text{strongly in} \quad C([0,T]; V) \cap L^2(0,T;(H^2_{per})^2),$$

$$\theta_\eta \to \theta \quad \text{strongly in} \quad C([0,T]; L^2(\Omega)) \cap L^2(0,T;H^1_{0,per}).$$

となる。

注意. $(1)_0$ に対する初期値境界値問題は

$$(L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;V)) \times L^{4/3}(0,T;\ L^2(\Omega))$$

で一意的な弱解をもつ。([4])

$H = V \times L^2(\Omega)$ とする。定理 1 の (ii),(iii) によって f が (6) を満たすときは、H 上の半群 $\{S(t)\}_{t\geq 0}$:

$$S(t)\ :\ H \ni \{u_0, \theta_0\} \longmapsto \{u(t), \theta(t)\} \in H$$

を考えることができる。

この半群 $\{S(t)\}_{t\geq 0}$ の $t \to \infty$ のときの挙動について次のことが成り立つ。

定理３. f は (6) を満たすとする。このとき、上で定めた半群 $\{S(t)\}_{t\geq 0}$ は次を満たす大域アトラクター $\mathcal{A}$ をもつ：

(i) $\mathcal{A}$ は $((H^2_{per})^2 \cap V) \times H^1_{0,per}(\Omega)$ において有界で、H でコンパクト。

(ii) $\mathcal{A}$ は H のすべての有界集合を引き寄せる。すなわち、H の任意の有界集合 $\mathcal{B}$ に対して、$\lim_{t\to\infty} d(S(t)\mathcal{B}, \mathcal{A}) = 0$. ここで、$d(\mathcal{B}_0, \mathcal{B}_1) = \sup_{y\in\mathcal{B}_0} \inf_{x\in\mathcal{B}_1} dist(x,y)$.

定理3で得た大域アトラクターの Hausdorff 次元に関しては次のような評価が得られる。

定理4. 定理3の大域アトラクター $\mathcal{A}$ の Hausdorff 次元は

$$c|\Omega|(1+Pr)(1+Gr+Gr^{1/2}Ra+O(\eta))$$

で上から評価される。

ここでは、2次元問題で散逸関数を考慮に入れた場合を取り扱ったが、$f(\theta)=\theta$ のときは解の一意性や漸近挙動などについてはわかっていない。3次元問題については、時間局所的強解の存在および十分小さな初期値に対する時間大域的強解の存在がわかっている程度であり、時間大域的弱解の存在はわかっていない。3次元問題における解の存在問題の難しさは Navier-Stokes 方程式の弱解の正則性の問題と密接に関係している。

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ASYMPTOTICS OF HEAVY MOLECULES IN HIGH MAGNETIC FIELDS

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ABSTRACT. We extend the results of Lieb, Solovej,Yngvason, that is, consider the asymptotic behavior of the properties of large molecules. We study the energy asymptotics of a molecule made of some atoms, and compare the each component of the energy.

0. INTRODUCTION

We study the Hailtonian of N-electrons interacting under Coulomb repulsion forces and under the attraction by K-nucleii submitting the uniform magnetic field:

$$H_N := \sum_{i=1}^{N}\{(\mathrm{p}^i + A(x^i))^2 + \sigma^i \cdot \mathbf{B} - \sum_{j=1}^{K} Z_j|x^i - R_j|^{-1}\}$$

$$+ \sum_{1\leq k<l\leq N} |x^k - x^l|^{-1} + \sum_{1<k<l<K} Z_k Z_l |R_k - R_l|^{-1} \quad on\, \mathcal{H}^N.$$

where, $\mathcal{H}^N := \Lambda^N L^2(\mathbf{R}^3; \mathbf{C}^2)$ is the Hilbert space of antisymmetric(fermionic) spinor valued functions, $\mathrm{p} = -i\nabla$, $\mathrm{A} = \frac{1}{2}\mathbf{B} \times x$ is a vector potential of a constant magnetic field $\mathbf{B} = (0,0,B)$ $(B > 0)$, and σ is the Pauli spin matrices. We investigate the ground state energy of $\mathcal{H}^N$,

$$E^Q(N, \{Z_j\}, \{R_j\}, B) := \inf\{< \Psi, H_N\Psi >: \Psi \in domain\ H_N, < \Psi, \Psi >= 1\}.$$

We are interested in the asymptotic behavior of E^Q as $Z_j = k_j Z, Z \to \infty$, N/Z fixed, and B depending on Z. It is considerd as the model of the surface of a neutoron star, and Lieb, Solovej, Yngvason([1]) studied this in case of $K = 1$ deeply. They considerd the 5-different regimes of the ratio B to Z. Among this, we treat the region 3,4,5 in [1] here, namely, $B/Z^{4/3} \to \infty$ as $Z \to \infty$, and try to extend the results of [1] to the K-nuclear case.

The first observation is, when the magnetic field is large enough, the coulomb force can be seen as the perturbation, and the ground state is confined to the lowst Landau band leading order. To be precise, we define,

$$E^Q_{conf} := \inf\{< \Psi, H_N\Psi >: \Pi_N\Psi = \Psi, < \Psi, \Psi >= 1\}$$

where $\Pi_N := \otimes\Pi_0$, the n-th tensorial product of Π_0 which is the projection onto the lowst Landau band.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Theorem 3. *Let $\lambda, \eta := B/2\pi Z^3$ fixed. Then when Z tends to infinity,*

$$Z^{-4}\rho^Q(Z^{-1}x; N, \{Z_j\}, B, \{R_j\}) \to \rho(x; \lambda, \{k_j\}, 2\pi\eta, \{\tilde{R}_j\}) \text{ weakly in } L^1_{loc}(\mathbf{R}^3)$$

$$Z^{-3}K^Q(N, \{Z_j\}, B, \{R_j\}) \to K^{DM}(\lambda, \{k_j\}, 2\pi\eta, \{\tilde{R}_j\})$$

$$Z^{-3}A^Q(N, \{Z_j\}, B, \{R_j\}) \to A^{DM}(\lambda, \{k_j\}, 2\pi\eta, \{\tilde{R}_j\})$$

$$Z^{-3}R_e^Q(N, \{Z_j\}, B, \{R_j\}) \to R_e^{DM}(\lambda, \{k_j\}, 2\pi\eta, \{\tilde{R}_j\})$$

1. Idea of proof

Unfortunately, the proofs are so complicated that I can only show the sketch of proof. The prof of Theorem 1 and Theorem 2 is mainly due to the method of [1].

Idea of proof of theorem 1. Let α is arbitrary subset of the integers $1, \ldots, N$. We define the projection $\Pi^\alpha := \prod_{i\in\alpha}\Pi_0^i \prod_{i\notin\alpha}\Pi_>^i$, where $\Pi_>^i := I - \Pi_0^i$. To prove Theorem 1, we separate the Hamiltonian in terms of the lowest Landau band, $H_N = \sum \Pi^\alpha H_N \Pi^\alpha$, and estimate each term by E_{conf} from below and estimate that the error term is lower order compared to E^Q.

Idea of proof of theorem 2. We study first the property of $\mathcal{E}^{DM}$, that is, the existence and uniwueness of the minimizer. This is done by the standard arguement of variational method(but the treatment of the kinetic energy is more sutle). To estimate E^Q from above by E^{DM}, we use the variational principle of Lieb[3],

$$E^Q \le Tr[(H_A - \sum_{j=1}^{K} Z_j|x-R_j|^{-1})K] + \frac{1}{2}\iint k(x,x)k(y,y)|x-y|^{-1}dxdy + \text{(Repulsion of nucleii)}$$

where, K is a density matrix such that $0 \le K \le 1$ and $0 \le TrK \le N$, and $k(x,x)$ is kernel of K. And for from below, we use the special bound of the exchange energy([1]).

$$H_N \ge \sum_{i=1}^{N}\left((1 - Z^{-1/3}H_A^i - \phi^{DM}(x^i)\right) - \frac{1}{2}\iint \frac{\rho_\Gamma(x)\rho_\Gamma(y)}{|x-y|}dxdy$$

$$-C_\lambda(1+\lambda^5)(1+Z^{8/3})(1+[ln(B/Z^3)]^2).$$

Idea of proof of theorem 3. Let $U(x)$ be a $C_0^\infty(\mathbf{R}^3)$ function. We define $H_N(\alpha) := H_N + \alpha\sum_{i=1}^N U(x^i)$ and $\mathcal{E}^{DM}(\alpha) := \mathcal{E}^{DM} + \alpha\int U(x)\rho_\Gamma dx$. As the proof of Theorem 2, we can prove $E^Q(\alpha)/E^{DM}(\alpha) \to 1$ as $Z \to \infty$. Using the concavity of α, we can conclude $\frac{\partial}{\partial\alpha}E^Q(\alpha)/\frac{\partial}{\partial\alpha}E^{DM}(\alpha) \to 1$. So we can obtain the weak convergence of ρ^Q by putting $\alpha = 0$. The convergence of each term of the energy, we follow the arguement of [2]. Define E_α^Q by E^Q replaced its kinetic energy K^{DM} by αK^{DM}. Similarly define $\mathcal{E}_\alpha^{DM}$, and we differentiate with respect to α.

Apology and Correction. Theorem 4 of Abstruct has been seemed to be *false!*(This is difficult problem...). I am deeply sorry for all the participants.

Theorem 1. *If $\lambda := N/Z \leq \Lambda$, and $\beta := B/Z^{4/3}$, there exits $\delta = \delta(\lambda^{2/3}\beta, \Lambda)$ with the property that $\delta \to 0$ as $\lambda^{2/3}\beta \to \infty$ such that, $(1-\delta)E^Q \geq E^Q_{conf}$.*

For a given $\Psi \in \mathcal{H}^N$, We define the density matrix associated to Ψ as follows.

$$\Gamma^{\Psi}_{x_\perp}(x_3, x_3') := \sum_{\sigma^{(i)}=\pm 1} \int \cdots \int \Psi(x_\perp, x_3; x^2, \cdots, x^N)\bar{\Psi}(x_\perp, x_3'; x^2, \cdots, x^N)\, dx^2 \cdots dx^N$$

$\Gamma^{\Psi}_{x_\perp}$ can by considerd as the bouded operator on $L^2(\mathbf{R})$ parametrized by $x_\perp$ and if the state Ψ lives in the lowest Landau band, and in the domain of H_N, $\Gamma^{\Psi}_{x_\perp}$ satisfies the following properties.

(1) For arbitrary $f \in \mathbf{L}^2(\mathbf{R})$, the map $x_\perp \to (f, \Gamma^{\Psi}_{x_\perp} f)$ is measurable.
(2) $\Gamma_{x_\perp}$ is a positive semidefinite, trace class operator for almost all $x_\perp \in \mathbf{R}^2$.
(3) $0 \leq \Gamma \leq B/2\pi$ for almost all $x_\perp \in \mathbf{R}^2$.
(4) $\int_{\mathbf{R}^2} Tr_{\mathbf{L}^2(\mathbf{R})}[(1-\partial_3^2)\Gamma] dx_\perp < \infty$.

$Tr_{\mathbf{L}^2(\mathbf{R})}[(-\partial_3^2)\Gamma]$ is defined to be the usual trace of $\partial_3^2 \Gamma \partial_3^2$. We define the set of density matrices, $\mathcal{G}_B^{DM} := \{\Gamma : \Gamma$ satisfies the properties above$\}$. The aim of this paper is to approximate E^Q by a suitable functional of $\Gamma \in \mathcal{G}_B^{DM}$. We define

$$\mathcal{E}^{DM}[\Gamma] := \int_{\mathbf{R}^2} Tr_{L^2(\mathbf{R})}\left[-\frac{\partial^2}{\partial x_3^2}\Gamma\right] dx_\perp - \sum_{j=1}^{K} \int \frac{Z_j \rho_\Gamma(x)}{|x - R_j|} dx$$

$$+\frac{1}{2}\iint \frac{\rho_\Gamma(x)\rho_\Gamma(y)}{|x-y|} dx dy + \sum_{k,l} Z_k Z_l |R_k - R_l|^{-1},$$

where $\rho_\Gamma(x) := \Gamma^{\Psi}_{x_\perp}(x_3, x_3)$. And we consider the infimum of $\mathcal{E}^{DM}$ on $\mathcal{G}_B^{DM}$ with a constraint.

$$E^{DM} := \inf\{\mathcal{E}^{DM}[\Gamma] : \Gamma \in \mathcal{G}_B^{DM}, \int_{\mathbf{R}^2} Tr_{L^2(\mathbf{R})}[\Gamma] dx_\perp \leq N\}$$

The main theorem of this paper is following,

Theorem 2. *Let $\lambda := N/Z$ fixed and $\beta := B/Z^{4/3}$ goes to infinity, then $E^Q/E^{DM} \to 1$.*

Next, we consider the convergence of the density function and each term of the Hamiltonian. To do this, we set $R_j = Z^{-1}\tilde{R}_j$ on account of the scaling property. We write $\rho^Q := \rho^Q(x; N, \{Z_j\}, B, \{R_j\})$ which is the density corresponding to the groung state and similarly $\rho^{DM} := \rho^{DM}(x; N, \{Z_j\}, B, \{R_j\})$. Moreover, we write each term of total energy,

$$E^Q = K^Q - A^Q + R_e^Q + R_n^Q.$$

And similarly for E^{DM},

$$E^{DM} = K^{DM} - A^{DM} + R_e^{DM} + R_n^{DM}.$$

We write $K^Q = K^Q(N, \{Z_j\}, B, \{R_j\})$ etc.

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On the extinction time of the motion by anisotropic mean curvature

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1. Introduction and statements of results The motion by anisotropic mean curvature appears in evolution of phase-boundaries such as the growth of crystals in supercooled water, grains in anealing metal and so on. This note is concerned with the extinction time of surfaces moving by its anisotropic mean curvature.

Let a continous function $\gamma : \mathbb{R}^n \to \mathbb{R}_+ = \{ \lambda \geq 0 \}$, which is often called a *energy density function*, be homogeneous of degree one and two times continuously differentiable except the origin. We suppose that γ is strictly convex, that is, the Hessian matrix of γ is non-negative and not degenerate without the direction to the origin. Namely,

$$\sum_{i,j=0}^{n} \frac{\partial^2 \gamma}{\partial p_i \partial p_j}(q)\xi^i \xi^j > 0 \qquad (1.1)$$

for linearly independent vectors q and ξ. Let $\{D_t\}_{t\geq 0}$ be a one-parameter family of open sets and let Γ_t be the boundary of D_t. We shall regard $\{D_t\}_{t\geq 0}$ as motion of open set and denote the inward normal velocity of Γ_t by $V(t,x)$.

We consider the following equations:

$$\beta(\nu)V = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\frac{\partial \gamma}{\partial p_i}(\nu) \qquad \text{on} \quad \Gamma_t, \qquad (1.2)$$

$$D_t|_{t=0} = D_0. \qquad (1.3)$$

Here ν is the inward unit normal vector of Γ_t and $\beta : S^{n-1} \to \mathbb{R}$ is a positive smooth function. In theory of the crystal growth such a β is called a *kinetic coefficient*.

The right hand side of (1.2) is called *anisotropic mean curvature* of Γ_t and the motion described by the equation (1.2)-(1.3) is called *anisotropic mean curvature flow*. In general we cannot expect smooth solutions of (1.2)-(1.3). Hence we adopt *the level sets approach*, which regards Γ_t as level sets of some suitable function u such that

$$\Gamma_t = \{ x \in \mathbb{R}^n ; \quad u(t,x) = 0 \quad \}, \quad t \geq 0.$$

Then u satisfies (1.2) the following equation provided each of its level sets moved by anisotropic mean curvature:

$$u_t = \frac{|\nabla u|}{\beta(\nabla u/|\nabla u|)} \sum_{i,j=1}^{n} \frac{\partial^2 \gamma}{\partial p_i \partial p_j}(\nabla u) \frac{\partial^2 u}{\partial x_i \partial x_j} \quad \text{in } (0,\infty) \times \mathbb{R}^n, \tag{1.4}$$

$$u(0,x) = g(x) \qquad \text{on} \quad \mathbb{R}^n, \tag{1.5}$$

where g is a smooth function satisfying

$$D_0 = \{ \; x \in \mathbb{R}^n \; ; \; g(x) > 0 \; \}$$

and $g + \alpha$ is compactlly supported for some constant α. The equation (1.4)-(1.5) is called *the level sets equation on anisotropic mean curvature flow.* It is known that $\{\Gamma\}_{t \geq 0}$ depends not on g but only on D_0.

The equation (1.4)-(1.5) admits the unique global solution in the sense of viscosity solutions. It is known that the solution D_t becomes empty for large enough t. We call the least upper bound of such t the *extinction time*, and we shall denote it by $t_*(D_0)$ or simply t_*.

Theorem 1. *Let D_0 be a bounded domain with smooth boundary and let t_* be the extinction time of the motion (1.2)-(1.3). Suppose that the energy density function multiplied by the kinetic coefficient is less than or equal to* 1. *Then,*

$$t_* \geq 2\left(L^n(D_0)/E(\Gamma_0)\right)^2. \tag{1.6}$$

Here $L^n(D_0)$ is n dimensional Lebesgue measure of D_0 and $E(\Gamma_0)$ is the surface energy of Γ_0 defined by

$$E(\Gamma_0) = \int_{\Gamma_0} \gamma(\nu) \, \mathrm{d} H^{n-1}.$$

Here H^{n-1} is $n-1$ dimensional Hausdorff measure.

It is not a essential assumption that the energy density function multiplied by the kinetic coefficient is not greater than 1. If α is the maximum of $\beta(q)\gamma(q)$ on the unit sphere we can see that the extinction time is dominated by the right hand side of (1.6) divided by α. Furthermore we can also see that the above theorem holds for $n-1$ rectifiable initial figures.

We next consider the motion under some conditions for the kinetic coefficient and the surface energy function.

In theory of the crystal growth and metallurial science many scientists often refer to the crystals or the grains whose volume is maximal for their surface energy. The figures of these crystals or grains are called *Wulff diagrams*. In the case that the kinetic coefficient is inversely proportional to the surface energy function, all the solutions of (1.2)-(1.3) are Wulff diagrams at any time if the initial figures are Wulff diagrams. Hence we now consider the case that the surface energy function multiplied by the kinetic coefficient equals some constants c on the unit sphere. Particularly we suppose the case that c equals 1, *i.e.*,

$$\beta(q)\gamma(q) = 1 \qquad \text{for} \quad |q| = 1. \tag{1.7}$$

In order to give the pertinency of the estimate in theorem 1 we need an assumption of the balance of surface energy. Namely, there is a orthogonal projection $\mathbb{P}$ from $\mathbb{R}^n$ to a (2 dimensional) plane in $\mathbb{R}^n$ such that

$$\mathbb{P}(\nabla\gamma(\mathbb{P}(\cdot))) = \nabla\gamma(\mathbb{P}(\cdot)). \tag{1.8}$$

For γ and γ satisfying the above assumptions we can show the optimality for the estimate given by Theorem 1 in the following sense.

Theorem 2. *Suppose that β and γ satisfy (1.7). If there is a projection satisfying (1.8), the constant 2 in the inequality (1.6) is optimal, i.e., for any constant $C > 2$, there is a bounded open set D with smooth boundary such that*

$$t_*(D) < C\,(L^n(D)/E(\partial D))^2. \tag{1.9}$$

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波動方程式の可制御性について

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1. 序

制御項を持つ発展方程式を、制御関数を変化させることによってその解を望ましい状態に変化させることが出来るかという問題を考える。ここでは発展方程式として波動方程式を考え、制御項は制御関数をデータに持つ非斉次 Dirichlet 境界条件で与える。すなわち、Ωを $R^n(n \geq 1)$ の有界領域、その境界Γ は C^∞級とする。この時、次の初期値境界値問題を考える。

$$Y'' - \Delta Y = 0 \quad \text{in } \Omega \times (0,T) \tag{1}$$

$$Y(0) = Y^0,\ Y'(0) = Y^1 \quad \text{in } \Omega \tag{2}$$

$$Y = \begin{cases} v & \text{on } \Gamma_0 \times (0,T) \\ 0 & \text{on } (\Gamma \setminus \Gamma_0) \times (0,T) \end{cases} \tag{3}$$

ただしΓ_0はΓの部分境界。更に、$\mathcal{X}, \mathcal{V}$を Hilbert 空間とした時、可制御性の問題は次のようになる。

$T > 0$ を与えた時、$\mathcal{X}$に属する任意の初期データ $\{Y^0, Y^1\}$ に対して $\mathcal{V}$に属する制御関数vが存在して、対応する (1)(2)(3) の解 $Y = Y(x,t;v)$ が終期条件：

$$Y(T) = Y'(T) = 0 \quad \text{in } \Omega \tag{4}$$

を満たすように出来る時、システム (1)(2)(3) は$\mathcal{X}$で制御$\mathcal{V}$により可制御であるという。すなわち、時刻 $t = 0$ で状態 $\{Y^0, Y^1\}$ のシステムを時間 Tの後に静止出来る。

Lions[4]、[5]、Ho[1]、Komornik[2] は HUM を用いて次の結果を得た。$x_0 = (x_1^0, \cdots, x_n^0)$ を R^nの任意の点として、$\nu(x) = (\nu_1(x), \cdots, \nu_n(x))$ をΩの外向き単位法線ベクトル、$m(x) = (x_1 - x_1^0, \cdots, x_n - x_n^0)$ とする。この時Γの部分境界$\Gamma_0 = \Gamma_0(x_0)$ を

$$\Gamma_0(x_0) = \{x \in \Gamma; (m(x) \circ \nu(x)) \geq 0\} \tag{5}$$

とする。ただし $(\cdot \circ \cdot)$ は R^nの内積。さらに$T > 2\sup\{|m(x)|; x \in \Omega\}$ とする。この時システム (1)(2)(3) は$\mathcal{X} = L^2(\Omega) \times H^{-1}(\Omega)$（$H^{-1}(\Omega)$ は $H_0^1(\Omega)$ の双対空間）で制御$\mathcal{V} = L^2(\Gamma_0 \times (0,T))$ により可制御である。

ここでは、初期状態 $\{Y^0, Y^1\}$ が属する状態空間 $\mathcal{X}$をより滑らかな空間に制限した時、制御空間 $\mathcal{V}$もより滑らかな空間にとりシステムが可制御になることが出来るかという状態空間と制御空間の regularity の関係を考える。1 次元の波動方程式の場合には Narukawa[7]、Narukawa-Suzuki[8] がある。

任意の正数εに対してΓの部分境界$\Gamma_\varepsilon = \Gamma_\varepsilon(x_0)$ を

$$\Gamma_\varepsilon(x_0) = \{x \in \Gamma; (m(x) \circ \nu(x)) \geq -\varepsilon\} \tag{6}$$

とする。また、$L^2(\Omega)$ 上の作用素 A を $A = -\Delta u, \quad D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ で定める。この時、次の結果を得た。

定理　任意の自然数 m に対して正数 T_mが存在して $T > T_m$の時、システム *(1)(2)(3)* は$\mathcal{X} = D(A^{\frac{m}{2}}) \times D(A^{\frac{m-1}{2}})$ で制御 $\mathcal{V} = H_0^m(0,T;L^2(\Gamma_\varepsilon)) \cap L^2(0,T;H^m(\Gamma_\varepsilon))$ により可制御である。

