第１３回

発展方程式若手セミナー

報告集

龍谷大学科学技術共同研究センター

序　　書

本報告集は、平成３年８月２１日～２４日に、龍谷大学龍谷荘において開催されたシンポジウム

"第１３回発展方程式若手セミナー"

における講演報告集である。

このシンポジウムは龍谷大学科学技術共同研究センターの「フラクタルを中心とした応用解析」の研究活動の一環として、科学研究費総合研究(A)（研究代表者・大阪大学田辺広城教授）の協力の下で実施された。

このシンポジウムの準備・運営に御協力をいただいた大阪大学・川中子正氏に厚く感謝いたします。

平成４年２月

研究代表者　山口　昌哉

目 次

発展方程式理論と"双曲型"偏微分方程式

東京大学理学部 堤誉志雄

§0. 序

放物型発展方程式の理論は、作用素の分数巾の理論と結びつき、線形及び非線形の放物型偏微分方程式の研究に大きな役割りを果たした。しかし、放物型でない発展方程式、いわゆる双曲型発展方程式の理論は、いまだに未完成と言って良い部分も多く、発展方程式理論の放物型でない偏微分方程式に対する有効性に疑問を投げかける意見もある。今回このノートにおいて、双曲型発展方程式理論の論文の中で、非線形偏微分方程式の研究に最も大きな影響を与えた論文の一つであるT. Katoに

よる[4]の論文の一部を概説し、双曲型発展方程式理論の有効性とその問題点を考えてみたい。

現在の双曲型発展方程式理論が適用できない物理的、工学的に重要な偏微分方程式はたくさんあるにもかかわらず、非線形双曲型偏微分方程式の分野で高い評価を受けている双曲型発展方程式理論の結果も少なからずある。T. Kato による [4] はその代表例である。

(例えば、[8] の 31ページの 9行目から12行目及び Remark 1 の Majda のコメントを参照)

しかし、双曲型発展方程式の理論は難解であり、[9], [11] などの日本語の秀れた解説書があるにもかかわらず、初心者が勉強をするにはかなりの苦労が必要なように思われる。(これは

筆者自身の経験から出て来た感想である。）このノートが双曲型発展方程式を勉強する際の一助となり、また今まであまり興味のなかった人がこのノートにより双曲型発展方程式に興味を持って下されば、筆者の喜びとするところである。

§1. [4] の内容の概略

[4] で T. Kato は次のような非線形発展方程式の時間局所解の構成理論を提出した。

$$(Q)\quad \frac{du}{dt} + A(t, u(t))u = f(t, u(t)),\quad 0 \le t \le T,$$

$$u(0) = \phi$$

$A(t, u(t))$ は $u(t)$ を1つ定めれば、双曲型発展作用素の生成作用素となるようなものである。

（ここで、双曲型発展作用素とは、放物型でないということであり、いわゆる偏微分方程式論における

双曲型とは意味が異なり、ずっと広いクラスである。)

[4]では(Q)を解くために、まず(Q)の非線形方程式を線形化して解き、その後に不動点定理(縮少写像の原理)によって(Q)の解を求めるという方針を取っている。そのため、[4]のPart 1では、まず線形双曲型発展方程式の理論が証明無しで述べられている。Part 2では、(Q)の線形化方程式によって写像を定義し、その不動点を求めるという方針で、(Q)の時間局所解の構成理論が示されている。Part 2で述べられている証明はそれ自身興味深い工夫がなされている。(例えば、このノートの§4のConcluding remark (i)を参照.) Part 3では、Part 2の結果の応用例が豊富に挙げられている。

T. Katoの着眼点の良さの一つは、問題を時間局所解の構成に限った点である。本来 発展方程式理論のような抽象論はその応用の広さが売り物の一つである。しかし、非線形方程式では時間大域解が存在するということ自体がその方程式の一つの特性と言って良く、時間大域解の存在はきわめて方程式自身の固有の性質に依存するものである。このため、時間大域解を考えて抽象論を作ろうとすると、適用範囲の狭い理論しかできないという状況におちいる可能性があるが、T. Katoは時間局所解に問題を限ることにより、非常に適用範囲の広い理論を作ることに成功した。

Part 3に挙げられている例のうちのいくつかを記しておく。

例 2.1 (非圧縮性 Euler 方程式)

$$\frac{\partial u}{\partial t} + (u, \nabla) u + \nabla p = 0, \quad 0 \le t \le T, \quad x \in \mathbb{R}^3,$$
$$\mathrm{div}\ u = 0,$$
$$u(0, x) = u_0(x),$$
$$u = (u^1(t, x), u^2(t, x), u^3(t, x)).$$