注意 1.　補間することにより定理は任意の実数 $m \geq 1$ に対して成立する。

2. 定理の証明

証明は Lions[4] に従って行われる。HUM は写像Λを考えることにより、非斉次初期値境界値方程式 (1)(2)(3) の可制御問題を対応する斉次初期値境界値方程式（ここでは (7) ）のアプリオリ評価の問題に帰着する。

m を任意の自然数とし、$\mathcal{X}_0 = D(A^{\frac{m+1}{2}}) \times D(A^{\frac{m}{2}})$ 、$\mathcal{X}_1 = D(A^{\frac{m-1}{2}}) \times D(A^{\frac{m}{2}})$ とする。この時、$\mathcal{X}_0$から $\mathcal{X}_1$の写像Λを次の斉次初期値境界値方程式 (7) と非斉次初期値境界値方程式 (9) で以下のように定める。

$\{\Phi^0, \Phi^1\} \in \mathcal{X}_0$に対して

$$\begin{aligned} &\Phi'' - \Delta\Phi = 0 \quad \text{in } \Omega \times (0,T) \\ &\Phi(0) = \Phi^0, \Phi'(0) = \Phi^1 \quad \text{in } \Omega \\ &\Phi = 0 \quad \text{on } \Gamma \times (0,T) \end{aligned} \tag{7}$$

を解き、$\frac{\partial\Phi}{\partial\nu}$を$\Phi$の法線方向微分とし、$\Gamma \times (0,T)$ 上の関数 vを

$$v = S(x)r(t)\frac{\partial\Phi}{\partial\nu} \tag{8}$$

で定める。ここで $S \in C^\infty(\overline{\Omega})$ 、$0 \leq S \leq 1$ を

$$S(x) = \begin{cases} 1 & x \in \Gamma_0 \\ 0 & x \in \Gamma \setminus \Gamma_\varepsilon \end{cases}$$

とし、$r \in C^\infty([0,T])$ 、$0 \leq r \leq 1$ を

$$\begin{aligned} &r^{(i)}(0) = r^{(i)}(T) = 0, i = 1, 2, \cdots, \\ &r(t) = 1 \quad t \in [\delta, T-\delta] \quad \delta > 0 \end{aligned}$$

とする。vを Dirichlet 境界データに持つ次の非斉次終期値境界値問題を解く。

$$\begin{aligned} &\Psi'' - \Delta\Psi = 0 \quad \text{in } \Omega \times (0,T) \\ &\Psi(T) = \Psi'(T) = 0 \quad \text{in } \Omega \\ &\Psi = v \quad \text{on } \Gamma \times (0,T) \end{aligned} \tag{9}$$

この時、作用素Λを

(10) $$\Lambda\{\Phi^0, \Phi^1\} = \{\Psi'(0), -\Psi(0)\}$$

で定める。

注意2.　　通常の HUM では (9) の Dirichlet 境界条件は

$$\Psi = \begin{cases} \frac{\partial\Phi}{\partial\nu} & \text{on } \Gamma_0 \times (0,T) \\ 0 & \text{on } (\Gamma \setminus \Gamma_0) \times (0,T) \end{cases}$$

で与えるが、このままであると $\{\Phi^0, \Phi^1\}$ の属する空間 $\mathcal{X}_0$の regularity を上げても、$\frac{\partial\Phi}{\partial\nu}$ の regularity は上がるが$\Psi|_{\Gamma\times(0,T)}$の regularity は上がらない。(8) で Sと rにより滑らかに $\Gamma_\varepsilon \times (0,T)$ 上に拡張することにより$\frac{\partial\Phi}{\partial\nu}$と$\Psi|_{\Gamma\times(0,T)} = v$は同じ regularity を持つ。□

注意3.　　Λは $\mathcal{X}_0$から $\mathcal{X}_1$への写像として Well-defined である。

$\Lambda\{\Phi^0, \Phi^1\} = \{\Psi'(0), -\Psi(0)\}$ が $D(A^{\frac{m-1}{2}}) \times D(A^{\frac{m}{2}})$ に属することを示せばよい。$\{\Phi^0, \Phi^1\} \in \mathcal{X}_0$なので$\frac{\partial\Phi}{\partial\nu} \in H^m(\Gamma \times (0,T))$ (Lasiecka-Lions-Triggiani[3])。よって $v = S(x)r(t)\frac{\partial\Phi}{\partial\nu} \in H^m(\Gamma \times (0,T))$ 。また、$v^{(j)}(T) = 0, j = 0, 1, \cdots, m-1$ なので、終期値問題 (9) の終期データ $\{0, 0\}$ に対する $t = T$での compatibility conditions はすべて満たされる。よって再び Lasiecka-Lions-Triggiani[3] より $\{\Psi(t), \Psi'(t)\} \in C(0,T; H^m(\Omega) \times H^{m-1}(\Omega))$ 。これより、特に $\{\Psi'(0), -\Psi(0)\} \in H^{m-1}(\Omega) \times H^m(\Omega)$ を得る。さらに$t = 0$での compatibility conditions から

$$0 = v|_{t=0} = \Psi(0)|_\Gamma, \quad 0 = v^{(2)}|_{t=0} = A\Psi(0)|_\Gamma, \cdots,$$
$$\begin{cases} 0 = v^{(m-2)}|_{t=0} = A^{\frac{m-2}{2}}\Psi(0)|_\Gamma & (m : \text{偶数}) \\ 0 = v^{(m-1)}|_{t=0} = A^{\frac{m-1}{2}}\Psi(0)|_\Gamma & (m : \text{奇数}) \end{cases}$$

となる。故に、$\Psi(0) \in D(A^{\frac{m}{2}})$ 。同様に、

$$0 = v^{(1)}|_{t=0} = \Psi'(0)|_\Gamma, \quad 0 = v^{(3)}|_{t=0} = A\Psi'(0)|_\Gamma, \cdots,$$
$$\begin{cases} 0 = v^{(m-1)}|_{t=0} = A^{\frac{m-2}{2}}\Psi'(0)|_\Gamma & (m : \text{偶数}) \\ 0 = v^{(m-2)}|_{t=0} = A^{\frac{m-3}{2}}\Psi'(0)|_\Gamma & (m : \text{奇数}) \end{cases}$$

となる。故に、$\Psi'(0) \in D(A^{\frac{m-1}{2}})$ 。従って $\{\Psi'(0), -\Psi(0)\} \in D(A^{\frac{m-1}{2}}) \times D(A^{\frac{m}{2}})$ を得る。
□

注意3より $\mathcal{X}_0$から $\mathcal{X}_1$への写像Λが定まるが、もしΛが全射であれば、システム (1)(2)(3) は$\mathcal{X}$で制御 $\mathcal{V} = H_0^m(0,T; L^2(\Gamma_\varepsilon)) \cap L^2(0,T; H^m(\Gamma_\varepsilon))$ で可制御となる。実際、Λが全射であれば、任意の $\mathcal{X}$の元 $\{Y^0, Y^1\}$ 、すなわち $\{Y^1, -Y^0\} \in \mathcal{X}_1$に対して、ある $\mathcal{X}_0$の元 $\{\Phi^0, \Phi^1\}$ が存在して、$\Lambda\{\Phi^0, \Phi^1\} = \{Y^1, -Y^0\}$ と出来る。Λの定義により、これは

(11) $$\Psi(0) = Y^0, \quad \Psi'(0) = Y^1 \quad \text{in } \Omega$$

を意味する。また、Ψは (9) の解であるから、改めて $Y=\Psi$と置けば、(9)(11) はシステム (1)(2)(3) が $\mathcal{X}$で可制御であることを示している。さらに、注意3より$\frac{\partial\Phi}{\partial\nu}\in H^m(\Gamma\times(0,T))$ であることと、(8) より

$$v|_{\Gamma_\epsilon}=Sr\frac{\partial\Phi}{\partial\nu}\in H_0^m(0,T;L^2(\Gamma_\epsilon))\cap L^2(0,T;H^m(\Gamma_\epsilon))$$
$$v|_{\Gamma\setminus\Gamma_\epsilon}=0$$

なので、制御空間$\mathcal{V}$として $H_0^m(0,T;L^2(\Gamma_\epsilon))\cap L^2(0,T;H^m(\Gamma_\epsilon))$ をとることが出来る。

3．写像Λの全射性

$\{\Psi^1,\Psi^0\}\in\mathcal{X}_1$に対して、$\mathcal{X}_0$上の有界線形汎関数：

$$\{\Phi^0,\Phi^1\}\mapsto\langle\{\Psi^1,\Psi^0\},\{\Phi^0,\Phi^1\}\rangle=(A^{\frac{m-1}{2}}\Psi^1,A^{\frac{m+1}{2}}\Phi^0)_{L^2(\Omega)}+(A^{\frac{m}{2}}\Psi^0,A^{\frac{m}{2}}\Phi^1)_{L^2(\Omega)}$$

（ここで $(\cdot,\cdot)_{L^2(\Omega)}$は $L^2(\Omega)$ の通常の内積を表す。）を対応させる写像をTとすると、Tは$\mathcal{X}_1$から$\mathcal{X}_0'$（$\mathcal{X}_0'$は$\mathcal{X}_0$の双対空間）への同型写像となる。したがって$T\Lambda$は$\mathcal{X}_0$から$\mathcal{X}_0'$への写像を定めるが、後で示す補題１、２により、次の不等式：

$$c_1\|\{\Phi^0,\Phi^1\}\|_{\mathcal{X}_0}\le\langle\Lambda\{\Phi^0,\Phi^1\},\{\Phi^0,\Phi^1\}\rangle\le c_2\|\{\Phi^0,\Phi^1\}\|_{\mathcal{X}_0}\tag{12}$$

が任意の $\{\Phi^0,\Phi^1\}\in\mathcal{X}_0$に対して示される。すなわち $\langle\Lambda\{\Phi^0,\Phi^1\},\{\Phi^0,\Phi^1\}\rangle$ は$\mathcal{X}_0$の内積を定める。よって Riesz の定理より$T\Lambda$は$\mathcal{X}_0$から$\mathcal{X}_0'$への同型写像となる。よって、Λも$\mathcal{X}_0$から$\mathcal{X}_1$への同型写像となる。

4．補題１・２

不等式 (12) は次の２つの補題により示される。

補題１． 任意の$\mathcal{X}_0$の元 $\{\Phi^0,\Phi^1\}$、$\{\tilde{\Phi}^0,\tilde{\Phi}^1\}$ に対して、次の等式が成り立つ。

$$\langle\Lambda\{\Phi^0,\Phi^1\},\{\tilde{\Phi}^0,\tilde{\Phi}^1\}\rangle=\int_0^T\int_\Gamma v^{(m)}\frac{\partial\tilde{\Phi}^{(m)}}{\partial\nu}d\Gamma dt\tag{13}$$

ただし$\tilde{\Phi}$は $\{\tilde{\Phi}^0,\tilde{\Phi}^1\}$ を初期データとした時の *(7)* の解とする。

証明. Ψを (9) の解とし、$\tilde{\Phi}''-\Delta\tilde{\Phi}=0$ をtに関してm回微分したものに$\Psi^{(m)}$を掛け$\Omega\times(0,T)$ 上で積分した式を部分積分することにより次の式を得る。（詳しくは Lions[4] を参照）

$$\begin{aligned}\int_0^T\int_\Gamma v^{(m)}\frac{\partial\tilde{\Phi}^{(m)}}{\partial\nu}d\Gamma dt &=\int_\Omega\Psi^{(m)}(T)\tilde{\Phi}^{(m+1)}(T)dx-\int_\Omega\Psi^{(m)}(0)\tilde{\Phi}^{(m+1)}(0)dx\\ &-\int_\Omega\Psi^{(m+1)}(T)\tilde{\Phi}^{(m)}(T)dx+\int_\Omega\Psi^{(m+1)}(0)\tilde{\Phi}^{(m)}(0)dx\end{aligned}$$

ここで、$\Psi'' - \Delta\Psi = 0$ より

$$\Psi^{(l)} = \begin{cases} \Delta^{\frac{l}{2}}\Psi & (l : \text{偶数}) \\ \Delta^{\frac{l-1}{2}}\Psi' & (l : \text{奇数}) \end{cases} \quad l = 2, 3, \cdots, m+1$$

であるから、特に

$$\Psi^{(l)}(T) = \begin{cases} \Delta^{\frac{l}{2}}\Psi(T) & (l : \text{偶数}) \\ \Delta^{\frac{l-1}{2}}\Psi'(T) & (l : \text{奇数}) \end{cases} \quad l = m, m+1$$

$$\Psi^{(l)}(0) = \begin{cases} \Delta^{\frac{l}{2}}\Psi(0) & (l : \text{偶数}) \\ \Delta^{\frac{l-1}{2}}\Psi'(0) & (l : \text{奇数}) \end{cases} \quad l = m, m+1$$

を得る。また、Ψの終期条件より$\Psi(T) = \Psi'(T) = 0$ であり、注意3より $\{\Psi(0), \Psi'(0)\} \in D(A^{\frac{m}{2}}) \times D(A^{\frac{m-1}{2}})$ であるから

$$\begin{aligned}
&\int_0^T \int_\Gamma v^{(m)} \frac{\partial \tilde{\Phi}^{(m)}}{\partial \nu} d\Gamma dt \\
&= \begin{cases} \int_\Omega A^{\frac{m}{2}}\Psi'(0) A^{\frac{m}{2}} \tilde{\Phi}^0 dx - \int_\Omega A^{\frac{m}{2}}\Psi(0) A^{\frac{m}{2}} \tilde{\Phi}^1 dx & (m : \text{偶数}) \\ \int_\Omega A^{\frac{m-1}{2}}\Psi'(0) A^{\frac{m+1}{2}} \tilde{\Phi}^0 dx - \int_\Omega A^{\frac{m+1}{2}}\Psi(0) A^{\frac{m-1}{2}} \tilde{\Phi}^1 dx & (m : \text{奇数}) \end{cases} \\
&= \int_\Omega A^{\frac{m-1}{2}}\Psi'(0) A^{\frac{m+1}{2}} \tilde{\Phi}^0 dx - \int_\Omega A^{\frac{m}{2}}\Psi(0) A^{\frac{m}{2}} \tilde{\Phi}^1 dx \\
&= \langle \Lambda\{\Phi^0, \Phi^1\}, \{\tilde{\Phi}^0, \tilde{\Phi}^1\} \rangle
\end{aligned}$$

を得る。□

補題2. 正の定数c_1, c_2が存在して、任意の $\{\Phi^0, \Phi^1\} \in \mathcal{X}_0$ に対して次の不等式が成り立つ。

(14)
$$c_1 \|\{\Phi^0, \Phi^1\}\|^2_{\mathcal{X}_0} \le \int_0^T \int_\Gamma v^{(m)} \frac{\partial \Phi^{(m)}}{\partial \nu} d\Gamma dt \le c_2 \|\{\Phi^0, \Phi^1\}\|^2_{\mathcal{X}_0}$$

証明. Lions[4] により、正の定数c_1、c_2、T_0が存在して、任意の $\{\Phi^0, \Phi^1\} \in H^1_0(\Omega) \times L^2(\Omega) = D(A^{\frac{1}{2}}) \times D(A^0)$ 、$T > T_0$に対して、不等式：

(15)
$$\begin{aligned}
c_1(T - T_0)\|\{\Phi^0, \Phi^1\}\|^2_{D(A^{\frac{1}{2}})\times D(A^0)} &\le \int_0^T \int_{\Gamma_0} |\frac{\partial \Phi}{\partial \nu}|^2 d\Gamma dt \\
&\le \int_0^T \int_\Gamma |\frac{\partial \Phi}{\partial \nu}|^2 d\Gamma dt \le c_2(T+1)\|\{\Phi^0, \Phi^1\}\|^2_{D(A^{\frac{1}{2}})\times D(A^0)}
\end{aligned}$$

が (7) の解Φに対して成り立つ。今、$\Phi^{(j)} (j = 1, 2, \cdots, m)$ は

$$\begin{aligned}
&\Theta'' - \Delta\Theta = 0 \quad \text{in } \Omega \times (0, T) \\
&\begin{cases} \Theta(0) = (-A)^{\frac{j}{2}}\Phi^0,\ \Theta'(0) = (-A)^{\frac{j}{2}}\Phi^1 & \text{in } \Omega \quad (j : \text{偶数}) \\ \Theta(0) = (-A)^{\frac{j-1}{2}}\Phi^1,\ \Theta'(0) = (-A)^{\frac{j+1}{2}}\Phi^0 & \text{in } \Omega \quad (j : \text{奇数}) \end{cases} \\
&\Theta = 0 \quad \text{on } \Gamma \times (0, T)
\end{aligned}$$

の解なので$\Theta = \Phi^{(j)}$に (15) を適用して

$$c_1(T-T_0)\|\{\Phi^0,\Phi^1\}\|^2_{D(A^{\frac{j+1}{2}})\times D(A^{\frac{j}{2}})} \le \int_0^T\int_{\Gamma_0}|\frac{\partial\Phi^{(j)}}{\partial\nu}|^2 d\Gamma dt$$
(16)
$$\le \int_0^T\int_{\Gamma}|\frac{\partial\Phi^{(j)}}{\partial\nu}|^2 d\Gamma dt \le c_2(T+1)\|\{\Phi^0,\Phi^1\}\|^2_{D(A^{\frac{j+1}{2}})\times D(A^{\frac{j}{2}})}$$

を任意の $\{\Phi^0,\Phi^1\}\in\mathcal{X}_0$と $j=1,2,\cdots,m$ に対して得る。(14) の右側の不等式を導く。

$$\int_0^T\int_\Gamma v^{(m)}\frac{\partial\Phi}{\partial\nu}^{(m)} d\Gamma dt$$
$$= \int_0^T\int_\Gamma S(x)\left(r(t)\frac{\partial\Phi}{\partial\nu}\right)^{(m)}\frac{\partial\Phi}{\partial\nu}^{(m)} d\Gamma dt$$
$$= \sum_{j=1}^m\binom{m}{j}\int_0^T\int_\Gamma S(x)r^{(m-j)}(t)\frac{\partial\Phi}{\partial\nu}^{(j)}\frac{\partial\Phi}{\partial\nu}^{(m)} d\Gamma dt$$
$$\le \sum_{j=1}^m\binom{m}{j}\int_0^T\int_\Gamma |S||r^{(m-j)}(t)||\frac{\partial\Phi}{\partial\nu}^{(j)}||\frac{\partial\Phi}{\partial\nu}^{(m)}| d\Gamma dt$$

$0\le S\le 1$、$r\in C^\infty([0,T])$ より

$$\le C_r\sum_{j=1}^m\int_0^T\int_\Gamma|\frac{\partial\Phi}{\partial\nu}^{(j)}||\frac{\partial\Phi}{\partial\nu}^{(m)}|d\Gamma dt$$
$$\le C_r\sum_{j=1}^m\left(\int_0^T\int_\Gamma|\frac{\partial\Phi}{\partial\nu}^{(j)}|^2 d\Gamma dt\right)^{\frac{1}{2}}\left(|\frac{\partial\Phi}{\partial\nu}^{(m)}|^2 d\Gamma dt\right)^{\frac{1}{2}}$$

(16) の右側の不等式より

$$\le C_rC_2(T+1)\sum_{j=1}^m\|\{\Phi^0,\Phi^1\}\|_{D(A^{\frac{j+1}{2}})\times D(A^{\frac{j}{2}})}\|\{\Phi^0,\Phi^1\}\|_{\mathcal{X}_0}$$
$$\le C\|\{\Phi^0,\Phi^1\}\|^2_{\mathcal{X}_0}$$

よって (14) の右側の不等式が得られた。次に (14) の左側の不等式を導く。

$$\int_0^T\int_\Gamma v^{(m)}\frac{\partial\Phi}{\partial\nu}^{(m)} d\Gamma dt$$
$$= \sum_{j=1}^m\binom{m}{j}\int_0^T\int_\Gamma S(x)r^{(m-j)}(t)\frac{\partial\Phi}{\partial\nu}^{(j)}\frac{\partial\Phi}{\partial\nu}^{(m)} d\Gamma dt$$

$S(x)=1, x\in\Gamma_0$と$r(t)=1, t\in[\delta, T-\delta]$ より

$$\ge \int_0^T\int_{\Gamma_0} r(t)|\frac{\partial\Phi}{\partial\nu}^{(m)}|^2 d\Gamma dt$$
$$-\sum_{j=1}^{m-1}\binom{m}{j}\left(\int_0^\delta+\int_{T-\delta}^T\right)\int_\Gamma|r^{(m-j)}(t)||\frac{\partial\Phi}{\partial\nu}^{(j)}||\frac{\partial\Phi}{\partial\nu}^{(m)}|d\Gamma dt$$
$$\ge \int_\delta^{T-\delta}\int_{\Gamma_0}|\frac{\partial\Phi}{\partial\nu}^{(m)}|^2 d\Gamma dt$$

$$-C_r \sum_{j=1}^{m-1} \left((\int_0^\delta + \int_{T-\delta}^T) \int_\Gamma |\frac{\partial \Phi^{(j)}}{\partial \nu}|^2 d\Gamma dt \right)^{\frac{1}{2}} \left((\int_0^\delta + \int_{T-\delta}^T) \int_\Gamma |\frac{\partial \Phi^{(m)}}{\partial \nu}|^2 d\Gamma dt \right)^{\frac{1}{2}}$$

(14) の不等式より

$$\geq C_1(T - 2\delta - T_0) \|\{\Phi^0, \Phi^1\}\|_{\mathcal{X}_0}^2$$

$$-C_r C_2 (2\delta + 1) \sum_{j=1}^{m-1} \|\{\Phi^0, \Phi^1\}\|_{D(A^{\frac{j+1}{2}}) \times D(A^{\frac{j}{2}})} \|\{\Phi^0, \Phi^1\}\|_{\mathcal{X}_0}$$

$$\geq C_1\{T - (2\delta + T_0 + C_1^{-1} C' C_2 (2\delta + 1))\} \|\{\Phi^0, \Phi^1\}\|_{\mathcal{X}_0}^2$$

よって $T_m = (2\delta + T_0 + C_1^{-1} C' C_2 (2\delta + 1))$ と置けば $T > T_m$ なる任意の T に対して、$C_1 = T - T_m$ として (14) の左側の不等式が成り立つ。□

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摩擦項を持つ波動方程式の解の挙動について

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1. 序と結果

$\mathcal{O}$ を $\mathbf{R}^N$ の有界な障害物として、$\Omega \equiv \mathbf{R}^N - \mathcal{O}$；障害物の外部領域（但し $\partial\Omega$ は滑らか）で次の波動方程式を考える：

$$
(1.1) \qquad \begin{cases} w_{tt} - \Delta w + b(x,t)w_t = 0 & \text{in} \quad \Omega \times (0,\infty) \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x) & \text{in} \quad \Omega \\ w(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,\infty) \end{cases}
$$

但し $b(x,t) \geq 0$ とする。従って $b(x,t)w_t$ は粘性型の摩擦抵抗を表す。また、初期値は $\{w_0, w_1\} \in \mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega)$ とする。(1.1) の解のエネルギーは

$$
\|W(t)\|_E^2 = \frac{1}{2}(\|\nabla w(t)\|_{L^2(\Omega)}^2 + \|w_t(t)\|_{L^2(\Omega)}^2)
$$

で与えられる（但し $W(t) = \{w(t), w_t(t)\}$ とする）。またエネルギー空間 E を次の様に定める：

$$
E = \{W(t) \quad | \quad \|W(t)\|_E < \infty\}.
$$

このとき次のエネルギー等式が成り立つ：

$$
(1.2) \qquad \|W(t)\|_E^2 + \int_s^t \int_\Omega b(x,\tau)w_t(\tau)^2 dx d\tau = \|W(s)\|_E^2 \qquad (0 \leq s < t) \quad .
$$

故に $b(x,t)$ の遠方での挙動によってエネルギーが減るか否かが問題となる。これに関しては以下の結果が知られている：

(Energy decay)　松村 [1]:Cauchy 問題 in $\mathbf{R}^N (N \geq 1)$ で

$$
\begin{cases} b_1(1+|x|+t)^{-1} \leq b(x,t) \leq b_2, \quad b_1, b_2 : \text{正定数} \\ b_t(x,t) \leq 0 \end{cases}
$$

なら解のエネルギーは $t \to +\infty$ で decay する。

(Energy nondecay)　望月 [2],[3]:Cauchy 問題 in $\mathbf{R}^N (N \neq 2)$ で

$$
\begin{cases} 0 \leq b(x,t) \leq b_3(1+|x|)^{-\delta}, \quad b_3 : \text{正定数} \\ \delta > 1 \end{cases}
$$

なら一般に解のエネルギーは $t \to +\infty$ で decay しない。

これらの結果を以下の様に拡張することができた [6]：

Theorem 1. $N \geq 1$ とし、$b(x,t)$ は

$$(1.3)\qquad \begin{cases} b_1(1+|x|+t)^{-1}\{\log(a+|x|+t)\}^{-\delta} \leq b(x,t) \leq b_2 \\ 0 \leq \delta \leq 1, \quad a > 1, \quad b_1, b_2 : \text{正定数} \\ b_t(x,t) \leq 0 \end{cases}$$

を満たす

⇒ (1.1) の解のエネルギーは $t \to +\infty$ で *decay* する;

$$0 < \exists\mu = \mu(\delta) < 1$$

$$s,t \quad \|W(t)\|_E^2 \leq \exists K(W(0),\varphi)\{\log(a+t)\}^{-\mu}$$

但し $\log a > \max\{1, 2\mu-1\}$ であり $K(W(0),\varphi) \in (0,+\infty)$ は初期値 $W(0)$ と関数 $\varphi(t) = \{\log(a+t)\}^{\mu}$ に依る量である。

$w_0(x,t)$ を摩擦項のない (free な) 波動方程式の解とする;

$$(1.4)\qquad \begin{cases} w_{0tt} - \triangle w_0 = 0 & \text{in} \quad \Omega \times (0,\infty) \\ w_0(x,0) = f_0(x), \quad w_{0t}(x,0) = f_1(x) & \text{in} \quad \Omega \\ w_0(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,\infty). \end{cases}$$

また $U_0(t)$ を $U_0(t)\{f_0, f_1\} = W_0(t) \equiv \{w_0(t), w_{0t}(t)\}$ なる E における unitary 作用素とする。

Theorem 2. $N \geq 3$ とし、$\mathcal{O}$ は原点に関して星状とする。$b(x,t)$ が

$$(1.5)\qquad \begin{cases} 0 \leq b(x,t) \leq b_3(1+|x|)^{-1}\{\log(a+|x|)\}^{-\delta} \\ \delta > 1, \quad a > 1, \quad b_3 : \text{正定数} \end{cases}$$

を満たす

⇒ (1.1) の解のエネルギーは一般に $t \to +\infty$ で *decay* せず、

$$\exists w_0(t) : (1.4)\text{の解} \quad s,t \quad \|W(t) - W_0(t)\|_E \to 0 \quad as \quad t \to +\infty$$

が成り立つ。

以下の節では証明の概略を与える。

2. Th1 の証明

重みの関数 $\varphi = \varphi(s) \quad (s \geq 0)$ を次のような関数とする;

$$(2.1)\qquad \begin{cases} \varphi(s) \geq 1, \quad \varphi'(s) > 0, \quad \varphi''(s) \leq 0, \quad \lim_{s\to+\infty} \varphi(s) = +\infty \\ \exists k > 1 \quad s,t \quad \varphi(s)\varphi''(s) + k^2\varphi'(s)^2 \leq 0. \end{cases}$$

このような φ として以下では

$$(2.2)\qquad \varphi(s) = \{\log(a+s)\}^{\mu}, \quad \mu > 0, \quad \log a > \max\{1, 2\mu-1\}$$

のみを考える。この φ に対して (1.1) の解の重みつきエネルギーを次で定める:

$$\|W(t)\|_{E_\varphi}^2 = \frac{1}{2}\int_\Omega [\varphi(|x|+t)\{w_t(t)^2 + |\nabla w(t)|^2\} - \varphi''(|x|+t)w(t)^2]dx.$$

そして重みつきエネルギー空間 E_φ を

$$E_\varphi = \{W(t) \quad | \quad \|W(t)\|_{E_\varphi}^2 < \infty\}$$

で定める。

(1.1) の第1式の両辺に $\{\varphi(|x|+t)w\}_t$ を乗じて発散形式を作ると次の等式が成り立つ:

$$X_T + \nabla\cdot Y + Z = 0$$

但し、

$$X = \frac{1}{2}\varphi(|x|+t)\{w_t^2 + |\nabla w|^2\} - \frac{1}{2}\varphi''(|x|+t)w^2 + \varphi'(|x|+t)w_t w$$

$$Y = -\{\varphi(|x|+t)w_t + \varphi'(|x|+t)w\}\nabla w + \frac{1}{2}\varphi''(|x|+t)w^2\frac{x}{r}$$

$$Z = b(x,t)\{\varphi(|x|+t)w_t^2 + \varphi'(|x|+t)w_t w\}$$

$$+\frac{1}{2}\varphi'(|x|+t)\{|\nabla w|^2 - 3w_t^2 + 2(\frac{x}{r}\cdot\nabla w)w_t\} - \frac{N-1}{2r}\varphi''(|x|+t)w^2.$$

これを $\Omega\times(0,t)$ 上で部分積分することにより次の補題が得られる:

Lemma 2.1. $N \geq 1$ とし、初期値は $\{w_0, w_1\} \in \mathbf{H}^2(\Omega)\times\mathbf{H}^1(\Omega)\cap E_\varphi$ とすると、次の不等式が成り立つ:

$$\frac{k-1}{k}\|W(t)\|_{E_\varphi}^2 + \int_0^t\int_\Omega \varphi(|x|+\tau)b(x,\tau)w_t^2 dxd\tau$$

$$\leq \frac{k+1}{k}\|W(0)\|_{E_\varphi}^2 + \int_0^t\int_\Omega \varphi'(|x|+\tau)\{2w_t^2 + b(x,\tau)|w_t w|\}dxd\tau.$$

ここに Young の不等式を用いて次が得られる:

Proposition 2.2. *Lemma 2.1 と同じ仮定をし、更に、以下を仮定する:*

(2.3)

$$\exists\varepsilon\in(0,1) \quad s.t \quad 2\int_0^t\int_\Omega \varphi'(|x|+\tau)w_\tau^2 dxd\tau \leq (1-\varepsilon)\int_0^t\int_\Omega \varphi(|x|+\tau)b(x,\tau)w_t^2 dxd\tau$$

(2.4) $$\exists K_1(W(0),\varphi)\in(0,\infty)$$

$$s.t \quad \int_0^t\int_\Omega |\varphi'(|x|+\tau)b(x,\tau)w_t w|dxd\tau$$

$$\leq K_1(W(0),\varphi)^{\frac{1}{2}}\left(\int_0^t\int_\Omega \varphi(|x|+\tau)b(x,\tau)w_t^2 dxd\tau\right)^{\frac{1}{2}}$$

このとき次が成り立つ:

$$\exists C>0 \quad s.t \quad \|W(t)\|_{E_\varphi}^2 \leq K(W(0),\varphi) = C\{\|W(0)\|_{E_\varphi}^2 + K_1(W(0),\varphi)\} < \infty$$

従って

$$\|W(t)\|_E^2 \leq K(W(0),\varphi)\varphi(t)^{-1}$$

が成り立つ。

(2.3) が満たされることは次による:

Lemma 2.3. μ を $0<\mu<\frac{b_1}{2}(\log a)^{1-\delta}$ $(0\le\delta\le 1)$ と選ぶと *(2.3)* が成り立つ。

これは $b(x,t)$ に対する仮定 (1.3) を用いて直接計算により得られる。一方 (2.4) が満たされることは次による:

Lemma 2.4. μ を $0<\mu<1$ と選ぶと *(2.4)* が成り立つ。但し

$$K_1(W(0),\varphi)=J(\varphi)\{\|W(0)\|_E^2+\|w_0\|_{L^2(\Omega)}^2\}$$

ここに

$$J(\varphi)=\int_0^\infty (a+\tau)^{-1}\{\log(a+\tau)\}^{\mu-2}d\tau<\infty.$$

Proof. Schwartz の不等式により

$$\begin{aligned}&\int_0^t\int_\Omega |\varphi'(|x|+\tau)b(x,\tau)w_t w|dxd\tau\\ &\le[\int_0^t\int_\Omega \varphi^{-1}(|x|+\tau)b(x,\tau)\{\varphi'(|x|+\tau)w\}^2dxd\tau]^{\frac12}\\ &\times[\int_0^t\int_\Omega \varphi(|x|+\tau)b(x,\tau)w_t^2dxd\tau]^{\frac12}\end{aligned}$$

であるが、μ の選び方から

$$\tag{2.5}\begin{aligned}&\int_0^t\int_\Omega \varphi^{-1}(|x|+\tau)b(x,\tau)\{\varphi'(|x|+\tau)w\}^2dxd\tau\\ &\qquad\le\int_0^t \varphi^{-1}(\tau)\{\varphi'(\tau)\}^2[\int_\Omega b(x,\tau)w^2dx]d\tau\end{aligned}$$

がわかる。故に

$$\tag{2.6}\int_0^t\int_\Omega \varphi^{-1}(|x|+\tau)b(x,\tau)\{\varphi'(|x|+\tau)w\}^2dxd\tau\le K_1(W(0),\varphi)$$

が示されればよい。(以下 (2.6) の証明) 微分積分学の基本定理と Schwartz の不等式、更に (1.2),(1.3) を用いると

$$\int_\Omega b(x,\tau)w^2dx\le\exists C(a+\tau)\{\|W(0)\|_E^2+b_2\|w_0\|_{L^2(\Omega)}\}$$

が成り立つ。これと (2.5) とから (2.6) が従う。 □

3. Th2 の証明

まず次の 2 つの等式が成り立つ (証明は望月 [4] 補題 26.2、26.4 を参照せよ):

Lemma 3.1. *(1.4)* の解 w_0 に対して次が成り立つ (但し $\psi = \psi(r)$ は実数値関数とし、また $r = |x|$ とする):

$$\frac{\partial}{\partial t}\{\frac{1}{2}(w_0t^2 + |\nabla w_0 + \frac{N-1}{2r}w_0\frac{x}{r}|^2 + \frac{(N-1)(N-3)}{4r^2}w_0^2)\}$$
$$-\nabla\cdot\{w_{0t}\nabla w_0 + \frac{N-1}{2r}w_0w_{0t}\frac{x}{r}\} = 0,$$

$$\frac{\partial}{\partial t}\{\psi(w_{0r} + \frac{N-1}{2r}w_0)w_{0t}\}$$
$$-\nabla\cdot\{\psi w_{0r}\nabla w_0 + \frac{\psi(N-1)}{2r}w_o\nabla w_0 + \frac{\psi(N-1)}{4r^2}w_0^2\frac{x}{r} + \frac{\psi}{2}(w_{0t}^2 - |w_0|^2)\frac{x}{r}\}$$
$$+\frac{\psi'}{2}(w_{0t}^2 + |\nabla w_0 + \frac{N-1}{2r}w_0\frac{x}{r}|^2)$$
$$+(\frac{\psi}{r} - \psi')(|\nabla w_0|^2 - w_{0r}^2) + (2\psi - r\psi')\frac{(N-1)(N-3)}{8r^3}w_0^2 = 0.$$

この等式を組み合わせて次の命題が得られる:

Proposition 3.2. $N \geq 3$ とし、$\mathcal{O}$ は原点に関して星状とする。このとき (1.4) の解に対して以下の不等式が成り立つ：

$$\|W_0(t)\|_E^2 + \frac{1}{2}\int_0^t\int_\Omega \psi'(w_{0,t}^2 + |\nabla w_0|^2)dxd\tau \leq \exists C\|W_0(0)\|_E^2$$

但し

$$\psi(|x|) = \frac{\gamma[\nu\{\log(a+|x|)\}^{\delta_0} - 1]}{\nu\{\log(a+|x|)\}^{\delta_0}}$$

$$(a \geq e, \quad 0 < \delta_0(= \delta - 1) < 1, \quad \nu > 1 + \frac{1}{\log a}, \quad 0 < \gamma < 1).$$

証明は望月 [4] の方法に従う。また Th2 の前半は次の命題による：

Proposition 3.3. *Th 2* と同じ仮定の下で、

$$F = \{f_0, f_1\} \in \mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega)$$

とし、$\sigma \geq 0$　を次の様に選ぶ；

$$\int_\sigma^\infty\int_\Omega b(x,\tau)w_{0t}^2dxd\tau < 4\|F\|_E^2.$$

また w^σ を次の波動方程式の解とする：

$$(3.1)\quad \begin{cases} w_{tt}^\sigma - \Delta w^\sigma + b(x,t)w_t^\sigma = 0 & in \quad \Omega\times(\sigma,\infty) \\ w^\sigma(x,\sigma) = w_0(x,\sigma), \quad w_t^\sigma(x,\sigma) = w_{0t}(x,\sigma) & in \quad \Omega \\ w^\sigma(x,t) = 0 & on \quad \partial\Omega\times(\sigma,\infty). \end{cases}$$

このとき $\|W^\sigma(t)\|_E$ は *decay* しない。

Proof. 背理法による。$\|W^\sigma(t)\|_E \to 0$ as $t \to +\infty$ と仮定する。(3.1) の第 1 式の両辺に w_{0t} を乗じて $\Omega \times (\sigma, t)$ $(0 \le \sigma < t)$ で部分積分して得られる式に Schwartz の不等式を用いると

$$
\begin{aligned}
2\|F\|_E^2 &= 2(W^\sigma(\sigma), W_0(\sigma))_E \\
&\le \{\int_\sigma^\infty \int_\Omega b(x,\tau) w_t^\sigma(\tau)^2 dx d\tau\}^{\frac{1}{2}} \{\int_\sigma^\infty \int_\Omega b(x,\tau) w_{0t}^2 dx d\tau\}^{\frac{1}{2}} \\
&= \|F\|_E \int_\sigma^\infty \int_\Omega b(x,\tau) w_{0t}^2 dx d\tau.
\end{aligned}
$$

ここに $(\ ,\ )_E$ は E における内積である。これより

$$4\|F\|_E^2 \le \int_\sigma^\infty \int_\Omega b(x,\tau) w_{0t}^2 dx d\tau$$

が従い矛盾を得る。 □

Th2 の後半の *Proof.* Prop3.2 に注意すると $\forall F \in E$ に対し

$$|(U_0(-t)W(t) - U_0(-t')W(t'), F)_E| \le \exists C\|F\|_E \{\int_{t'}^t \int_\Omega b(x,\tau)([U_0(\tau)F]_2)^2 dx d\tau\}^{\frac{1}{2}}$$

が得られるので

$$\|U_0(-t)W(t) - U_0(-t')W(t')\|_E \le C\{\int_{t'}^t \int_\Omega b(x,\tau)([U_0(\tau)F]_2)^2 dx d\tau\}^{\frac{1}{2}}.$$

これより結果が従う。 □

4. 注意

境界条件が $w_n(x,t) = 0$ on $\partial\Omega$ の場合にも $\mathcal{O}$ が星状ならば Th1 と同じ結果が成り立つ。また外部問題では境界上の摩擦はエネルギーの decay にほとんど影響を与えないこともわかる。

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A CONSTRUCTION OF A FUNCTION f SATISFYING $\Delta f \in L^1_{loc}$ AND $\partial_{ij} f \notin L^1_{loc}$ (WOLFGANG ROTHER (1987) の論文紹介)

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1.Introduction

既知の結果として $1 < p < \infty$ に対し $f \in W^{1,p}_{loc}(R^n)$ であり $\Delta f \in L^p_{loc}(R^n)$ ならば、$\partial_{ij} f \in L^p_{loc}(R^n)$ $(1 \leq i,j \leq n)$ であることが知られているが、p=1 の時 $f \in W^{1,1}_{loc}(R^n)$ であり $\Delta f \in L^1_{loc}(R^n)$ かつ $\partial_{ij} f \notin L^1_{loc}$ $(1 \leq i,j \leq n)$ である f が存在する。そのような f を構成することが、この論文の目的である。

2.The construction of a counterexample

$0 < \delta < 1$, $B_\delta := \{x \in R^n |\, |x| < \delta\}$, ω_n を n 次元単位球の体積とし、$g(x), \Gamma(x)$ を以下で定義する。

$$g(x) := \begin{cases} |x|^{-n}(\log|x|)^{-2} & (0 < |x| \leq \delta) \\ 0 & (|x| = 0 \ , \ |x| > \delta) \end{cases}$$

$$\Gamma(x) := \begin{cases} \dfrac{1}{n(2-n)\omega_n}|x|^{2-n} & (n \geq 3) \\ \dfrac{1}{2\pi}\log|x| & (n = 2) \end{cases}$$

更に $f := g * \Gamma$ とすると $f \in W^{1,1}_{loc}(R^n)$ $(\because g \in L^1(R^n), \Gamma \in L^1_{loc}(R^n))$, $\Delta f = g$ in $\mathcal{D}'(R^n)$ となる。g はその定義より $g \in L^1(R^n)$ なので $\Delta f \in L^1(R^n)$ が言える。

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Lemma 1.

$$\text{a)}\quad \int_{S^{n-1}} |x-\omega|^{2-n} dS_\omega = n\omega_n \cdot [\max|x|, 1]^{2-n} \quad (n \geq 3)$$

$$\text{b)}\quad \int_{S^{n-1}} \log|x-\omega| dS_\omega = 2\omega_2 \cdot \max\{\log|x|, 0\} \quad (n = 2)$$

この証明の $n=3$ のときについては、[J.Wermer,Potential Theory.LMN 408,Berlin-Heidelberg-New York 1974.] を参照していただきたい。また、$n \neq 3$ の時もほぼ同様に示される。

Lemma 2.

$0 < |x| \leq \delta$ に対し、以下が成り立つ。

$$a)\quad f(x) = (2-n)^{-1}\left\{|x|^{2-n}\int_0^{|x|} s^{-1}(\log s)^{-2}ds + \int_{|x|}^{\delta} s^{1-n}(\log s)^{-2}ds\right\} \quad (n \geq 3)$$

$$b)\quad f(x) = -1 + \log(-\log\delta) - \log(-\log|x|) \quad (n = 2)$$

Proof a) $0 \leq r = |x|$ に対し $F(r) := f(x), G(r) := g(x)$ とする。

$$\begin{aligned} f(x) &= g * \Gamma \\ &= (n(2-n)\cdot\omega_n)^{-1}\int g(y)\ |x-y|^{2-n}dy \\ &= (n(2-n)\cdot\omega_n)^{-1}\int_0^{\delta}\int_{S^{n-1}} g(r\omega)\ |x-r\omega|^{2-n}dS_\omega\ r^{n-1}\ dr \\ &= (n(2-n)\cdot\omega_n)^{-1}\int_0^{\delta} G(r)\ r\int_{S^{n-1}} |r^{-1}x-\omega|^{2-n}\ dS_\omega\ dr \end{aligned}$$

ここで、Lemma 1. を用いて

$$\begin{aligned} f(x) &= (n(2-n)\cdot\omega_n)^{-1}\int_0^{\delta} G(r)\ r\cdot n\omega_n\cdot[\max\{|r^{-1}x|, 1\}]^{2-n}\ dr \\ &= (2-n)^{-1}\left\{\int_0^{|x|} G(r)r|r^{-1}x|^{2-n}\ dr\ +\ \int_{|x|}^{\delta} G(r)r\ dr\right\} \\ &= (2-n)^{-1}\left\{|x|^{2-n}\int_0^{|x|} G(r)r^{n-1}\ dr\ +\ \int_{|x|}^{\delta} G(r)r\ dr\right\} \\ &= (2-n)^{-1}\left\{|x|^{2-n}\int_0^{|x|} r^{-n}(\log r)^{-2}\cdot r^{n-1}\ dr\ +\ \int_{|x|}^{\delta} r^{-n}(\log r)^{-2}r\ dr\right\} \\ &= (2-n)^{-1}\left\{|x|^{2-n}\int_0^{|x|} r^{-1}(\log r)^{-2}dr\ +\ \int_{|x|}^{\delta} r^{1-n}(\log r)^{-2}\ dr\right\} \end{aligned}$$

b) についても、ほぼ同様の計算により示される。 ■

Lemma 3.

$a)\quad F'(r) = -r^{1-n}\cdot(\log r)^{-1} \quad (0 < r < \delta)$

$b)\quad F''(r) = r^{-n}\cdot(\log r)^{-2} + (n-1)\cdot r^{-n}\cdot(\log r)^{-n} \quad (0 < r < \delta)$

Proof

$$
\begin{aligned}
F(r) &= (2-n)^{-1}\left\{ r^{2-n}\int_0^r s^{-1}(\log s)^{-2}ds + \int_r^\delta s^{1-n}(\log s)^{-2}ds \right\} \\
F'(r) &= r^{1-n}\int_0^r s^{-1}(\log s)^{-2}ds \\
&= r^{1-n}\int_{-\infty}^{\log r} t^{-2}dt \\
&= r^{1-n}\cdot -(\log r)^{-1} \\
&= -r^{1-n}\cdot(\log r)^{-1} \\
F''(r) &= -(1-n)\cdot r^{-n}\cdot(\log r)^{-1} + \{-r^{1-n}\cdot -1\cdot(\log r)^{-2}\cdot \frac{1}{r}\} \\
&= r^{-n}\cdot(\log r)^{-2} + (n-1)\cdot r^{-n}\cdot(\log r)^{-1} \quad \blacksquare
\end{aligned}
$$

Lemma 4.

$1 \le i, j \le n$ に対し $\partial_{ij}f \notin L^1(B_\delta^*) \quad (B_\delta^* := B_\delta \backslash \{0\})$

Proof

$i \neq j$の時、$x \in B_\delta^*$ に対し、

$$
\begin{aligned}
\partial_{ij}f(x) &= \partial_{ij}F(|x|) \\
&= (F''(|x|) - F'(|x|)\,|x|^{-1})x_i x_j |x|^{-2}
\end{aligned}
$$

ここで、Lemma 3. を用いて

$$
\partial_{ij}f(x) = |x|^{-n}\cdot(\log|x|)^{-2}\cdot x_i x_j |x|^{-2} + n\cdot|x|^{-n}(\log|x|)^{-1}\cdot x_i x_j |x|^{-2}
$$

ここで、

$$f_1 := |x|^{-n} \cdot (\log|x|)^{-2} \cdot x_i x_j |x|^{-2}$$

$$f_2 := n \cdot |x|^{-n} \cdot (\log|x|)^{-1} \cdot x_i x_j |x|^{-2}$$

とすると、$f_1 \in L^1(B_\delta^*)$ であるが、

$$\int_{B_\delta^*} |f_2(x)| dx = n \cdot \int_0^\delta r^{-1} \cdot |\log r|^{-1} dr \cdot \int_{S^{n-1}} |\omega_i \cdot \omega_j| dS_\omega$$

ここで、

$$\int_0^\delta r^{-1} |\log r|^{-1} dr = +\infty$$

$$\int_{S^{n-1}} |\omega_i \cdot \omega_j| dS_\omega > 0$$

なので、$f_2 \notin L^1(B_\delta^*)$、よって、$\partial_{ij} f \notin L^1(B_\delta^*)$

$i = j$の時

$$\partial_{ii} f(x) = |x|^{-n} \cdot (\log|x|)^{-2} \cdot x_i^2 \cdot |x|^{-2} \ + \ |x|^{-n} (\log|x|)^{-1} \cdot (n \cdot x_i^2 \cdot |x|^{-2} - 1)$$

となり、$i = j$の時と同様に$\partial_{ii} \notin L^1(B_\delta^*)$ である。

よって、$1 \leq i, j \leq n$ に対して、$\partial_{ij} f \notin L^1(B_\delta^*)$ である。 ■

以上により、ここで構成した fは、$\partial_{ij} f \notin L^1_{loc}(R^n)$ $(1 \leq i, j \leq n)$ を満たすことが示された。

Reference

Wolfgang Rother, Arch.Math.,Vol.48,88-91(1987)

AN INTRODUCTION TO THE TREATISE "THE FUNCTIONAL CALCULUS" AND ITS SIMPLE APPLICATION.

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This is a paper which introduce to the treatise

E.BRIAN DAVIES.

"THE FUNCTIONAL CALCULUS"(1992)

Institut Mittag-Leffler.

and is added its simple application.

I added the simple application because I wanted to show an advantage that $f(H)$ (2) could be given by an integral formula and I thought it would be boring only an introduction of the treatise.

I omitted all the proofs in the treatise.

Finally, I am very glad that I have attended at this seminar and felt men's aurae of whom I had heard.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1.I Introduction to the treatise "THE FUNCTIONAL CALCULUS".

Notation (N).

X:Banach space

H:Densely defined closed operator on X with spectrum $spH \subset \mathbb{R}$ and

$\exists c, \alpha \geq 0$

$s.t$

$$\| (z-H)^{-1} \|_{\mathbb{B}(X)} \quad \leq \quad c \mid Imz \mid^{-1} \left(\frac{<z>}{\mid Imz \mid}\right)^{\alpha} \qquad (\quad \forall z \in \mathbb{C} \backslash \mathbb{R} \quad)$$

$$(\quad <z> \quad := (1+ \mid z \mid^{2})^{\frac{1}{2}} \quad)$$

$$\mathcal{A} : \{ \ f \in C^{\infty}(\mathbb{R}, \mathbb{C}) \quad | \quad \exists \beta < 0 \quad \text{s.t} \quad \forall r = 0,1,2,\cdots \quad \mid f^{(r)}(x) \mid \leq O(<x>^{\beta - r}) \ \}$$

Remark. The above resolvent norm inequality holds in the following cases.

(1) $X = L^{1}(\mathbb{R})$
$H = \dfrac{d^2}{dx^2}$.
In this case it holds with $c = 2, \alpha = 0$.

(2) $X = L^{p}(\mathbb{R}^{n}) \quad (1 \leq p < \infty)$

$H = \Delta$.

(3) X:a Hilbert space.

H:a self-adjoint operator.

Clearly with $c = 1, \alpha = 0$.

(4) $X = \mathbb{C}^{n}$

H:a $n \times n$ matrix with real eigenvalues.

(5) If H satisfies the above inequality then $a + bH \quad (a, b \in \mathbb{R}, \quad 0 < b \leq 1)$ satisfies it too with the same c and α.

Our purpose is to construct a homomorphism $\mathcal{H}$ from $\mathcal{A}$ to $\mathbb{B}(X)$.

Let τ be a following function.

$$\tau \in C_0^\infty(\mathbb{R},\mathbb{C})$$

$$\tau(x) = \begin{cases} 1 & |x|<1 \\ 0 & |x|>2 \end{cases}$$

With above τ,given $f \in \mathcal{A}$ and $k \geq 0$ we define an "*almost analytic extention*"of f to $\mathbb{C}$ by the formula

$$\tilde{f}(x,y) := \left(\sum_{r=0}^{k} f^{(r)}(x)(iy)^r/r! \right)\tau\left(\frac{y}{<x>}\right). \tag{1}$$

$$\left(\quad <x>:= \quad (1+|x|^2)^{\frac{1}{2}} \quad \right)$$

So we define a function $\mathcal{H}$ from $\mathcal{A}$ to $\mathbb{B}(X)$ as below.

For $f \in \mathcal{A}$,

$$\mathcal{H}(f) \equiv f(H) \equiv \quad -\frac{1}{\pi}\int_{\mathbb{C}\backslash\mathbb{R}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z-H)^{-1}dxdy \tag{2}$$

$$\left(\quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad \right).$$

This is uniquely defined independent of k and τ if $k > \alpha$, so that we proceed hereafter as $k > \alpha$.

The function $\mathcal{H}$ has following properties.

$$\alpha \in \mathbb{C}, \quad f,g \in \mathcal{A}, \quad suppf : \text{support of } f$$

$$\begin{aligned} (\alpha f)(H) &= \alpha f(H) \\ (f+g)(H) &= f(H)+g(H) \\ (fg)(H) &= f(H)\circ g(H) \\ suppf \cap spH = \emptyset \quad &\Rightarrow \quad f(H)=0 \\ (z-\cdot)^{-1}(H) &= (z-H)^{-1} \qquad (\forall z \in \mathbb{C}\backslash\mathbb{R}) \end{aligned} \tag{3}$$

2. A simple application

In this section,we keep in mind to prove the next theorem.The notation $\frac{d}{dt}$ means the strong derivative.This will be used without notice as the same hereafter.

Theorem 1.

Let $1 \leq p < \infty$ fix, Ω be an open set in $\mathbb{R}^n$ which satisfies $sp\Delta \subset (-\infty, 0]$ and the resolvent inequality in (N) *holds as $X = L^p(\Omega), H = \Delta$ (for example $\Omega = \mathbb{R}^n$,ball,cube) and $a, b \in \mathbb{R}$ $(0 < b \leq 1)$ fix.*

Under these assumptions,for arbitrary $u \in L^p(\Omega)$ there exists $\{v(t)\}_{t>0} \subset D(\Delta)$ uniquely which satisfies the following equations.

$$\begin{cases} \dfrac{dv(t)}{dt} = (a + b\Delta)v(t) \quad (t > 0) \\ \lim_{t\downarrow 0} v(t) = u \end{cases}$$

To deal with the problem generally,we start basing on the new assumptions (N') which is only added to (N) that spectrum $spH \subset (-\infty, a]$.

Step1. *(The construction of C^0- semigroup $\{e^{tH}\}_{t\geq 0}$)*

(construction). Let $\varepsilon > 0$ fix and χ_ε be a function which satisfies

$$\chi_\varepsilon \in C^\infty(\mathbb{R}, \mathbb{C})$$

$$\chi_\varepsilon(x) = \begin{cases} 1 & x < a + \varepsilon \\ 0 & a + 2\varepsilon < x \end{cases}$$

We denote f_ε for $f \in \mathcal{A}$

$$f_\varepsilon(x) := f(x)\chi_\varepsilon(x) \quad x \in \mathbb{R}.$$

At this time, $e_\varepsilon^{t\cdot}(t > 0)$ is in $\mathcal{A}$. So we could get a subset $\{\mathcal{H}e_\varepsilon^{t\cdot}\}_{t>0}$ in $\mathbb{B}(X)$.

If we denote

$e^{tH} \equiv \mathcal{H}e_{\varepsilon}^{t\cdot} \quad (t > 0)$

$e^{0H} \equiv I \quad (I : \text{identity mapping })$

then the $\{e^{tH}\}_{t\geq 0}$ becomes C^0–semigroup [See [1],p16∼17]

and it satisfies $e^{(t_1+t_2)H} = e^{t_1 H}e^{t_2 H}$ for $t_1, t_2 \geq 0$ [See (3)].

Step2. *(H is the generator of $\{e^{tH}\}_{t\geq 0}$.)*

Let A be the generator of the C^0- semigroup $\{e^{tH}\}_{t\geq 0}$. Then

$$A \quad = \quad H.$$

This will be proved by using the **Theorem** 10.14 which is described in [2] (p236) and the definition of e^{tH}.

Step3.

By using the Hille–Yosida thorem[See [2], p240],we can get the following result from **Step1** *and* **Step2**.

Thorem2

For arbitrary $u_0 \in D(H)$,

$\{e^{tH}u_0\}_{t\geq 0}$ is the solution of the following equations.

$$\begin{cases} \dfrac{dv(t)}{dt} = Hv(t) \quad t \geq 0 \\ \lim_{t\downarrow 0} v(t) = u_0 \end{cases}$$

Remark. In fact, $\{e^{tH}u_0\}_{t>0}$ is the unique solution of the above equations. But I omit the proof.

Step4. *(Thorem which involves* **Thoreml** *)*

Here we start adding more assumptions to (N')

i.e $\quad 1 \leq \forall p < \infty$:fix

Ω : open set in $\mathbb{R}^n$,

$X = L^p(\Omega)$,

$C_0^\infty(\Omega) \subset D(H)$

and

$u(\in X)$ is in $D(H)$

if and only if

$\exists v \in X$ s.t $\quad \int_\Omega uH\varphi \ = \ \int_\Omega v\varphi \quad \forall\varphi \in C_0^\infty(\Omega)$

and at this time v coincide with Hu.

These added assumptions come from the properties of Δ in **Theorem1** mainly.

Thoerem3

For arbitrary $u \in X$,

$$e^{tH}u \quad \in \quad D(H) \qquad (t > 0)$$

and $\{e^{tH}u\}_{t>0}$ is the unique solution of the following equations.

$$\begin{cases} \dfrac{dv(t)}{dt} = Hv(t) \\ \lim_{t\downarrow 0} v(t) = u \end{cases}$$

Except the uniqueness of the solution,this will be confirmed soon by using the result in **Step3** if the next lemma is proved.

Lemma

For arbitrary $u \in X$, $\quad e^{tH}u \in D(H) \quad (t > 0)$.

This will be proved by using the added assumptions.

On the uniqueness of the solution,I omit.

Remark. If we put $H = a + b\Delta$,then **Theorem1** is proved by **Theorem3**.

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THE STEFAN PROBLEM WITH A KINETIC CONDITION AT THE FREE BOUNDARY （ XIE WEIQING の論文紹介 ）

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1.Introduction

以下において、$u = u(x,t)$ は温度を表し、$\Gamma_t : x = s(t)$ は自由境界を表すものとする。次の system を考える：

(1.1) $$u_t = k_L u_{xx} \quad \text{in} \quad (0, s(t)) \times (0, T]$$

(1.2) $$u_t = k_S u_{xx} \quad \text{in} \quad (s(t), 1) \times (0, T]$$

(1.3) $$u^- = u^+ = u^1 \quad \text{on} \quad \Gamma_t$$

(1.4) $$k_L u_x^- - k_S u_x^+ = -L\dot{s}(t) \quad \text{on} \quad \Gamma_t$$

(1.5) $$u^1 = \varepsilon \dot{s}(t) \quad \text{on} \quad \Gamma_t$$

(1.6) $$s(0) = b \quad , 0 < b < 1$$

(1.7) $$u(x,0) = \varphi_1(x) \quad : \text{given} \quad , x \in [0, b]$$

(1.8) $$u(x,0) = \varphi_2(x) \quad : \text{given} \quad , x \in [b, 1]$$

(1.9) $$u(i-1, t) = f_i(t) \quad : \text{given} \quad , t \in [0, T] \quad (i = 1, 2)$$

ここで、k_L, k_Sは、各々液体部分、固体部分での熱伝導係数で、L は潜熱、$\varepsilon > 0$である。また、(1.7) ∼ (1.9) において、$\varphi_1 \in C^1[0,b]$ ，$\varphi_2 \in C^1[b,1]$ ，$f_i \in C^1(R) \cap L^\infty(R)$ である。

以下、問題 (1.1) ∼ (1.9) を、（Ｐ）で表す。

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2. Existence and Uniqueness

Theorem. 問題（Ｐ）の解は一意に存在する。

（証明の概略）

《存在性》Schauder の不動点定理を用いる。

$K(T_0, M) := \{s(t) \in C^1[0, T_0]; s(0) = b, 0 < s(t) < 1, |\dot{s}(t)| \leq M\}$ とする。ここで、Mは fixed、T_0は十分小で、$MT_0 \leq min\{b, 1-b\}$ とする。写像 $Fs = h$ を $h(t) := b + \frac{1}{\epsilon}\int_0^t u(s(\tau), \tau)dt$ で定義し、この F が不動点 s を持つことを示せば良い。以下、F が、compact 写像となることを示せば、存在性についての証明が完成する。

《一意性》2つの解の組 $(u(x,t), s(t))$、$(\tilde{u}(x,t), \tilde{s}(t))$ があったとして、それらの差が、0 になることを証明すれば十分。

まず、自由境界を直線化させるための変数変換をおこなう。新しい空間の変数 ξ を、

$$\xi = \alpha(x, s(t)) \quad (x, s(t)) \in [0,1] \times [\delta, 1-\delta]$$

で、定義する。この変数変換は、以下の性質をもつ：

$$(2.1) \qquad \alpha(i, s) = i \quad (i = 0, 1) \quad , \alpha(s, s) = \frac{1}{2} \quad , \alpha_x(s, s) = 1 \quad (\delta \leq s \leq 1-\delta)$$

$$(2.2) \qquad \alpha_x(x, s) \geq \alpha_0 > 0 \quad , |D^\beta \alpha| \leq c \quad (|\beta| \leq 3) \quad \text{in} \quad [0,1] \times [\delta, 1-\delta]$$

（ただし、$\alpha_0, c > 0 : constant$ ）この変換によって、x, t の関数は、ξ, t の関数へと書き換えられる。そこで、変換後の関数を、$v(\xi, t) := u(x, t) \quad , w(\xi, t) := \tilde{u}(x, t)$ で、表すことにする。いままで、$u, \tilde{u}$で、記述された system は、v, wで記述されることになる。そこで、$z(\xi, t) := v(\xi, t) - w(\xi, t)$ とおけば、$z \equiv 0$ を示すことが、最終目標になる。以下、v, wで記述された system を、zの system に書き換え、

$$\int_0^1 z^2(\xi, T)d\xi = 0$$

を示せば、一意性の証明が完成する。 (Q.E.D.)

3. 自由境界の Regularity について

ここでは、自由境界 $s(t)$ が C^∞ となっていることを示す。

まず、変数変換

$$\xi = x - s(t) \tag{3.1}$$

を行ない、自由境界を直線化する。また、$v(\xi,t) = u(\xi + s(t), t)$ for N_1 とおく。

ここで、$(s(t), u(x,t))$ は（P）の一意解であり、

$N_1 := \{(\xi,t) | -\delta < \xi < \delta, 0 < t < T\}$ である。

このとき、$v = v(\xi,t)$ は、以下の system を満たす：

$$v_t - k_L v_{\xi\xi} = \dot{s}(t) v_\xi \quad \text{in} \quad (-\delta, 0) \times (0,T) \tag{3.2}$$

$$v_t - k_S v_{\xi\xi} = \dot{s}(t) v_\xi \quad \text{in} \quad (0, \delta) \times (0,T) \tag{3.3}$$

$$v(\xi, 0) = \tilde{\varphi}(\xi) \quad , \xi \in [-\delta, \delta] \tag{3.4}$$

ここで、$\tilde{\varphi} := \begin{cases} \hat{\varphi}_1(\xi) = \varphi_1(\xi + b) & , \quad \xi \in [-\delta, 0] \\ \hat{\varphi}_2(\xi) = \varphi_2(\xi + b) & , \quad \xi \in [0, \delta] \end{cases}$

$$v^-(0,t) = v^+(0,t) = \varepsilon \dot{s}(t) \quad , 0 < t < T \tag{3.5}$$

$$k_L v_\xi^-(0,t) - k_S v_\xi^+(0,t) = -L\dot{s}(t) \quad , 0 < t < T \tag{3.6}$$

<u>仮定：</u> $k_L \geq k_S$

<u>Def.</u>

$$w(\xi,t) := v(\xi,t) - v(-\tilde{k}\xi, t) \tag{3.7}$$

$$\tilde{w}(\xi,t) := v(\xi,t) + \tilde{k} v(-\tilde{k}\xi, t) \tag{3.8}$$

ここで、

$$\tilde{k} := \left(\frac{k_S}{k_L}\right)^{\frac{1}{2}} \leq 1 \tag{3.9}$$

このとき、$w = w(\xi,t)$ は以下の system を満たす：

$$w_t - k_L w_{\xi\xi} = \dot{s}(t)\{v_\xi(\xi,t) - v_\xi(-\tilde{k}\xi, t)\} \quad \text{in} \quad N_1^- \tag{3.10}$$

$$w(0,t) = 0 \quad , 0 \leq t \leq T \tag{3.11}$$

$$w(\xi, 0) = \hat{\varphi}_1(\xi) = \hat{\varphi}_2(-\tilde{k}\xi) \quad , -\delta \leq \xi \leq 0 \tag{3.12}$$

また、$\tilde{w}=\tilde{w}(\xi,t)$ は以下の system を満たす：

(3.13)
$$\tilde{w}_t - k_S\tilde{w}_{\xi\xi} = \dot{s}(t)\{v_\xi(\xi,t)+\tilde{k}v_\xi(-\tilde{k}\xi,t)\} \quad \text{in} \quad N_1^-$$

(3.14)
$$k_L\tilde{w}_\xi(0,t)+\frac{1}{(\tilde{k}+1)\varepsilon}\tilde{w}(0,t)=0 \quad ,0\le t\le T$$

(3.15)
$$\tilde{w}(\xi,0)=\hat{\varphi}_1(\xi)+\tilde{k}\hat{\varphi}_2(-\tilde{k}\xi) \quad ,-\delta\le\xi\le 0$$

ただし、$N_1^- := \{(\xi,t); -\delta<\xi<0 \quad ,0<t<T\}$ である。

ここで、[1] の埋め込み定理により、

(3.16)
$$w_\xi,\tilde{w}_\xi \in C^{\alpha,\frac{\alpha}{2}}(N_2) \qquad (N_2\subset N_1, 0<\alpha<1)$$

がわかる。(3.7),(3.8) より、

(3.17)
$$v(\xi,t)=\frac{1}{\tilde{k}+1}\{w(\xi,t)+\tilde{w}(\xi,t)\}$$

(3.18)
$$v(-\tilde{k}\xi,t)=\frac{1}{\tilde{k}+1}\{\tilde{w}(\xi,t)-w(\xi,t)\}$$

以上で準備ができた。あとは、bootstrap と呼ばれる方法を用いる。

まず、(3.16) より $w_\xi,\tilde{w}_\xi\in C^{\alpha,\frac{\alpha}{2}}$ である。

これと、(3.17),(3.18) より、$v_\xi\in C^{\alpha,\frac{\alpha}{2}}$ がわかる。

よって、(3.5) により、$\dot{s}(t)\in C^{0,\frac{\alpha}{2}}$ がわかる。

これらを (3.10),(3.13) に代入すると、各々の右辺は $C^{\alpha,\frac{\alpha}{2}}$。ここで、[1] の Regularity theorem により、$w,\tilde{w}\in C^{2+\alpha,\frac{2+\alpha}{2}}$ がわかる。

このことと、(3.16) により、$w_\xi,\tilde{w}_\xi\in C^{1+\alpha,\frac{1+\alpha}{2}}$ がわかる。これと、(3.17),(3.18) によって、$v_\xi\in C^{1+\alpha,\frac{1+\alpha}{2}}$ がわかる。

以下、この操作を任意の有限回続けると、$w,\tilde{w}\in C^\infty, v\in C^\infty, s\in C^\infty$ がわかる。

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Asymptotic Stability for a Modified Penrose-Fife Model of Phase Transitions

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1. Introduction and assumptions

We consider a one-dimensional solid-liquid phase transition problem. This model is described by two parameters θ and w; θ is the absolute temperature and w is a non-conserved order parameter which indicates the physical situation and is constrained by

$$-1 \le w \le 1 \quad \text{on } Q := (0, +\infty) \times (-L, L).$$

Now, we introduce a new parameter u which is given by the relation $\theta := -\dfrac{1}{u}$ $(-\infty < u < 0)$. Following Penrose-Fife approach [5], we obtain the coupled system P:=$\{$(1.1)-(1.6)$\}$:

$$\left[-\frac{1}{u} + \lambda(w)\right]_t - u_{xx} = f \quad \text{in } Q, \tag{1.1}$$

$$w_t - \kappa w_{xx} + g(w) + \xi - \lambda'(w)u = 0 \quad \text{in } Q, \tag{1.2}$$

$$\xi \in \partial I_{[-1,1]}(w) \quad \text{in } Q, \tag{1.3}$$

$$\pm u_x(t, \pm L) + u(t, \pm L) = h_\pm(t) \quad \text{for } t \ge 0, \tag{1.4}$$

$$w_x(t, \pm L) = 0 \quad \text{for } t \ge 0, \tag{1.5}$$

$$u(0,x) = u_0(x), \quad w(0,x) = w_0(x) \quad \text{for } x \in [-L, L]. \tag{1.6}$$

Here, L is a positive number; κ is a sufficiently small positive constant; $\partial I_{[-1,1]}$ is the subdifferential of the indicator function of the compact interval $[-1,1]$.

We are interested in the large-time behaviour of the three regions $\Omega_-(t)$, $\Omega_+(t)$ and $\Omega_m(t)$, where $\Omega_-(t) := \{x \in [-L, L]; w(t,x) = -1\}$, $\Omega_+(t) := \{x \in [-L, L]; w(t,x) = 1\}$ and $\Omega_m(t) := \{x \in [-L, L]; -1 < w(t,x) < 1\}$ are respectively the pure solid, pure liquid and mushy regions.

Next, we consider the following system P$^\infty$:=$\{$(1.7)-(1.11)$\}$ which describes the steady-state fór P when f converges to 0 and $h_\pm$ converges to h^∞ as $t \to +\infty$ in some sense:

$$-u^\infty_{xx} = 0 \quad \text{in } (-L, L), \tag{1.7}$$

$$-\kappa w^\infty_{xx} + g(w^\infty) + \xi - \lambda'(w^\infty)u^\infty = 0 \quad \text{in } (-L, L), \tag{1.8}$$

$$\xi \in \partial I_{[-1,1]}(w^\infty) \quad \text{in } (-L, L), \tag{1.9}$$

$$\pm u^\infty_x(\pm L) + u^\infty(\pm L) = h^\infty, \tag{1.10}$$

$$w_x^\infty(\pm L) = 0, \tag{1.11}$$

where h^∞ is a negative constant.
Then, we note that from (1.7) and (1.10) u^∞ is a constant h^∞.

In this paper, we discuss P and P^∞ under the following assumptions:

(A1) $g(w) = w^3 + w$;

(A2) $\lambda(w) = -\frac{1}{2}w^2$;

(A3) $f \in W^{1,2}_{loc}(R_+; L^2(-L,L)) \cap L^2(R_+; L^2(-L,L))$ such that

$$\sup_{t\geq 0}\{|f|_{W^{1,2}(t,t+1;L^2(-L,L))}\} < +\infty;$$

(A4) $h_\pm \in W^{1,2}_{loc}(R_+)$ such that

$$\sup_{t\geq 0}\{|h_+|_{W^{1,2}(t,t+1)} + |h_-|_{W^{1,2}(t,t+1)}\} < +\infty$$

and

$$h_\pm - h^\infty \in L^2(R_+);$$

(A5) $u_0 \in H^1(-L,L)$ with $-\frac{1}{u_0} \in L^2(-L,L)$, and $w_0 \in H^2(-L,L)$ such that

$$w_{0x}(\pm L) = 0, \quad -1 \leq w_0 \leq 1 \quad \text{on } [-L,L].$$

Notations: For simplicity, we use the following notations:

$$H^1(-L,L)^* : \text{ the dual space of } H^1(-L,L),$$

$$\langle\cdot,\cdot\rangle : \text{ the duality pairing between } H^1(-L,L)^* \text{ and } H^1(-L,L),$$

$$(v,z) := \int_{-L}^{L} v(x)z(x)dx \quad \text{for all } v,z \in L^2(-L,L),$$

$$a(v,z) := \int_{-L}^{L} \nabla v(x)\cdot\nabla z(x)dx \quad \text{for all } v,z \in H^1(-L,L).$$

2.Results

At first, we give the variational formulation for P.

Definition 2.1. A couple of functions u and w is called a solution of P, if the following conditions (w1)-(w3) are fulfilled:

(w1) For every finite $T > 0$, $u \in L^\infty(0,T;H^1(-L,L))$, $-\frac{1}{u} \in L^\infty(0,T;L^2(-L,L)) \cap W^{1,2}(0,T;H^1(-L,L)^*)$ and $w \in L^\infty(0,T;H^1(-L,L)) \cap W^{1,2}(0,T;L^2(-L,L))$.

(w2) For all $z \in H^1(-L,L)$ and for a.e. $t \geq 0$,

$$\left\langle \left(-\frac{1}{u}\right)'(t), z \right\rangle + ((\lambda(w))'(t), z) + a(u(t), z)$$

$$+(u(t,-L) - h_-(t))z(-L) + (u(t,L) - h_+(t))z(L) = (f(t), z)$$

and $u(0) = u_0$, where the prime " $'$ " denotes the derivative $\frac{d}{dt}$.

(w3) For all $z \in H^1(-L,L)$ and a.e. $t \geq 0$,

$$(w'(t), z) + \kappa a(w(t), z) + (g(w(t)) + \xi(t), z) = (\lambda'(w(t))u(t), z)$$

and $w(0) = w_0$, where $\xi \in L^2_{loc}(R_+; L^2(-L,L))$ is such that $\xi \in \partial I_{[-1,1]}(w)$ a.e. on Q.

Next, we give the variational formulation for {(1.8),(1.9),(1.11)}.

Definition 2.2. For a given constant h^∞, a function v is called a solution of {(1.8),(1.9),(1.11)}, if $v \in H^2(-L,L)$ and

$$-\kappa v_{xx} + \gamma + g(v) - \lambda'(v)h^\infty = 0 \quad \text{a.e.in } (-L,L), \tag{2.1}$$

$$\gamma \in L^2(-L,L), \quad \gamma \in \partial I_{[-1,1]}(v) \quad \text{a.e.in } (-L,L), \tag{2.2}$$

$$v_x(\pm L) = 0. \tag{2.3}$$

For simplicity, we denote by P^∞_v the problem {(2.1)-(2.3)} and put $q(h^\infty; v) := g(v) - \lambda'(v)h^\infty = v^3 + (1 + h^\infty)v$.

From the results of [4], we obtain the following theorem.

Theorem 2.1. *Under the conditions (A1)-(A6) and $h^\infty \in (-\infty, 0)$, problem P admits one and only one solution $\{u, w\}$ which satisfies the following (a)-(d):*

(a) $u \in L^\infty(R_+; H^1(-L,L))$, $w \in L^\infty(R_+; H^2(-L,L))$ *and* $w' \in L^\infty(R_+; L^2(-L,L))$.

(b) $u - u^\infty \in L^2(R_+; H^1(-L,L))$ *with* $u^\infty = h^\infty$ *and* $w' \in L^2(R_+; L^2(-L,L))$.

(c) $u(t) \longrightarrow h^\infty$ *weakly in* $H^1(-L,L)$ *as* $t \to +\infty$.

(d) *The omega limit set* $\omega(u_0, w_0)$ *of the order parameter* w, *defined by*

$$\omega(u_0, w_0) := \{v \in H^1(-L,L); w(t_n) \longrightarrow v \text{ in } H^1(-L,L)$$
$$\text{for some } t_n \text{ with } t_n \uparrow +\infty\},$$

is non-empty, bounded and closed in $H^2(-L,L)$ *as well as connected in* $H^1(-L,L)$. *Moreover, any function* $w^\infty \in \omega(u_0, w_0)$ *is a solution of* P^∞_v.

From Theorem 2.1 (c), it is enough to investigate the large-time behaviour of the order parameter w only.

Our main results are the following theorems.

Theorem 2.2. *If $-1 \le h^\infty < 0$, then P_v^∞ admits one and only one solution 0, that is, the omega limit set $\omega(u_0, w_0)$ is $\{0\}$; hence the order parameter w converges to 0 as $t \to +\infty$. (See Figure 1)*

Theorem 2.3. *If $-2 \le h^\infty < -1$, then the omega limit set $\omega(u_0, w_0)$ is a singleton $\{w^\infty\}$, that is, the order parameter w converges to w^∞ as $t \to +\infty$. (See Figure 2) Moreover, w^∞ is a solution for the following ordinary differential equation:*

$$-\kappa v_{xx} + q(h^\infty; v) = 0 \quad in\ (-L, L), \tag{2.4}$$

$$v_x(\pm L) = 0. \tag{2.5}$$

These theorems say that that if $-2 \le h^\infty < 0$, the constraints give no influence upon the order parameter w after a large time. And this case is essentially same as in [2].

The most interesting result is stated as follows.

Theorem 2.4. *[cf. 3] If $h^\infty < -2$, there are the following two possibilities (1) and (2) where $G(h^\infty; v) := \int_0^v q(h^\infty; s)ds$ (See Figure 3 and 4):*

(1) *$\omega(u_0, w_0)$ is a singleton $\{w^\infty\}$. In this case, the order parameter w converges to w^∞ in $H^1(-L, L)$ as $t \to +\infty$.*
Moreover, if $G(h^\infty; \pm 1) < G(h^\infty; w^\infty(-L)) < 0$, then w^∞ is a non-constant solution of $\{(2.4), (2.5)\}$.

(2) *$\omega(u_0, w_0)$ contains a continuum of solutions of P_v^∞. In this case, the following statements (e1)-(e3) hold:*

(e1) *$G(h^\infty; v(\pm L)) = G(h^\infty; \pm 1)$, that is, $v(-L), v(L) \in \{-1, 1\}$, for all $v \in \omega(u_0, w_0)$.*

(e2) *For the order parameter w it holds that*

$$\lim_{t \to +\infty} |w_x(t)|_{L^2(-L,L)} = |v_x|_{L^2(-L,L)}$$

and

$$\lim_{t \to +\infty} \int_{-L}^{L} G(h^\infty; w(t,x))dx = \int_{-L}^{L} G(h^\infty; v(x))dx$$

for all $v \in \omega(u_0, w_0)$.

(e3) *The number of all points $x \in [-L, L]$ with $v(x) = 0$ is finite and independent of the choice of $v \in \omega(u_0, w_0)$.*

3.Construction of solutions ($h^\infty < -2$)

We note that the set of constant solutions is $\{-1, 0, 1\}$.

In this section, we consider non-constant solutions of P_v^∞. We observe that $q(h^\infty; v)$ satisfies (q1) and (q2) below:

(q1) (oddness) $q(h^\infty; -v) = -q(h^\infty; v)$ for all $v \in R$.

(q2) (convexity-concavity)

$$q(h^\infty; 0) = q\left(h^\infty; \pm\sqrt{-(1+h^\infty)}\right) = 0,$$

$$q'(h^\infty; \cdot)\left(= \frac{d}{dv}q(h^\infty; \cdot)\right) > 0 \quad \text{on } (-\infty, -\sqrt{\tfrac{-(1+h^\infty)}{3}}) \cup (\sqrt{\tfrac{-(1+h^\infty)}{3}}, +\infty),$$

$$q'\left(h^\infty; \pm\sqrt{\frac{-(1+h^\infty)}{3}}\right) = 0,$$

$$q'(h^\infty; \cdot) < 0 \quad \text{on } (-\sqrt{\tfrac{-(1+h^\infty)}{3}}, \sqrt{\tfrac{-(1+h^\infty)}{3}}),$$

$$q''(h^\infty; \cdot)\left(= \frac{d^2}{dv^2}q(h^\infty; \cdot)\right) \leq 0 \quad \text{on } (-\infty, 0),$$

$$q''(h^\infty; 0) = 0,$$

$$q''(h^\infty; \cdot) \geq 0 \quad \text{on } (0, +\infty).$$

To construct a non-constant solution, we use the solution v^* of following ordinary differential equation denoted by O_v^∞:

$$-\kappa v^*_{xx} + q(h^\infty; v^*) = 0 \quad \text{in } R,$$

$$v^*(0) = 0, \quad v^*_x(0) = \left(\frac{-2b}{\kappa}\right)^{\frac{1}{2}}.$$

where $G(h^\infty; \pm 1) \leq b < 0$. From the general theory of ordinary differential equations, O_v^∞ has a unique solution v^* and there exist two constants x_+ and x_- such that

$$x_- = -x_+, \tag{3.1}$$

$$x_- < 0 < x_+,$$

$$v_x(x) > 0 \quad \text{on } (x_-, x_+),$$

$$v_x(x_\pm) = 0, \quad v(x_\pm) = \eta_\pm(b)$$

and

$$x_+ - x_- = \left(\frac{\kappa}{2}\right)^{\frac{1}{2}} \int_{\eta_-(b)}^{\eta_+(b)} \frac{dv}{\{G(h^\infty; v) - b\}^{\frac{1}{2}}} =: I(b),$$

where $v = \eta_\pm(b)$ are the roots of the algebraic equation $G(h^\infty; v) = b$ with $-1 \le \eta_-(b) < 0 < \eta_+(b) \le 1$ and $\eta_-(b) = -\eta_+(b)$.
Also, let $\bar{v}^*(x) := v^*(-x)$.

Finally, we construct non-constant solutions of P_v^∞ using v^* and $\bar{v}^*$.

Theorem 3.1. *[cf. 3] Assume $G(h^\infty; \pm 1) < b < 0$, and ($\star$) there exists a natural number N such that $2L = NI(b)$. Then P_v^∞ has a non-constant solution w^∞ such that $b = G(h^\infty; w^\infty(-L)) = G(h^\infty; w^\infty(L))$, i.e. $w^\infty(-L), w^\infty(L) \in \{\eta_-(b), \eta_+(b)\}$.*
Moreover, for example, when $N = 3$, the non-constant solution w^∞ of P_v^∞ is expressed by v or $-v$, where

$$v(x) = \begin{cases} v^*(x_- + L + x) & \text{for } x \in [-L, -L + I(b)], \\ \bar{v}^*(-x_+ + L - I(b) + x) & \text{for } x \in (-L + I(b), -L + 2I(b)], \\ v^*(x_- + L - 2I(b) + x) & \text{for } x \in (-L + 2I(b), L] \end{cases} \qquad (\textit{See Figure } 5)$$

Remark 3.1. (1) From Theorems 2.1 (d) and 3.1, we see that $\omega(u_0, w_0)$ is a singleton $\{w^\infty\}$, that is, the order parameter w converges to w^∞ as $t \to +\infty$.
(2) In Theorem 3.1 (2), any non-constant solution associated to N satisfying ($\star$) is similarly expressed, too.
(3) The properties of function $I(b)$ is referred to [1].

Theorem 3.2 *[cf. 3] Assume $G(h^\infty; \pm 1) = b$ and let $N^* := \max\{N \in N; 2L \ge NI(b)\}$. Then, for any natural number $N \le N^*$ P_v^∞ has a non-constant solution w^∞ such that*

$$(\star\star) \quad 2L = NI(b) + |\Omega_-| + |\Omega_+|,$$

where $\Omega_- := \{x \in [-L, L]; w^\infty(x) = -1\}$ and $\Omega_+ := \{x \in [-L, L]; w^\infty(x) = 1\}$.
Moreover, the following (1) and (2) hold:

(1) $v(-L), v(L) \in \{-1, 1\}$.

(2) *For example, when $N = 3$, any non-constant solution w^∞ is expressed by v or $-v$, where*

$$v(x) = \begin{cases} -1 & \text{for } x \in [-L, -L + \delta_1], \\ v^*(x_- + L - \delta_1 + x) & \text{for } x \in (-L + \delta_1, -L + \delta_1 + I(b)), \\ 1 & \text{for } x \in [-L + \delta_1 + I(b), -L + \delta_1 + \delta_2 + I(b)], \\ \bar{v}^*(-x_+ + L - \delta_1 - \delta_2 - I(b) + x) & \\ \qquad \text{for } x \in (-L + \delta_1 + \delta_2 + I(b), -L + \delta_1 + \delta_2 + 2I(b)), & \\ -1 & \text{for } x \in [-L + \delta_1 + \delta_2 + 2I(b), -L + \delta_1 + \delta_2 + \delta_3 + 2I(b)], \\ v^*(x_- + L - \delta_1 - \delta_2 - \delta_3 - 2I(b) + x) & \\ \qquad \text{for } x \in (-L + \delta_1 + \delta_2 + \delta_3 + 2I(b), -L + \delta_1 + \delta_2 + \delta_3 + 3I(b)), & \\ 1 & \text{for } x \in [-L + \delta_1 + \delta_2 + \delta_3 + 3I(b), -L + \delta_1 + \delta_2 + \delta_3 + \delta_4 + 3I(b)] \end{cases}$$

and δ_i $(i = 1,2,3,4)$ *are any non-negative constants which satisfy* $2L = 3I(b) + \delta_1 + \delta_2 + \delta_3 + \delta_4$.
Furthermore, once a non-constant solution is given, any other solution is obtained by translating the non-constant parts without changing the order of the non-constant parts, when $2L > 3I(b)$. *(See figure 6)*

Remark 3.2. From Theorem 3.3, we see Theorem 2.4 (2). This means that the pure regions may drift very slowly.

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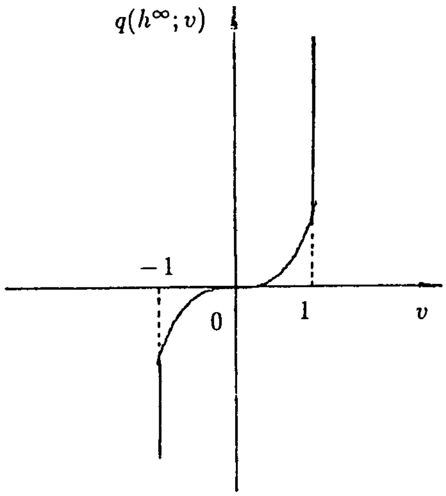

Figure 1

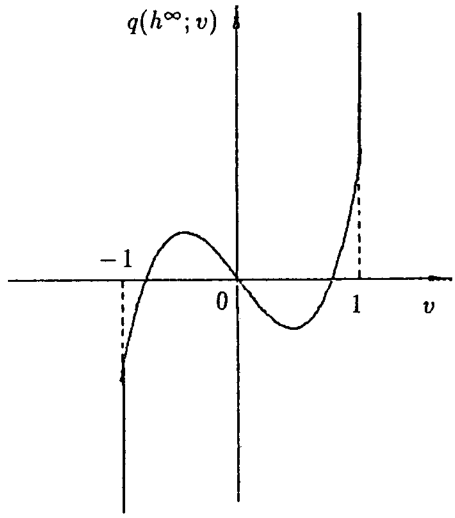

Figure 2

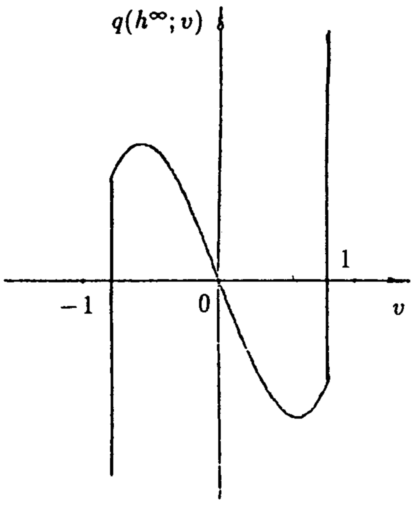

Figure 3

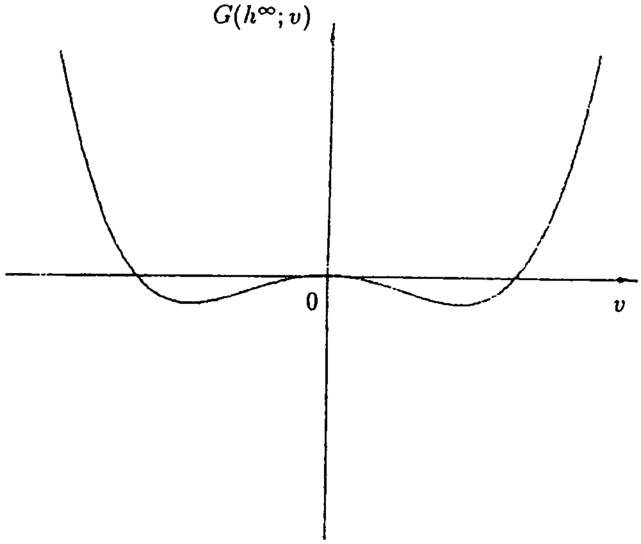

Figure 4

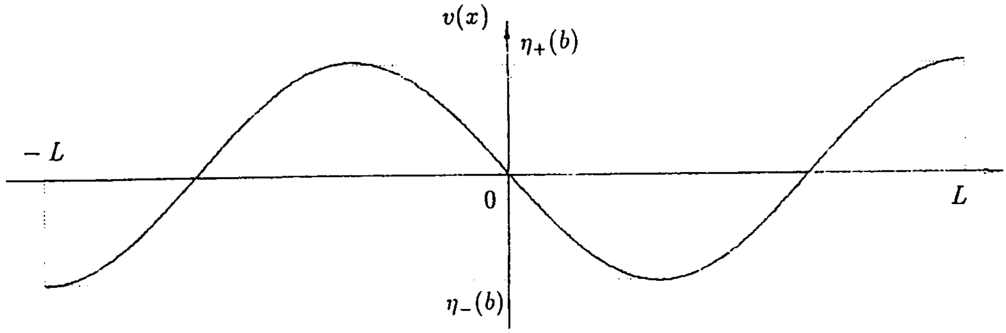

Figure 5

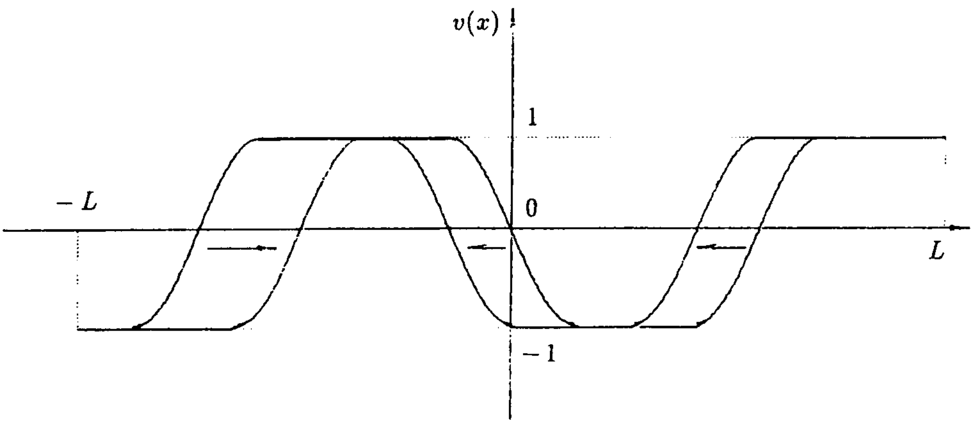

Figure 6

相転移問題に対する数値シュミレーション

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1. Asymptotic behavior as $\kappa \to 0$.

相転移問題を記述する, 次の非線形偏微分方程式の system を考える.

$$(P_\kappa)\begin{cases} (u+w)_t - \Delta u = f(t,x) & \text{in } Q := (0,+\infty)\times\Omega, \\ \nu w_t - \kappa\Delta w + \beta(w) + g(w) \ni u & \text{in } Q, \\ \dfrac{\partial u}{\partial n} + n_0 u = h(t,x) & \text{on } \Sigma := (0,+\infty)\times\Gamma, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \Sigma, \\ u(0,\cdot) = u_0, \quad w(0,\cdot) = w_0 & \text{in } \Omega. \end{cases}$$

$$(P_0)\begin{cases} (u+w)_t - \Delta u = f(t,x) & \text{in } Q, \\ \nu w_t + \beta(w) + g(w) \ni u & \text{in } Q, \\ \dfrac{\partial u}{\partial n} + n_0 u = h(t,x) & \text{on } \Sigma, \\ u(0,\cdot) = u_0, \quad w(0,\cdot) = w_0 & \text{in } \Omega. \end{cases}$$

ここで $\Omega \subset \mathbf{R}^N$ $(1 \le N \le 3)$: 有界, $\Gamma = \partial\Omega$: smooth とし, 次を仮定する.

(A1) β : maximal monotone graph in $\mathbf{R}\times\mathbf{R}$ s.t. $\overline{D(\beta)} = [\sigma_*, \sigma^*]$ for some constants σ_*, σ^* with $-\infty < \sigma_* < \sigma^* < +\infty$.

(A2) g : Lipschits cont. function.

(A3) $f \in L^2_{loc}(\mathbf{R}_+; L^2(\Omega))$.

(A4) $h \in W^{1,2}_{loc}(\mathbf{R}_+; L^2(\Gamma)) \cap L^\infty(\mathbf{R}_+; L^\infty(\Gamma))$.

(A5) ν, n_0 : positive constants.

(A6) $u_0, w_0 \in L^2(\Omega)$, $\sigma_* \le w_0 \le \sigma^*$.

ここで問題 (P_κ) と (P_0) の解の定義を与える.

Definition 1.1 $0 < T < +\infty$, $\kappa > 0$ とする. $u_\kappa : [0,T] \to H^1(\Omega)^*$, $w_\kappa : [0,T] \to L^2(\Omega)$ の組 $\{u_\kappa, w_\kappa\}$ が次の条件 (N1-1) から (N1-3) を満たすとき, $\{u_\kappa, w_\kappa\}$ を問題 (P_κ) の解と呼ぶ.

(N1-1) $u_\kappa \in C([0,T]; H^1(\Omega)^*) \cap W^{1,2}_{loc}((0,T]; H^1(\Omega)^*) \cap L^2(0,T; L^2(\Omega)) \cap L^2_{loc}((0,T]; H^1(\Omega))$,
$w_\kappa \in C([0,T]; L^2(\Omega)) \cap W^{1,2}_{loc}((0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$,
and $\hat{\beta}(w_\kappa) \in L^1(0,T; L^1(\Omega))$;

(N1-2) $\langle u'_\kappa(t) + w'_\kappa(t), z\rangle + \int_\Omega \nabla u_\kappa(t) \cdot \nabla z dx + \int_\Gamma (n_0 u_\kappa(t) - h(t)) z d\Gamma = (f(t), z)$
for all $z \in H^1(\Omega)$ and a.e. $t \in [0,T]$, where $(\cdot,\cdot)$: the inner product in $L^2(\Omega)$ and $\langle \cdot,\cdot \rangle$: the duality pairing between $H^1(\Omega)^*$ and $H^1(\Omega)$.

(N1-3) there exists $\xi \in L^2_{loc}((0,T]; L^2(\Omega))$ s.t. $\xi \in \beta(w_\kappa)$ a.e. in $Q_T := (0,T) \times \Omega$ and

$$\nu(w'_\kappa(t), z) + \kappa \int_\Omega \nabla w_\kappa(t) \cdot \nabla z dx + (\xi(t), z) + (g(w_\kappa(t)), z) = (u_\kappa(t), z)$$

for all $z \in H^1(\Omega)$ and a.e. $t \in [0,T]$.

Definition 1.2 $0 < T < +\infty$, $\kappa = 0$ とする. $\tilde{u} : [0,T] \to H^1(\Omega)^*$, $\tilde{w} : [0,T] \to L^2(\Omega)$ の組 $\{\tilde{u}, \tilde{w}\}$ が次の条件 (N2-1) から (N2-3) を満たすとき, $\{\tilde{u}, \tilde{w}\}$ を問題 (P_0) の解と呼ぶ.

(N2-1) $\tilde{u} \in C([0,T]; H^1(\Omega)^*) \cap W^{1,2}_{loc}((0,T]; H^1(\Omega)^*) \cap L^2(0,T; L^2(\Omega)) \cap L^2_{loc}((0,T]; H^1(\Omega))$,
$\tilde{w} \in C([0,T]; L^2(\Omega)) \cap W^{1,2}_{loc}((0,T]; L^2(\Omega))$, and $\hat{\beta}(\tilde{w}) \in L^1(0,T; L^1(\Omega))$;

(N2-2) = (N1-2)

(N2-3) there exists $\xi \in L^2_{loc}((0,T]; L^2(\Omega))$ s.t. $\xi \in \beta(\tilde{w})$ a.e. in Q_T and

$$\nu(\tilde{w}'(t), z) + (\xi(t), z) + (g(\tilde{w}(t)), z) = (\tilde{u}(t), z)$$

for all $z \in H^1(\Omega)$ and a.e. $t \in [0,T]$.

解の存在と一意性について, 次の定理が成り立つ.

Theorem 1.1 (Damlamian-Kenmochi-Sato[4])
(A1) から (A6) を仮定する. $0 < T < +\infty$, $\kappa > 0$ とすると, 問題 (P_κ) は一意解 $\{u_\kappa, w_\kappa\}$ をもつ.

Theorem 1.2 (Damlamian-Kenmochi-Sato[4])
(A1) から (A6) を仮定する．$0 < T < +\infty$, $\kappa = 0$ とすると，問題 (P_0) は一意解 $\{\tilde{u}, \tilde{w}\}$ をもつ．

今回は，以下の定理に対応する数値シュミレーションを行った．

Theorem 1.3 (Colli-Sprekels[2])
(P_κ), (P_0) の解 $\{u_\kappa, w_\kappa\}, \{\tilde{u}, \tilde{w}\}$ に対して

$$u_\kappa \to \tilde{u},\ w_\kappa \to \tilde{w} \text{ in } C([0,T]; L^2(\Omega)) \text{ as } \kappa \to 0 \text{ for any } T < +\infty. \quad \text{(Fig.1)}$$

2. Asymptotic behavior as $t \to +\infty$.

ここでは，問題 (P_κ), (P_0) の解の漸近挙動について述べるため，次の steady-state problems を考える．

$$(P_\kappa^\infty)\begin{cases} -\Delta u^\infty = f^\infty & \text{in } \Omega, \\ -\kappa\Delta w^\infty + \beta(w^\infty) + g(w^\infty) \ni u^\infty & \text{in } \Omega, \\ \dfrac{\partial u^\infty}{\partial n} + n_0 u^\infty = h^\infty & \text{on } \Gamma, \\ \dfrac{\partial w^\infty}{\partial n} = 0 & \text{on } \Gamma. \end{cases}$$

$$(P_0^\infty)\begin{cases} -\Delta u^\infty = f^\infty & \text{in } \Omega, \\ \beta(w^\infty) + g(w^\infty) \ni u^\infty & \text{in } \Omega, \\ \dfrac{\partial u^\infty}{\partial n} + n_0 u^\infty = h^\infty & \text{on } \Gamma. \end{cases}$$

ここで，$f^\infty \in L^2(\Omega)$ と $h^\infty \in L^2(\Gamma)$ はそれぞれ $f(t)$ と $h(t)$ の $t \to +\infty$ としたときの極限である．実際には，$f - f^\infty \in L^2(\mathbf{R}_+; L^2(\Omega))$, $h - h^\infty \in L^2(\mathbf{R}_+; L^2(\Gamma))$ を仮定すれば十分である．

Theorem 2.1 (Sato-Shirohzu-Kenmochi [9])
$(P_0), (P_0^\infty)$ の解をそれぞれ $\{u, w\}, \{u^\infty, w^\infty\}$ とする．
(1) $u \to u^\infty := \dfrac{h^\infty}{n_0}$ weakly in $H^1(\Omega)$ as $t \to +\infty$.

(2) $w \to \exists w^\infty$ in $L^2(\Omega)$ as $t \to +\infty$ s.t. $\beta(w^\infty(x)) + g(w^\infty(x)) \ni u^\infty(x)$ a.e. for $x \in \Omega$.
(Fig.2, Fig.3)

Theorem 2.2 (Ito–Kenmochi [8])
(P_κ) の解を $\{u, w\}$ とする.

(1) $u \to u^\infty := \dfrac{h^\infty}{n_0}$ weakly in $H^1(\Omega)$ as $t \to +\infty$.

(2) w は一般に収束しないが, ω–limit set を

$$\omega(u_0, w_0) := \{v \in H^1(\Omega); \exists t_n \to +\infty \text{ s.t. } w(t_n, \cdot) \to v \text{ in } H^1(\Omega)\}$$

とおくと, $\omega(u_0, w_0)$ は次を満たす.

(a) $\omega(u_0, w_0)$: non–empty, closed, bounded in $H^2(\Omega)$, and compact, connected in $H^1(\Omega)$.

(b) $\forall v \in \omega(u_0, w_0)$ は次を満たす;

$$\begin{cases} -\kappa\Delta v + \beta(v) + g(v) \ni u^\infty & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \Gamma. \end{cases}$$

Remark 2.1 ω–limit set は, 一般に連続体を含む. (Fig.4)

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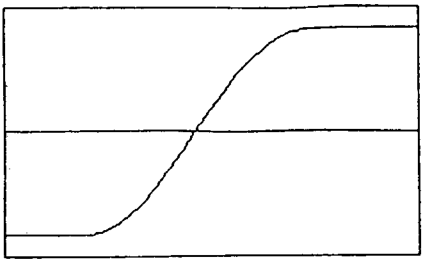
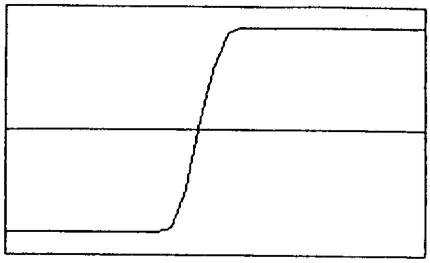


$\kappa = 0.1$



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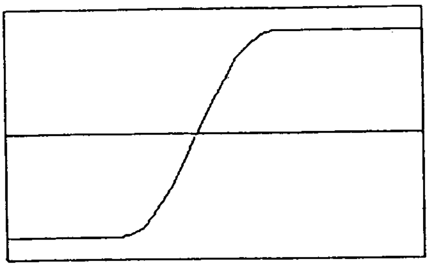

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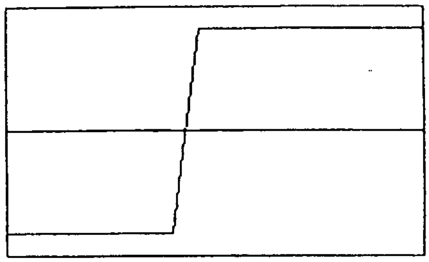
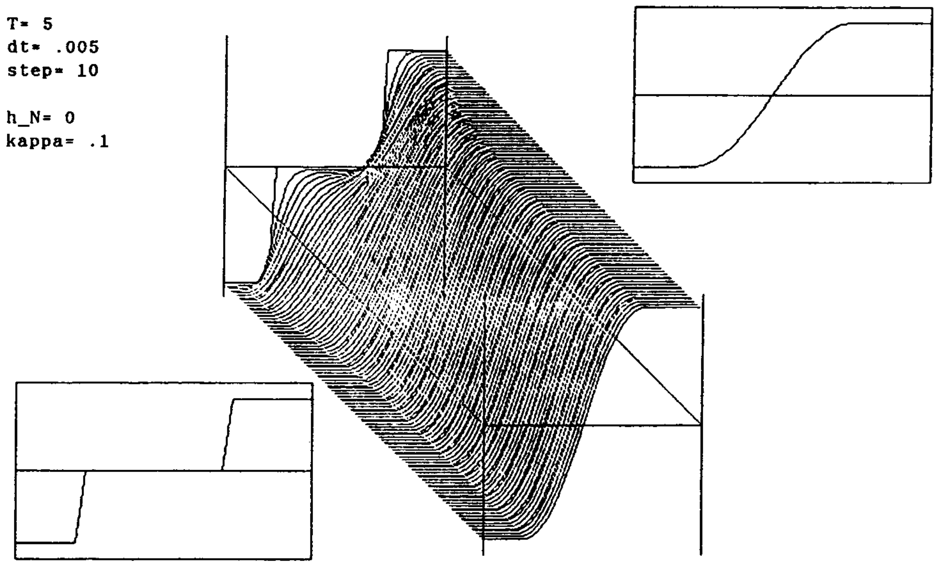
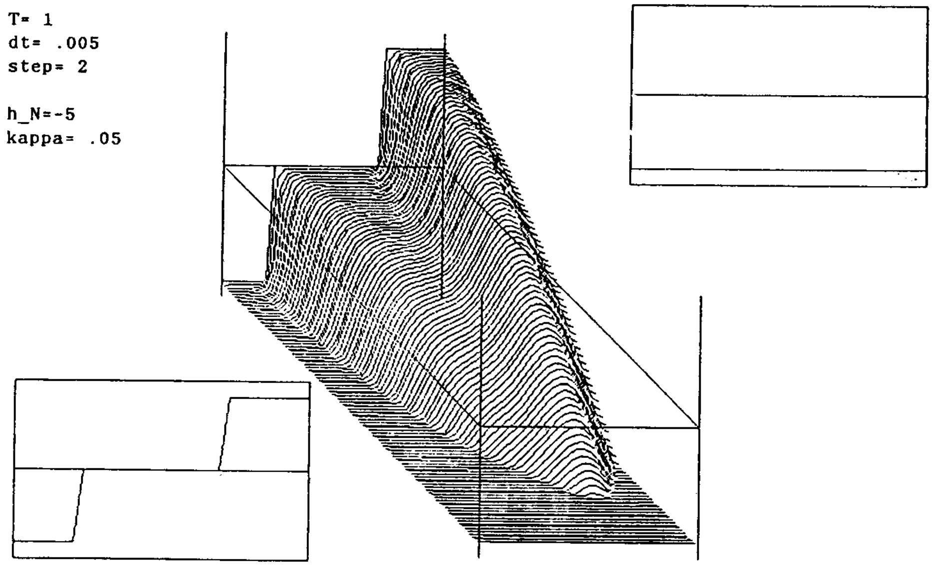


$\kappa = 0$

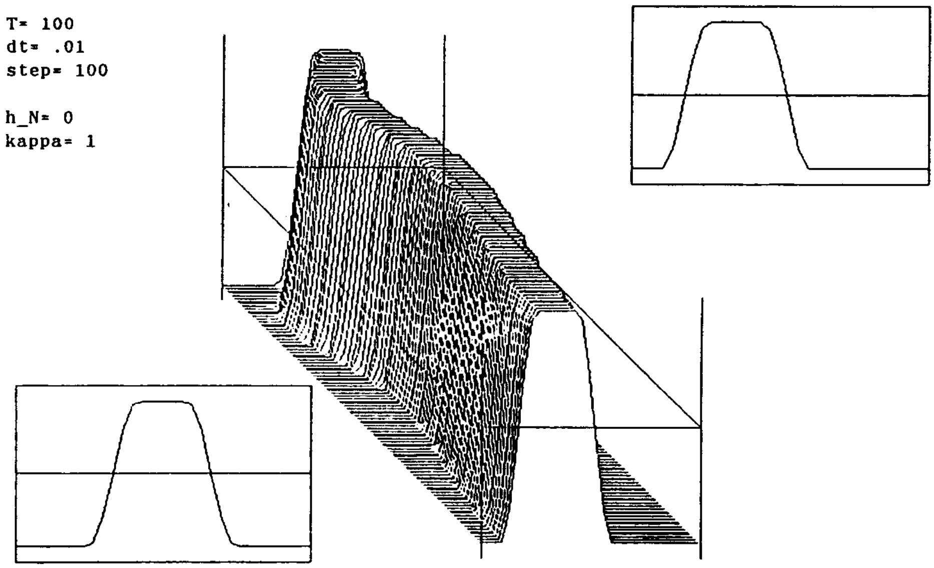
(Fig.1)



(Fig.2)



(Fig.3)



(Fig.4)

ASYMPTOTIC COMPLETENESS FOR LONG-RANGE MANY-PARTICLE SYSTEMS WITH STARK EFFECT, II

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§1. Introduction

The present paper is a continuation to the work [AT] where we have proved the asymptotic completeness of the Graf-type modified wave operators for many-particle Stark Hamiltonians with a class of long-range potentials. We here study the problem of the asymptotic completeness for many-particle Stark Hamiltonians with a larger class of long-range potentials.

We consider a system of N particles moving in a given constant electric field $\mathcal{E} \in \mathbf{R}^3$, $\mathcal{E} \neq 0$. Let m_j, e_j and $r_j \in \mathbf{R}^3$, $1 \leq j \leq N$, denote the mass, charge and position vector of the j-th particle, respectively. The N particles under consideration are supposed to interact with one another through the pair potentials $V_{jk}(r_j - r_k)$, $1 \leq j < k \leq N$. Then the total Hamiltonian for such a system is described by

$$\tilde{H} = \sum_{1 \leq j \leq N} \left\{ -\frac{1}{2m_j} \Delta_{r_j} - e_j \mathcal{E} \cdot r_j \right\} + V,$$

where $\xi \cdot \eta = \sum_{j=1}^{3} \xi_j \eta_j$ for ξ, $\eta \in \mathbf{R}^3$ and the interaction V is given as the sum of the pair potentials

$$V = \sum_{1 \leq j < k \leq N} V_{jk}(r_j - r_k).$$

As usual, we consider the Hamiltonian $\tilde{H}$ in the center-of-mass frame. We introduce the metric $\langle r, \tilde{r} \rangle = \sum_{j=1}^{N} m_j r_j \cdot \tilde{r}_j$ for $r = (r_1, \ldots, r_N)$ and $\tilde{r} = (\tilde{r}_1, \ldots, \tilde{r}_N) \in \mathbf{R}^{3 \times N}$. We use the notation $|r| = \langle r, r \rangle^{1/2}$. Let X and X_{cm} be the configuration spaces equipped with the metric $\langle \cdot, \cdot \rangle$, which are defined by

$$X = \left\{ r \in \mathbf{R}^{3 \times N} : \sum_{1 \leq j \leq N} m_j r_j = 0 \right\},$$

$$X_{\mathrm{cm}} = \left\{ r \in \mathbf{R}^{3 \times N} : r_j = r_k \text{ for } 1 \leq j < k \leq N \right\}.$$

These two subspaces are mutually orthogonal. We denote by $\pi : \mathbf{R}^{3\times N} \to X$ and $\pi_{\rm cm} : \mathbf{R}^{3\times N} \to X_{\rm cm}$ the orthogonal projections onto X and $X_{\rm cm}$, respectively. For $r \in \mathbf{R}^{3\times N}$, we write $x = \pi r$ and $x_{\rm cm} = \pi_{\rm cm} r$, respectively. Let $E \in X$ and $E_{\rm cm} \in X_{\rm cm}$ be defined by

$$E = \pi\left(\frac{e_1}{m_1}\mathcal{E}, \dots, \frac{e_N}{m_N}\mathcal{E}\right), \quad E_{\rm cm} = \pi_{\rm cm}\left(\frac{e_1}{m_1}\mathcal{E}, \dots, \frac{e_N}{m_N}\mathcal{E}\right),$$

respectively. Then the total Hamiltonian $\tilde{H}$ is decomposed into $\tilde{H} = H \otimes Id + Id \otimes T_{\rm cm}$, where Id is the identity operator, H is defined by

$$H = -\Delta/2 - \langle E, x\rangle + V \qquad \text{on } L^2(X),$$

$T_{\rm cm}$ denotes the free Hamiltonian $T_{\rm cm} = -\Delta_{\rm cm}/2 - \langle E_{\rm cm}, x_{\rm cm}\rangle$ acting on $L^2(X_{\rm cm})$, and Δ (resp. $\Delta_{\rm cm}$) is the Laplace-Beltrami operator on X (resp. $X_{\rm cm}$). We assume that $|E| \neq 0$. This is equivalent to saying that $e_j/m_j \neq e_k/m_k$ for at least one pair (j,k). Then H is called an N-particle Stark Hamiltonian in the center-of-mass frame.

A non-empty subset of the set $\{1, \dots, N\}$ is called a cluster. Let C_j, $1 \le j \le m$, be clusters. If $\cup_{1\le j\le m} C_j = \{1, \dots, N\}$ and $C_j \cap C_k = \emptyset$ for $1 \le j < k \le m$, $a = \{C_1, \dots, C_m\}$ is called a cluster decomposition. We denote by $\#(a)$ the number of clusters in a. We denote by $\tilde{\mathcal{A}}$ the set of cluster decompositions and set $\mathcal{A} = \{a \in \tilde{\mathcal{A}} : \#(a) \ge 2\}$. We let a, $b \in \tilde{\mathcal{A}}$. If b is obtained as a refinement of a, that is, if each cluster in b is a subset of a cluster in a, we say $b \subset a$, and its negation is denoted by $b \not\subset a$. We note that $a \subset a$ is regarded as a refinement of a itself. If, in particular, b is a strict refinement of a, that is, if $b \subset a$ and $b \neq a$, this relation is denoted by $b \subsetneq a$. We denote by $\alpha = (j,k)$ the $(N-1)$–cluster decomposition $\{(j,k), (1), \dots, (\widehat{j}), \dots, (\widehat{k}), \dots, (N)\}$.

Next we define the two subspaces X^a and X_a of X as

$$X^a = \left\{r \in X : \sum_{j\in C} m_j r_j = 0 \text{ for each cluster } C \text{ in } a\right\},$$
$$X_a = \{r \in X : r_j = r_k \text{ for each pair } \alpha = (j,k) \subset a\}.$$

We note that X^α is the configuration space for the relative position of j-th and k-th particles. Hence we can write $V_\alpha(x^\alpha) = V_{jk}(r_j - r_k)$. These spaces are mutually orthogonal and span the total space $X = X^a \oplus X_a$, so that $L^2(X)$ is decomposed as the tensor product $L^2(X) = L^2(X^a) \otimes L^2(X_a)$. We also denote by $\pi^a : X \to X^a$ and $\pi_a : X \to X_a$ the orthogonal projections onto X^a and X_a, respectively, and write $x^a = \pi^a x$ and $x_a = \pi_a x$ for a generic point $x \in X$. The intercluster interaction I_a is defined by

$$I_a(x) = \sum_{\alpha \not\subset a} V_\alpha(x^\alpha),$$

and the cluster Hamiltonian

$$H_a = H - I_a = -\Delta/2 - \langle E, x\rangle + V^a, \quad V^a(x^a) = \sum_{\alpha\subset a} V_\alpha(x^\alpha),$$

governs the motion of the system broken into non-interacting clusters of particles. Let $E^a = \pi^a E$ and $E_a = \pi_a E$. Then the operator H_a acting on $L^2(X)$ is decomposed into

$$H_a = H^a \otimes Id + Id \otimes T_a \qquad \text{on } L^2(X^a)\otimes L^2(X_a),$$

where H^a is the subsystem Hamiltonian defined by

$$H^a = -\Delta^a/2 - \langle E^a, x^a\rangle + V^a \qquad \text{on } L^2(X^a),$$

T_a is the free Hamiltonian defined by

$$T_a = -\Delta_a/2 - \langle E_a, x_a\rangle \qquad \text{on } L^2(X_a),$$

and Δ^a (resp. Δ_a) is the Laplace-Beltrami operator on X^a (resp. X_a). By choosing the coordinates system of X, which is denoted by $x = (x^a, x_a)$, appropriately, we can write $\Delta^a = |\nabla^a|^2$ and $\Delta_a = |\nabla_a|^2$, where $\nabla^a = \partial_{x^a} = \partial/\partial x^a$ and $\nabla_a = \partial_{x_a} = \partial/\partial x_a$ are the gradients on X^a and X_a, respectively. We note that we denote by x^a (resp. x_a) a vector in X^a (resp. X_a) as well as the coordinates system of X^a (resp. X_a).

We now state the precise assumption on the pair potentials. Let c be a maximal element of the set $\{a \in \mathcal{A} : E^a = 0\}$ with respect to the relation $\subset$. As is easily seen, such a cluster decomposition uniquely exists and it follows that $E^\alpha = 0$ if $\alpha \subset c$, and $E^\alpha \neq 0$ if $\alpha \not\subset c$. Thus the potential V_α with $\alpha \not\subset c$ (resp. $\alpha \subset c$) describes the pair interaction between two particles with $e_j/m_j \neq e_k/m_k$ (resp. $e_j/m_j = e_k/m_k$). If, in particular, $e_j/m_j \neq e_k/m_k$ for any $j \neq k$, then c becomes the N-cluster decomposition. We make different assumptions on V_α according as $\alpha \not\subset c$ or $\alpha \subset c$. We assume that :

(V) $V_\alpha(x^\alpha) \in C^\infty(X^\alpha)$ is a real-valued function and has the decay property

$$\partial^\beta_{x^\alpha} V_\alpha(x^\alpha) = O(|x^\alpha|^{-(\rho+|\beta|)/2}) \quad \alpha \not\subset c, \qquad \partial^\beta_{x^\alpha} V_\alpha(x^\alpha) = O(|x^\alpha|^{-(\rho+|\beta|)}) \quad \alpha \subset c$$

for some $\sqrt{3} - 1 < \rho \leq 1$.

Under this assumption, all the Hamiltonians defined above are essentially self-adjoint on C_0^∞. We denote their closures by the same notations. Throughout the whole exposition, the notations c and ρ are used with the meanings described above. If V_α satisfies this decay assumption, then V_α is called a long-range potential. To formulate the obtained result precisely, we define the modified wave operators. The definition requires several new notations. We assume that $a \subset c$. Then the subsystem operator H^a does not have a uniform electric field, that is, $E^a = 0$. Hence it may have bound states in $L^2(X^a)$. We denote by $P^a : L^2(X^a) \to L^2(X^a)$ the eigenprojection associated with H^a. We also denote the direction of E by $\omega = E/|E|$ and write $z = \langle x, \omega \rangle$. We should note that $z = \langle x_a, \omega \rangle$ because of $\omega^a = 0$. We set $x_{\|} = z\omega$ and $x_{\perp} = x - x_{\|}$, and write $x_{a,\perp} = \pi_a x_{\perp}$. Then we can write $x_a = (x_{a,\perp}, x_{\|})$. We also write $p_a = (p_{a,\perp}, p_{\|})$ for the coordinates dual to $x_a = (x_{a,\perp}, x_{\|})$ and denote by $D_a = -i\nabla_a = (D_{a,\perp}, D_{\|})$ the corresponding velocity operator. If we write $\partial_{\|} = \omega\partial_z$, we see that $D_{\|} = -i\partial_{\|}$ and $D_{a,\perp} = D_a - D_{\|}$. Let I_a^c be the intercluster interaction obtained from H^c:

$$I_a^c(x) = I_a^c(x^c) = \sum_{\alpha \subset c,\, \alpha \not\subset a} V_\alpha(x^\alpha).$$

We consider the time-dependent Hamiltonian

$$H_{aD}(t) = H_a + I_a^c(tD_{a,\perp}) + I_c(tD_{a,\perp} + t^2E/2) \qquad \text{on } L^2(X). \tag{1.1}$$

Since $D_{a,\perp}$ commutes with H_a, the three operators on the right-hand side of (1.1) commute with one another. We note that $I_a^c(tD_{a,\perp}) = I_a^c(tD_a)$ for $I_a^c(tp_{a,\perp}) = I_a^c(t\pi^c p_{a,\perp}) = I_a^c(tp_a)$. Then we denote by $U_{aD}(t)$ the propagator which is generated by $H_{aD}(t)$, that is, $\{U_{aD}(t)\}_{t\in\mathbf{R}}$ is a family of unitary operators such that for $\psi \in D(H_{aD}(0))$, $\psi_t = U_{aD}(t)\psi$ is a strong solution of $id\psi_t/dt = H_{aD}(t)\psi_t$, $\psi_0 = \psi$. $U_{aD}(t)$ is explicitly represented by

$$U_{aD}(t) = \exp(-itH_a)\exp\left(-i\int_0^t \{I_a^c(sD_{a,\perp}) + I_c(sD_{a,\perp} + s^2E/2)\}ds\right). \tag{1.2}$$

With these notations, the Dollard-type modified wave operators in question are now defined by

$$W_{aD}^{\pm} = s - \lim_{t\to\pm\infty} \exp(itH)U_{aD}(t)(P^a \otimes Id), \quad a \subset c. \tag{1.3}$$

It can be easily proved that if these wave operators exist, their ranges are all closed and they have the intertwining property $\exp(itH)W_{aD}^{\pm} = W_{aD}^{\pm}\exp(itH_a)$ for $t \in \mathbf{R}$.

The main result of this paper is the following theorem.

Theorem 1.1. *Assume that (V) is fulfilled. Let c be as above. Then the Dollard-type wave operators $W_{aD}^{\pm}$, $a \subset c$, exist, have the intertwining property and are asymptotically complete:*

$$L^2(X) = \sum_{a \subset c} \oplus \, Range\, W_{aD}^{\pm}.$$

If, in particular, c is the N-cluster decomposition, that is, no subsystem has zero reduced charge, the asymptotic completeness of the Dollard-type modified wave operators can be also proved under the assumption (V) with $\rho > 1/2$. For we need not apply the argument of Dereziński [D] to this situation. Furthermore, we can introduce the modifiers which are different from the Dollard-type ones, so that the asymptotic completeness of such modified wave operators can be proved under the assumption

(V) with $\rho > 0$. This result is an extension of the result for two-particle systems of Jensen-Yajima [JY] and White [W1,W2] to the case of many-particle systems.

The problem of the asymptotic completeness for many-particle quantum systems has made great progress for the past several years. For the systems without electric fields, this problem was first solved by Sigal-Soffer [SS1] for a large class of short-range pair potentials. After that work, alternative proofs have been given by several authors (cf. [Gr1], [Ki], [T1], [Y] and [Z]). On the other hand, for the long-range case, Enss [E] first proved the completeness for three-particle systems with the pair potentials decaying like $O(|x^\alpha|^{-\nu})$ at infinity for some $\nu > \sqrt{3}-1$. This result has been extended by Dereziński [D] and Zielinski [Z] to N-particle systems and also the case of potentials decaying more slowly has been dealt with by Gérard [G] and Wang [Wa] for three-particle systems. We should note that the condition $\rho > \sqrt{3}-1$ in our assumption (V) is assumed in order to apply the argument of [D].

For the systems with uniform electric fields, if the assumption (V) is satisfied for some $\rho > 1$, V_α is called a short-range potential. For the class of short-range pair potentials, the ordinary wave operators

$$W_a^\pm = s - \lim_{t\to\pm\infty} \exp(itH)\exp(-itH_a)(P^a \otimes Id)$$

exist without adding the time-dependent modifiers $I_a^c(tD_{a,\perp}) + I_c(tD_{a,\perp} + t^2E/2)$ to the cluster Hamiltonians H_a. The asymptotic completeness in the short-range case has been proved by Tamura [T3] and Møller [Mø] for N-particle systems. However it is known that such wave operators do not generally exist for the class of long-range potentials which we consider here (see [JO] and [O] for the case of two-particle systems).

In the previous work [AT], we have considered the class of long-range potentials such that

$$\partial_{x^\alpha}^\beta V_\alpha(x^\alpha) = O(|x^\alpha|^{-(\rho+\mu|\beta|)}) \quad \alpha \not\subset c,$$

for some ρ, $\mu > 0$ such that $\rho + \mu > 1$ ($V_\alpha(x^\alpha)$, $\alpha \subset c$, satisfy the same assumption as in (V)) and we have proved the asymptotic completeness of the Graf-type modified wave operators

$$W_{aG}^\pm = s - \lim_{t\to\pm\infty} \exp(itH)U_{aG}(t)(P^a \otimes Id), \quad a \subset c,$$

where the propagators $U_{aG}(t)$ are generated by the time-dependent Hamiltonians

$$H_{aG}(t) = H_a + I_a^c(tD_a) + I_c(t^2E/2),$$

and are concretely represented by

$$U_{aG}(t) = \exp(-itH_a)\exp\left(-i\int_0^t \{I_a^c(sD_a) + I_c(s^2E/2)\}\, ds\right).$$

This type of wave operators was first introduced by Graf [Gr2] for two-particle systems (also see [JO]). However it is known that such wave operators do not exist for the class of long-range potentials which we consider here (see [JO] for the case of two-particle systems). Therefore we need introduce the Dollard-type modifiers (1.2).

§2. An outline of the proof of Theorem 1.1

In this section, we give an outline of the proof of Theorem 1.1. For brevity, we omit the proof of the existence of the Dollard-type modified wave operators $W_{aD}^{\pm}$, $a \subset c$.

We define a conical neighborhood of $\omega = E/|E|$ by $\Gamma(\omega, \epsilon_1, r) = \{x \in X : \langle \omega, x/|x| \rangle \geq 1 - \epsilon_1, |x| > r\}$ for $\epsilon_1 > 0$ and $r > 0$. Let $\tilde{q}_c \in S_0(X) = \{q \in C^\infty(X) : |\partial_x^\beta q(x)| \leq C_\beta \langle x \rangle^{-|\beta|}\}$ be such that $\tilde{q}_c = 1$ in $\Gamma(\omega, 2\epsilon_1, |E|/4)$, and $\tilde{q}_c = 0$ outside $\Gamma(\omega, 3\epsilon_1, |E|/5)$. We consider the time-dependent Hamiltonian

$$H_c(t) = H_c + W_c(t), \qquad W_c(t) = F(z/t^2 \geq |E|/4)\tilde{q}_c(x) I_c(x),$$

and denote by $U_c(t)$ the propagator generated by $H_c(t)$, that is, $\{U_c(t)\}_{t \geq 1}$ is a family of unitary operators such that for $\psi \in D(H_c(1))$, $\psi_t = U_c(t)\psi$ is a strong solution of $id\psi_t/dt = H_c(t)\psi_t$, $\psi_1 = \psi$. Then we need the following proposition, which is a key step for the proof of Theorem 1.1. Its proof is completed some propagation estimates, but we omit it here.

Theorem 2.1 (Asymptotic clustering). *Let the notation be as above. Then for $\psi \in L^2(X)$, there exists $\psi_c^{\pm} \in L^2(X)$ such that*

$$\exp(-itH)\psi = U_c(t)\psi_c^{\pm} + o(1) \qquad ast \to \pm\infty.$$

This theorem implies that we have only to study the scattering theory for each cluster decomposition $a \subset c$, where the constant electric field $\mathcal{E}$ has no influence. Hence, we can apply the result due to Dereziński [D] to prove Theorem 1.1. Now we prove the asymptotic completeness. By Theorem 2.1 and Dereziński's result, we have as $t \to \pm\infty$,

$$\begin{aligned} \exp(-itH)\psi &= U_c(t)\psi_c^{\pm} + o(1) \\ &= \sum_{a \subset c} U_{aD}(t)\tilde{\psi}_c^{a,\pm} + o(1) \end{aligned}$$

for some $\psi_c^{a,\pm} \in Range\,(P^a \otimes Id)$. This implies

$$\psi \in \sum_{a \subset c} \oplus\, Range\, W_{aD}^{+},$$

which completes the proof of Theorem 1.1. □

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Stability of solitary waves for coupled nonlinear Schrödinger equations

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ここでは Coupled nonlinear Schrödinger equations (CNLS) の孤立波解の安定性について考える. まず, この問題に関する歴史を簡単に振り返る. 単独の 非線形 Schrödinger 方程式 (NLS) の孤立波解の安定性・不安定性に関する数学的研究は '８０年代前半に Berestycki and Cazenave ('81), Cazenave and Lions ('82) などにより始められた. また, 単独の非線形 Klein–Gordon 方程式 (NLKG) に関しても Shatah ('83), Shatah and Strauss ('85) などにより研究された. その後 NLS, NLKG, KdV 方程式などの孤立波解の安定性・不安定性の問題は Grillakis, Shatah and Strauss ('87) により抽象的な枠組みにまとめられた. また, Grillakis, Shatah and Strauss ('90) では ある種の対称性をもった方程式系の孤立波解の安定性・不安定性を扱えるように その枠組みは一般化されたが, 例えば, Klein–Gordon 方程式と Schrödinger 方程式とがカップルした Coupled Klein–Gordon–Schrödinger equations (KGS) や プラズマ物理に現れる Zakharov 方程式系など物理的に重要な多くの方程式系は彼らの枠組みには あてはまらない. ここで 考える CNLS は

$$\begin{cases} iu_t + \Delta u + (a|u|^2 + |v|^2)u = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^N, & (1) \\ iv_t + \Delta v + (|u|^2 + a|v|^2)v = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^N & (2) \end{cases}$$

である. ここで u, v は $t \in \mathbb{R}$, $x \in \mathbb{R}^N$ を変数とする複素数値の未知関数で $a \in \mathbb{R}$ は定数. 特に $a = 1$ のとき (1)–(2) は

$$i\overrightarrow{u} + \Delta \overrightarrow{u} + |\overrightarrow{u}|^2 \overrightarrow{u} = 0, \quad \overrightarrow{u} = (u, v) \tag{3}$$

と形式的に単独の形に書けるが, このときに限り (1)–(2) は Grillakis, Shatah and Strauss ('90) の枠組みにあてはまり, それ以外のときにはあてはまらない. 先ほど述べた ある種の対称性とはこのような意味である. 単独の方程式に対する理論が完成した現在, KGS や Zakharov 方程式系のような Schrödinger 方程式と他の波動方程式とがカップルした方程式系の孤立波解の安定性を調べることが一つの問題として残っているが, ここでは, その

第一歩として単独の NLS の自然な拡張と考えられる (1)-(2) を取り上げる. もちろん, $a=1$ とは限らない場合を考える. (1)-(2) は $N=1$ のとき光ファイバー中の光パルスの伝播を記述するモデルとして現れる (Wadati, Iizuka and Hisakado ('92)). この場合, 定数 a としては $a>0$ かつ $a\neq 1$ であるものが現れる. ここで, (u,v) は横波である光の 2 成分に対応している. (1)-(2) で $u=v$ とおくと通常の単独の NLS になる:

$$iu_t+\Delta u+(a+1)|u|^2u=0, \quad t\in\mathbb{R}, \quad x\in\mathbb{R}^N. \tag{4}$$

光ファイバー中の光パルスの伝播を記述するモデルとしては 単独の NLS (4) が使われることが多い (ただし $N=1$ のとき) が, CNLS (1)-(2) では さらに, 2 つの横波の相互作用が考慮されている. $1\le N\le 3$, $a>-1$ のとき (4) は孤立波解 $u_\omega(t,x)=e^{i\omega t}\varphi_\omega(x)$ をもつ. ここで $\omega>0$ で φ_ω は

$$\begin{cases} -\Delta\varphi+\omega\varphi-(a+1)|\varphi|^2\varphi=0, \quad x\in\mathbb{R}^N, \\ \varphi\in H^1(\mathbb{R}^N) \end{cases} \tag{5}$$

の一意的な正値球対称解とする. このとき, $(u_\omega(t,x),\, u_\omega(t,x))$ は CNLS (1)-(2) の孤立波解となるが, この単独の方程式の解からつくられた孤立波解の安定性について考える. その前に, 単独の NLS に対する結果を振り返る. 考える方程式は

$$iu_t+\Delta u+|u|^{p-1}u=0, \quad t\in\mathbb{R}, \quad x\in\mathbb{R}^N, \tag{6}$$

$p>1$, $N\ge 1$, ただし $N\ge 3$ のときは $p<(N+2)/(N-2)$ とする. また, φ_ω を

$$\begin{cases} -\Delta\varphi+\omega\varphi-|\varphi|^{p-1}\varphi=0, \quad x\in\mathbb{R}^N, \\ \varphi\in H^1(\mathbb{R}^N) \end{cases} \tag{7}$$

の一意的な正値球対称解とする.

$1<p<1+4/N$ のとき任意の $\omega>0$ に対して (6) の孤立波解 $u_\omega(t,x)=e^{i\omega t}\varphi_\omega(x)$ は次の意味で軌道安定である (Cazenave and Lions ('82)): 任意の $\varepsilon>0$ に対して $\delta>0$ があり、もし $u_0\in H^1(\mathbb{R}^N)$ が $\|u_0-\varphi_\omega\|_{H^1}<\delta$ を満たせば u_0 を初期値とする (6) の解 $u(t,x)$ は

$$\sup_{t\in\mathbb{R}}\inf_{\theta\in\mathbb{R},y\in\mathbb{R}^N}\|u(t)-e^{i\theta}\tau_y\varphi_\omega\|_{H^1}<\varepsilon$$

を満たす. ここで $\tau_y v(x) = v(x-y)$ とする.

$p \geq 1 + 4/N$ のとき任意の $\omega > 0$ に対して (6) の孤立波解 $u_\omega(t,x)$ は不安定である ($p > 1 + 4/N$ のとき Berestycki and Cazenave ('81), $p = 1 + 4/N$ のとき Weinstein ('83)).

これら単独の場合の結果から $N = 2, 3$ のとき CNLS (1)–(2) の孤立波解 $(u_\omega(t,x), u_\omega(t,x))$ は不安定であることが直ちに分かる. そこで, 以下 $N = 1$ とする. $N = 1$ のとき, (5) の一意的な正値球対称解 φ_ω は

$$\varphi_\omega(x) = \sqrt{\frac{2\omega}{a+1}} \operatorname{sech} \sqrt{\omega} x$$

と初等関数を用いて表される. すでに述べたように単独の NLS の孤立波解 $u_\omega(t,x) = e^{i\omega t}\varphi_\omega(x)$ は軌道安定である. Wadati, Iizuka and Hisakado ('92) では, (1)–(2) の孤立波解として $(u_\omega(t), u_\omega(t))$ が存在することは示されているが, その安定性に関しては考察されていない. 今回, 次の結果を得た。

<u>定理 1</u>. $a > -1$ とする。このとき任意の $\omega > 0$ に対して (1)–(2) の孤立波解 $(u_\omega(t), u_\omega(t))$ は次の意味で軌道安定である: 任意の $\varepsilon > 0$ に対して $\delta > 0$ があり、もし $(u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ が

$$\|(u_0, v_0) - (\varphi_\omega, \varphi_\omega)\|_{H^1 \times H^1} < \delta$$

を満たせば (u_0, v_0) を初期値とする (1)–(2) の解 $(u(t), v(t))$ は

$$\sup_{t \in \mathbb{R}} \inf_{\alpha, \beta, y \in \mathbb{R}} \|(u(t), v(t)) - (e^{i\alpha}\tau_y \varphi_\omega, e^{i\beta}\tau_y \varphi_\omega)\|_{H^1 \times H^1} < \varepsilon$$

を満たす. ここで $\tau_y v(x) = v(x-y)$ とする.

<u>注 2</u>. 任意の $(u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ に対して $(u(0), v(0)) = (u_0, v_0)$ なる (1)–(2) の解 $(u(t), v(t))$ が $C(\mathbb{R}; H^1(\mathbb{R}) \times H^1(\mathbb{R}))$ の中に一意的に存在し, 次の保存則を満たす:

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad \|v(t)\|_{L^2} = \|v_0\|_{L^2}, \quad t \in \mathbb{R}, \tag{8}$$

$$E(u(t), v(t)) = E(u_0, v_0), \quad t \in \mathbb{R}. \tag{9}$$

ここで

$$E(u,v)=\left\|\frac{\partial u}{\partial x}\right\|_{L^2}^2+\left\|\frac{\partial v}{\partial x}\right\|_{L^2}^2-\frac{a}{2}(\|u\|_{L^4}^4+\|v\|_{L^4}^4)-\int_{-\infty}^{\infty}|u(x)|^2|v(x)|^2dx$$

定理 1 の証明では Cazenave and Lions ('82) に従い次の最小化問題:

$$I(\lambda)=\inf\{E(u,v):(u,v)\in H^1(\mathbb{R})\times H^1(\mathbb{R}),\ \|u\|_{L^2}^2=\|v\|_{L^2}^2=\lambda\},\quad \lambda>0 \qquad (10\lambda)$$

を考える. また, $\mathcal{G}(\lambda)$ を (10λ) の解全体の集合とする, すなわち

$$\mathcal{G}(\lambda)=\{(u,v)\in H^1(\mathbb{R})\times H^1(\mathbb{R}):E(u,v)=I(\lambda),\ \|u\|_{L^2}^2=\|v\|_{L^2}^2=\lambda\}$$

とおく. 定理 1 の証明では $\mathcal{G}(\lambda)$ が 次のように特徴付けられることが カギとなる:

$$\mathcal{G}(\lambda(\omega))=\{(e^{i\alpha}\tau_y\varphi_\omega,\ e^{i\beta}\tau_y\varphi_\omega):\alpha,\ \beta,\ y\in\mathbb{R}\},\quad \omega>0. \qquad (11)$$

ここで $\lambda(\omega)=\|\varphi_\omega\|_{L^2}^2=4\sqrt{\omega}/(a+1)$. これから単独の NLS の解 $u_\omega(t,x)=e^{i\omega t}\varphi_\omega(x)$ からつくられた CNLS の孤立波解 $(u_\omega(t,x),u_\omega(t,x))$ は各時刻 t において最小化問題 $(10\lambda(\omega))$ の解であることが分かる. あとは Lions の concentration compactness method を用いて次の補題 3 を示し, それを使って, 任意の $\lambda>0$ に対して $\mathcal{G}(\lambda)$ が CNLS (1)-(2) に対して安定であることを示せばよい.

<u>補題 3</u>. $a>-1,\lambda>0$ とする. $\{(u_j,v_j)\}\subset H^1(\mathbb{R})\times H^1(\mathbb{R})$ が $\|u_j\|_{L^2}^2\to\lambda,\ \|v_j\|_{L^2}^2\to\lambda$ $E(u_j,v_j)\to I(\lambda)$ を満たせば,

$$(\tau_{y_{j'}}u_{j'},\ \tau_{y_{j'}}v_{j'})\to(\varphi^1,\ \varphi^2)\quad in\ H^1(\mathbb{R})\times H^1(\mathbb{R})$$

となる 部分列 $\{(u_{j'},v_{j'})\}$, $\{y_{j'}\}\subset\mathbb{R}$ と $(\varphi^1,\ \varphi^2)\in\mathcal{G}(\lambda)$ が存在する.

<u>命題 4</u>. 任意の $\lambda>0$ に対して $\mathcal{G}(\lambda)$ は *CNLS (1)-(2)* に対して安定である.

<u>命題 4 の証明</u>. 背理法で示す. もしも $\mathcal{G}(\lambda)$ が安定でないとすると, ある $\varepsilon_0>0$ と初期値の列 $\{(u_{0j},v_{0j})\}\subset H^1(\mathbb{R})\times H^1(\mathbb{R})$ で

$$\inf_{(\varphi^1,\varphi^2)\in\mathcal{G}(\lambda)}\|(u_{0j},v_{0j})-(\varphi^1,\varphi^2)\|_{H^1\times H^1}\to 0, \qquad (12)$$

$$\sup_{t\in\mathbb{R}} \inf_{(\varphi^1,\varphi^2)\in\mathcal{G}(\lambda)} \|(u_j(t), v_j(t)) - (\varphi^1, \varphi^2)\|_{H^1\times H^1} \geq \varepsilon_0 \tag{13}$$

となるものが存在する. ここで, $(u_j(t), v_j(t))$ は (u_{0j}, v_{0j}) を初期値とする CNLS (1)-(2) の解. (13) から

$$\inf_{(\varphi^1,\varphi^2)\in\mathcal{G}(\lambda)} \|(u_j(t_j), v_j(t_j)) - (\varphi^1, \varphi^2)\|_{H^1\times H^1} = \varepsilon_0 \tag{14}$$

となる $t_j \in \mathbb{R}$ が存在する. (12) と保存則 (8)-(9) から

$$\|u_j(t_j)\|_{L^2}^2 = \|u_{0j}\|_{L^2}^2 \to \lambda, \quad \|v_j(t_j)\|_{L^2}^2 = \|v_{0j}\|_{L^2}^2 \to \lambda, \tag{15}$$

$$E(u_j(t_j),\, v_j(t_j)) = E(u_{0j},\, v_{0j}) \to I(\lambda) \tag{16}$$

を得る. (15)-(16) と補題 3 より

$$(\tau_{y_{j'}} u_{j'}(t_{j'}),\, \tau_{y_{j'}} v_{j'}(t_{j'})) \to (\varphi^1,\, \varphi^2) \quad \text{in} \ \ H^1(\mathbb{R}) \times H^1(\mathbb{R})$$

となる部分列 $\{(u_{j'}(t_{j'}),\, v_{j'}(t_{j'}))\}$ と $\{y_{j'}\} \subset \mathbb{R}$ と $(\varphi^1,\, \varphi^2) \in \mathcal{G}(\lambda)$ が存在する. しかし, $(\tau_{-y_{j'}}\varphi^1,\, \tau_{-y_{j'}}\varphi^2) \in \mathcal{G}(\lambda)$, だから これは (14) と矛盾する. 故に, $\mathcal{G}(\lambda)$ は安定である. □

$\mathcal{G}(\lambda)$ が (11) のように特徴付けられることを示すには次の補題 5 を用いる.

<u>補題 5</u>. $a > -1, \lambda > 0$ とする. $(u, v) \in \mathcal{G}(\lambda)$ とすると, $|u(x)| = |v(x)|$ が 任意の $x \in \mathbb{R}$ に対して成り立つ.

この補題 5 により問題はカップルしていない場合に帰着される.

その後の結果

若手セミナーの予稿を提出した後で, 冒頭に述べた Coupled Klein–Gordon–Schrödinger equations (KGS) と Zakharov 方程式系の孤立波解の安定性も定理 1 とほぼ同様に証明できることが分かった. 以下 KGS について簡単に説明する. 考える方程式は

$$
\begin{cases}
i\dfrac{\partial}{\partial t}\psi + \dfrac{1}{2}\Delta\psi = -\phi\psi, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^3, & (17) \\
\dfrac{\partial^2}{\partial t^2}\phi - \Delta\phi + m^2\phi = |\psi|^2, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^3. & (18)
\end{cases}
$$

ここで ψ は複素数値, ϕ は実数値関数で m は正定数とする. KGS (17)-(18) は核子場と中間子場の相互作用を記述する古典的なモデルで, ψ は核子場を, ϕ は中間子場を表し, m は中間子の質量を表す (Yukawa 1935).

KGS (17)-(18) は次の保存則を満たす:

$$\|\psi(t)\|_{L^2}^2 = \|\psi_0\|_{L^2}^2, \quad t \in \mathbb{R}, \tag{19}$$

$$\|\frac{\partial}{\partial t}\phi(t)\|_{L^2}^2 + E(\psi(t), \phi(t)) = \|\phi_1\|_{L^2}^2 + E(\psi_0, \phi_0), \quad t \in \mathbb{R}, \tag{20}$$

ここで $(\psi(t), \phi(t))$ は $(\psi(0), \phi(0), \frac{\partial}{\partial t}\phi(0)) = (\psi_0, \phi_0, \phi_1)$ なる (17)-(18) の解で,

$$E(u, v) = \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + m^2\|v\|_{L^2}^2 - 2\int_{\mathbb{R}^3} |u(x)|^2 v(x)dx.$$

CNLS と違い $\phi(t)$ の L^2 ノルムは保存しないことに注意する. CNLS のときと同様に最小化問題

$$I(\mu) = \inf\{E(u, v) : (u, v) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3), \ \|u\|_{L^2}^2 = \mu\} \tag{21}$$

を考える. CNLS のときと違い v については制約条件がないことに注意する.

$$\Sigma(\mu) = \{(u, v) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) : E(u, v) = I(\mu), \ \|u\|_{L^2}^2 = \mu\}$$

とおくと ある正定数 C_0 が存在して$\mu > C_0 m$ であれば$\Sigma(\mu)$ は空でなく$\Sigma(\mu) \times \{0\}$ は エネルギー空間 $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ の中で KGS (17)-(18) に対して安定であることが示される. 単独の NLKG の時間に依存しない定常解はすべて不安定である (Shatah ('85)) ことを注意する. 今回の結果は Klein-Gordon 方程式は Schrödinger 方程式とカップルすることにより安定な定常解をもつことを主張している.

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Pohozaev-type inequalities and nonexistence results for some quasilinear elliptic equations in exterior domains

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In this note, we are concerned with the nonexistence of nontrivial solutions of quasilinear elliptic equations of the form:

$$(\mathrm{E})\begin{cases} -\Delta_p u &= |u|^{q-2}u \quad \text{in } \Omega, \qquad 1 < p, q < \infty, \\ \quad u &= 0 \qquad\quad\ \ \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p := \mathrm{div}(|\nabla \cdot|^{p-2}\nabla \cdot)$ is the p-Laplace operator, and Ω is an exterior domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. This equation arises from the minimizing problem for the Rayleigh quotient $R(v) = \|\nabla v\|_{L^p}/\|v\|_{L^q}$, i.e., the minimizing element u of $R(v)$ in $W_0^{1,p}(\Omega) \setminus \{0\}$, normalized in a proper way, gives a nontrivial solution for (E).

When Ω is a general unbounded domain, the significance of our equation from this aspect might fade away, since the infimum of $R(v)$ could be zero in general. On the other hand, quite recently, the existence of nontrivial solution for $-\Delta_p u = f(u)$, $u|_{\partial\Omega} = 0$ has been studied vigorously by many peoples under various conditions on $f(\cdot)$ and Ω (not necessarily bounded). For instance we refer to [4, 5], [10], [3].

Therefore, from this view-point of partial differential equations, it would be meaningful to investigate the nonexistence of nontrivial solutions for these equations in unbounded domains. This effort in this direction has been done by several peoples such as in [1], [7], [6]. However these are all restricted to the case where solutions are classical, i.e., in $C^2(\Omega) \cap C^1(\overline{\Omega})$, and it should be noted that the solution of (E) with $p \neq 2$ does not always belong to $C^2(\Omega)$. Hence, in order to establish persuasive nonexistence results, we should work in much wider class of weak solutions. This kind of attempt was already done in [8] for the case where Ω is bounded. The purpose of this note is to show that the nonexistence can be discussed in a class of weak solutions analogous to that for bounded domains in [8]. The details of proof is shown in a forthcoming paper [2].

Our main result reads as follows.

Theorem 1 *Let $\Omega = \mathbb{R}^N \setminus \Omega_0$, and let Ω_0 be a bounded starshaped domain. Put*

$$\mathcal{P} = \{u \in L^q(\Omega); |u|^{q-1} \in L_{loc}^{\frac{p}{p-1}}(\overline{\Omega}), \nabla u \in L^p(\Omega), u|_{\partial\Omega} = 0\}.$$

Then the following hold.

(i) *Let* $1 < q < p^*$ *with* $p^* = \infty$ *if* $p \geq N$ *and* $p^* = Np/(N-p)$ *if* $p < N$, *then* (E) *has no nontrivial weak solution belonging to* P.

(ii) *Let* $p < N$ *and* $q = p^*$, *then* (E) *has no nontrivial weak solution of definite sign belonging to* P.

Remark

Above results together with our previous results in [8] suggest the following duality between the interior problems and the exterior problems for starshaped domains. Although it seems that the existence of nontrivial positive solutions for the exterior problems with $q > p^*$ is not yet proved, we strongly believe that it should hold true.

duality between interior and exterior problems			
domain	$q < p^*$	$q = p^*$	$q > p^*$
interior	∃ positive solution	no positive solution	no nontrivial solution
exterior	no nontrivial solution	no positive solution	∃ positive solution ?

To prove the theorem, we introduce a "Pohozaev-type inequality" valid for weak solutions u belonging to $\mathcal{P}$. To do this, we need some approximation procedures. We can apply the same basic idea as in [8]. However there appear several difficulties to be overcome which comes from the unboundedness nature of the domain Ω.

First of all, we prepare a sequence of bounded domains $\Omega_n = \Omega \cap B_{R_n}$ (B_{R_n}: ball centered at the origin with radius R_n which tends to ∞ as $n \to \infty$) and the cut-off functions $g_n(\cdot) \in C^1(\mathbb{R})$ such that $0 \leq g_n'(s) \leq 1$ $\forall s \in \mathbb{R}$, $g_n(s) = s$ for $|s| \leq n$, $g_n(s) = (n+1)\text{sign } s$ for $|s| \geq n+1$.
Let u be a weak solution of (E) belonging to $\mathcal{P}$, and let $u_n = g_n(u)$ and let $\underline{u_n} := u_n|_{\Omega_n}$. For $\varepsilon \in (0,1]$ and $n \in \mathbb{N}$, we take a function $v_n^\varepsilon \in C_0^\infty(\Omega_n)$ such that $\|v_n^\varepsilon\|_{L^\infty} \leq C$ and $v_n^\varepsilon \to 2|\underline{u_n}|^{q-2}\underline{u_n}$ strongly in $L^r(\Omega_n)$ for all $r \in [1,\infty)$ as $\varepsilon \to 0$.

We introduce the approximate equations:

$$(\mathrm{E})_n^\varepsilon \begin{cases} |w_n^\varepsilon|^{q-2}w_n^\varepsilon + A_\varepsilon w_n^\varepsilon = v_n^\varepsilon & \text{in } \Omega_n, \\ w_n^\varepsilon = 0 & \text{on } \partial\Omega_n, \end{cases}$$

where $A_\varepsilon w(x) = -\text{div}\{(|\nabla w|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla w(x)\}$.

In order to obtain a Pohozaev's type identity, we arrange the cut-off function $\chi_R(|x|)$ satisfying $\chi_R(|x|) \in C^\infty(\Omega)$, $2R \leq R_n$; $\chi_R(|x|) = 1$ for $|x| \leq R$; $\chi_R(|x|) = 0$ for $|x| \geq 2R$; $0 \leq \chi_R(|x|) \leq 1$; $|\chi_R'(|x|)| \leq \frac{1}{R}$; for as to the integrability of $u \in \mathcal{P}$ in Ω, we only assume that $u \in L^q(\Omega)$ and $\nabla u \in L^p(\Omega)$, therefore we encounter serious difficulties concerning the integrability of various integrants in procedures of deriving the Pohozaev-type inequality.

We multiply $(\mathrm{E})_n^\varepsilon$ by $\chi_R(|x|)\sum_{j=1}^N x_j \dfrac{\partial w_n^\varepsilon}{\partial x_j}$ and carry out integration by parts several times and pass to the limit in the order of $\varepsilon \to 0, n \to \infty$ and $R \to \infty$. Then by delicate arguments on the convergence of w_n^ε with the aid of the convex analysis, we finally deduce the following lemma:

Lemma 2 *Let u be a weak solution of* (E) *belonging to $\mathcal{P}$. Then the following Pohozaev-type inequality holds.*

$$\left(\frac{N}{q} + \frac{p-N}{p}\right) \int_\Omega |u|^q dx + \mathcal{R} \le 0, \tag{1}$$

where

$$\mathcal{R} = \overline{\lim_{n\to\infty}}\, \overline{\lim_{\varepsilon\to 0}} \frac{\bar{p}-1}{p} \int_{\partial\Omega} (|\nabla w_n^\varepsilon|^2 + \varepsilon)^{\frac{p}{2}} (-x\cdot\vec{n}) dS, \ \bar{p} = \min(p, 2).$$

Proof of Theorem 1

The first assertion (i) is a direct consequence of (1), since $\mathcal{R} \ge 0$. As for the assertion for the critical case (ii), we need additional delicate arguments. The relation $q = p^*$ together with (1) implies that $\mathcal{R} = 0$. Hence, for any $\eta > 0$, there exist N and $\varepsilon_0 > 0$ such that

$$\int_{\partial\Omega} (|\nabla w_n^\varepsilon|^2 + \varepsilon)^{\frac{p}{2}} dS < \eta \quad \forall n \ge N, \ 0 < \forall\varepsilon < \varepsilon_0.$$

On the other hand, since

$$N|\Omega_0| = \int_{\Omega_0} \operatorname{div} x\, dx = \int_{\partial\Omega_0} x\cdot(-\vec{n}(x)) dS = \int_{\partial\Omega} (-x\cdot\vec{n}) dS,$$

there exist a positive number ρ and a relatively open subset $\Gamma_0 \subset \partial\Omega$ such that $(-x\cdot\vec{n}) \ge \rho > 0$ on $\overline{\Gamma_0}$.
Thus we have

$$\int_{\Gamma_0} |\nabla w_n^\varepsilon|^p dS < \eta \quad \forall n \ge N, \ \forall\varepsilon \in (0, \varepsilon_0). \tag{2}$$

Recall that $\nabla w_n^\varepsilon \to \nabla w_n$ strongly in $L^p(\Omega_n)$ as $\varepsilon \to 0$, in particular, $w_n^\varepsilon \to w_n$ strongly in $L^p(\Gamma_0)$. Then by (2), there exists a subsequence $w_n^{\varepsilon_k}$ such that $\nabla w_n^{\varepsilon_k} \rightharpoonup \nabla w_n$ weakly in $L^p(\Gamma_0)$, whence follows

$$\int_{\Gamma_0} |\nabla w_n|^p dS < \eta \quad \forall n \ge N. \tag{3}$$

Let x_0 be arbitary point in Γ_0, then there exist a sufficiently small number $\ell > 0$ and $y \in \Omega$ such that $x_0 \in \partial B_{2\ell}(y) \cap \partial\Omega$. Put $v(x) = \alpha(3\ell - r)^\delta - \alpha\ell^\delta$, $r = |x-y|$, $\alpha > 0$, $\delta > 0$. Then a simple calculation shows that for a sufficiently small ℓ, α, and sufficiently large δ, v satisfies

$$-\Delta_p v + v^{q-1} \le 0. \quad \text{in } \Omega_\ell = B_{2\ell}(y) \setminus \overline{B_\ell(y)} \tag{4}$$

On the other hand, recalling that $u_n(x) \uparrow u(x)$ in Ω and $w_n(x) \ge 0$ in Ω_n, so $w_{n+1}|_{\partial\Omega_n} \ge 0 = w_n|_{\partial\Omega_n}$, we can apply a comparison theorem (see [9, Lemma 3]), to deduce $\bar{w}_n(x) \uparrow w(x)$ on Ω.
Hence, since the Harnack principle (see [11, Trudinger]) assures that $w_n(x) > 0$ for all $x \in \Omega_n$, there exists a sufficiently small α such that $v|_{\partial B_\ell(y)} = \alpha(2^\delta - 1)\ell^\delta \le w_n|_{\partial B_\ell(y)}$

hold for all n. Again by the comparison theorem, we see that $v(x) \leq w_n(x)$ in Ω_ℓ for all n.

In particular we get

$$v(x_0 + t\vec{n}(x_0)) - v(x_0) \leq w_n(x_0 + t\vec{n}(x_0)) - w_n(x_0).$$

Dividing both sides by $t < 0$ and letting $t \to -0$, we finally obtain

$$0 > -\alpha\delta\ell^{\delta-1} = \frac{\partial v}{\partial n}(x_0) \geq \frac{\partial w_n}{\partial n}(x_0) \quad \text{for all } n,$$

which contradict (3).

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ある準線型拡散方程式系の正値定常解について

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以下の研究は山田義雄先生との共同研究です.

1 問題

考える問題は次の準線型拡散方程式系からきている.

$$
(1) \qquad \begin{cases} u_t = \Delta\{(1+\alpha v)u\} + au(1-u-bv) & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta\{(1+\beta u)v\} + dv(1+cu-v) & \text{in } \Omega \times (0,\infty), \\ u = v = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0, v(\cdot,0) = v_0 & \text{in } \Omega, \end{cases}
$$

ただしΩはなめらかな境界$\partial\Omega$をもつ有界領域, α,βは非負定数, a,b,c,d は正定数である.

この方程式は数理生物学に現れ, えじきと捕食者の関係にある２つの生物 A,B の個体数変化を表したものである. uをＡの個体数密度vをＢの個体数密度とする. １番目の方程式に着目すると, $b>0$ であるから, ＡはＢに食われて減少し, また仲間同士の競争により, 自分自身の増加によっても増殖率は減少する. 一方２番目の方程式に着目すると（$c>0$ に注意), ＢはＡを食って増加し, 競争により自分自身の増加は増殖率に負の効果を及ぼす. 最も広く知られている prey-predator（餌食と捕食者）model は

$$
(2) \qquad \begin{cases} u_t = \Delta u + au(1-u-bv) & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta v + dv(1+cu-v) & \text{in } \Omega \times (0,\infty), \\ u = v = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0, v(\cdot,0) = v_0 & \text{in } \Omega. \end{cases}
$$

であるから, (1) ではu,vの拡散係数にu,vが関与しているという, より現実に近い model になっている. 詳しくは [3]. (2) の方程式系の正値解の存在, 一意性については [1][2] 参照のこと.

(1) の方程式系の正値定常解について調べる；すなわち

$$
(3) \qquad \begin{cases} \Delta\{(1+\alpha v)u\} + au(1-u-bv) = 0 & \text{in } \Omega, \\ \Delta\{(1+\beta u)v\} + dv(1+cu-v) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}
$$

をみたす正値解 (u,v)（ただし $u>0, v>0$）の存在と一意性について調べたい.

まず記号などを定義する.

任意の $q(x)\in C(\bar{\Omega})$ をあたえたとき，次の固有値問題の最小固有値を$\lambda_1(q)$ であらわす.

$$\text{(4)}\qquad \begin{cases} -\Delta u + q(x)u = \lambda u & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases}$$

$\lambda_1(q)$ は以下のように特徴付けられることが知られている.

$$\lambda_1(q) = \inf_{u\in H_0^1, \|u\|=1}\{\|\nabla u\|^2 + \int_\Omega q(x)u^2dx\}$$

特に $q(x)\equiv 0$ のとき，$\lambda_1(0)$ のかわりにλ_1とかく.

$a>\lambda_1$とすると，次の (5) は正値解をただひとつもつことが知られている．この解をϕ_aとする.

$$\text{(5)}\qquad \begin{cases} \Delta u = au(1-u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases}$$

ただし $a\le\lambda_1$のときは，$\phi_a=0$ と約束する.

このように準備すると，以下の定理が証明できる.

Theorem 1 $(i) a\le\lambda_1$ならば，(3) の正値解は存在しない.
$(ii) a>\lambda_1$とする．$\lambda_1(\dfrac{a(b\phi_d-1)}{1+\alpha\phi_d})<0,\quad \lambda_1(\dfrac{-d(c\phi_a+1)}{1+\beta\phi_a})<0$
あるいは
$\lambda_1(\dfrac{a(b\phi_d-1)}{1+\alpha\phi_d})>0,\quad \lambda_1(\dfrac{-d(c\phi_a+1)}{1+\beta\phi_a})>0$
の条件が満たされていれば，(3) の正値解が存在する.

Theorem 2 $a>\lambda_1$かつ$\alpha\le b$ とする.
$(i)\lambda_1(\dfrac{-d(c\phi_a+1)}{1+\beta\phi_a})>0$ であれば (3) の正値解は存在しない.
(ii) 特に $\beta\le c$ のとき，$\lambda_1(\dfrac{a(b\psi_d-1)}{1+\alpha\psi_d})>0$ であれば，(3) の正値解は存在しない．ただし $\psi_d=\dfrac{\phi_d}{1+\beta\phi_a}$.

Remark 1 a,d をパラメターとみる．$\alpha\le b$ のとき，Theorem 1,Theorem 2 の図を描くと Fig.1 のようになる.

Theorem 3 $N=1$ とする。$\alpha<b$, $\beta<c$, $\alpha+b\le\frac{\alpha}{\beta}$ ならば，(3) の正値解は一意である.

S_1, S_2を次のように定義する.

$$S_1 = \{(a,d) \in \mathbf{R}^2; \lambda_1(\frac{-d(c\phi_a + 1)}{1+\beta\phi_a}) = 0\},$$

$$S_2 = \{(a,d) \in \mathbf{R}^2; \lambda_1(\frac{a(b\psi_d - 1)}{1+\alpha\psi_d}) = 0\}.$$

$\beta - c$ を正にとると S_1は単調増加になる (Fig 2). さらに$\beta - c$を大きくとれば, S_1, S_2は入れ替わる (Fig 3).

Theorem 4 $(a, d^*) \in S_1$とする. a を fix すると次のいずれかが成り立つ.
(I) あるδ_1が存在して $d \in (d^*, d^* + \delta_1)$ について $(u_a, 0)$ の近くに安定な正値解が存在する.
(II) あるδ_2が存在して $d \in (d^* - \delta_2, d^*)$ について $(u_a, 0)$ の近くに不安定な正値解が存在する.

Remark 2 S_2についても同様の結果が得られる.
Remark 3 もしも Fig.2,Fig.3 で領域 Vの外側の (a, d) に対し正の分岐解が存在する場合, この分岐解の他に少なくとももう 1 つ正値解が存在することが証明できる. しかも $a, b, c, d, \alpha, \beta$がある条件を満たすようにとれば, Remark 3 で述べたことが実際におこり, 正値解が少なくとも 2 つ存在する.

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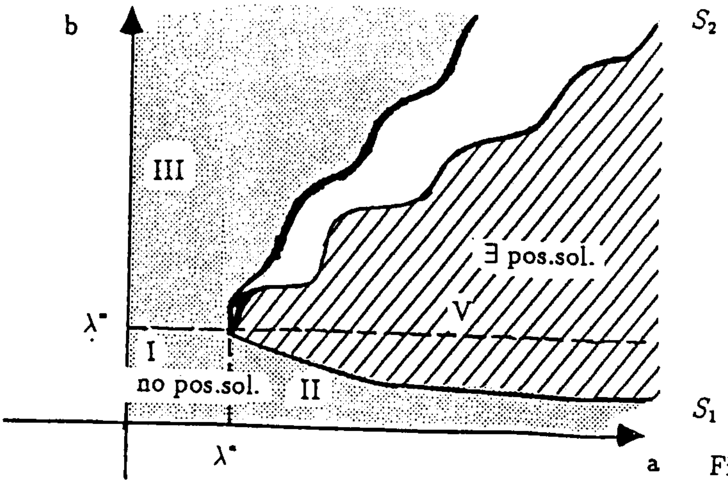

Fig.1 $d \geq \beta$ $\alpha \leq b$

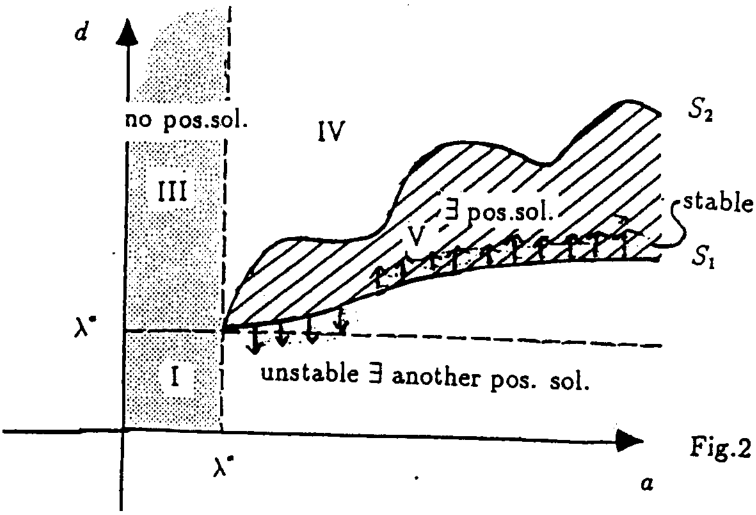

Fig.2 $d < \beta$

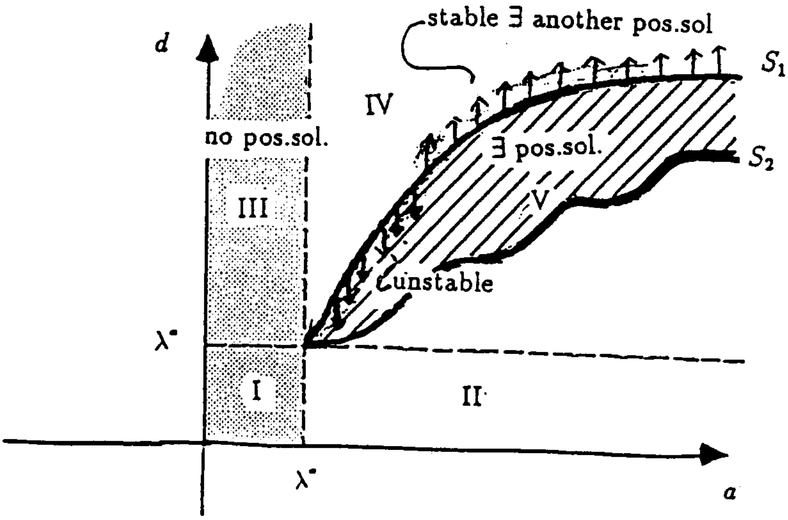

Fig.3 $d < \beta$