例 2.2 (一般化された K-dV 方程式)

$$\frac{\partial u}{\partial t} + u_{xxx} + a(u) u_x = 0, \quad 0 \le t \le T, \quad x \in \mathbb{R},$$
$$u(0, x) = u_0(x),$$
$$a \in C^\infty(\mathbb{R}).$$

§3. 線形双曲型発展方程式

このセクションでは、[4]の Part I で述べられている線形双曲型発展方程式の理論について、検討してみたい。次のような線形発展方程式を考える。

(L) $\dfrac{du}{dt} + A(t) u = f(t), \quad 0 \le t \le T$

$u(0) = \phi.$

次のように記号を定義する。

X, Y ; Banach空間で、$\|\cdot\|_X$, $\|\cdot\|_Y$をそれぞれ XとYのノルムとする。

$B(X,Y)$; XからYへの有界線形作用素全体からなるBanach空間とし、そのノルムを $\|\cdot\|_{B(X,Y)}$と書く。

$B(X)$; $B(X, X)$をこのように略記する。

$G(X, M, \beta)$, $M>0$, $\beta \geq 0$; Xにおける次のような半群の生成作用素Aの集合とする。

$$\|e^{-tA}\|_{B(X)} \leq M e^{\beta t}, \quad t \geq 0.$$

<u>Remark</u> 3.1 Hille-Yosidaの定理

より、$G(X, M, \beta)$の要素Aは次のように、Aのレゾルベントを使って特徴付けられる。

$A \in G(X, M, \beta)$

$\Longleftrightarrow$ A ; Xで稠密に定義された閉作用素

$\lambda > \beta$ならλはAのレゾルベント集合に属し、

$$\|(A+\lambda)^{-k}\|_{B(X)} \leq M(\lambda-\beta)^{-k}, \quad \lambda > \beta, \quad k \in \mathbb{N}.$$

3.1. 発展作用素の存在定理

次のような仮定をする。

[A.1] ある $M, \beta > 0$ に対し

$$\{A(t)\}_{0 \le t \le T} \subset G(X, M, \beta)$$

かつ $\{A(t)\}$ は次のような意味で安定(stable)である。即ち、任意の $0 \le t_1 < \cdots < t_k \le T$ となる有限列 $\{t_j\}_{j=1}^{k}$ に対して、

(3.1) $\left\| \prod_{j=1}^{k} (A(t_j) + \lambda)^{-1} \right\|_{B(X)} \le M(\lambda - \beta)^{-k}, \quad \lambda > \beta,$

が成立する。但し、(3.1)の積は時間の順番通りに作用するものとする。

[A.2] あるBanach空間Yと同型写像 $S; Y \mapsto X$ が存在し、次を満たす。

(3.2) $SA(t)S^{-1} = A(t) + B(t), \quad B(t) \in B(X),$
$0 \le t \le T.$

ここで、$B(t)$ は強可測(即ち、$t \mapsto B(t)x$ が強可測)

で、$\|B(t)\|_{B(X)}$ は t の関数として $[0,T]$ 上で upper integrable であるとする。

[A.3] [A.2] で与えられる Banach 空間 Y に対して、

$$Y \subset D(A(t)), \quad 0 \le t \le T$$

(これと閉グラフ定理より、$A(t) \in B(Y, X)$ となる。)

さらに、$A(t)$ は $t \longmapsto A(t) \in B(Y, X)$ の関数として強連続であるとする。

<u>Remark</u> 3.2. (i) (3.1) の条件は次の (3.3) と同値である。

(3.3) $$\left\| \prod_{j=1}^{k} e^{-s_j A(t_j)} \right\|_{B(X)} \le M e^{\beta(s_1 + \cdots + s_k)},$$

$$0 \le t_1 < \cdots < t_k \le T, \quad s_j \ge 0.$$

([9], [11] を参照)

(ii) 各 $t \in [0,T]$ に対して、$A(t) \in G(X, 1, \beta)$

なら、(1.1)の条件は自動的に満たされる。なぜなら、

$$\| \prod_{j=1}^{k} (A(t_j)+\lambda)^{-k} \|_{B(X)} \leq \prod_{j=1}^{k} \|(A(t_j)+\lambda)^{-1}\|_{B(X)}$$

$$\leq (\lambda-\beta)^{-k}, \quad \lambda > \beta.$$

(iii) [A.2]の upper integrable の定義については、[13]を参照せよ。

双曲型発展作用素の存在定理を述べる。

<u>Theorem</u> 3.1. [A.1]～[A.3]を仮定する。そのとき、$\Delta = \{(t,s) \in \mathbb{R}^2 ;\ T \geq t \geq s \geq 0\}$上で以下の性質を持つ発展作用素$U(t,s)$が唯一つ存在する。

(a) $U(t,s)$は$\Delta \mapsto B(X)$の写像として強連続で、$U(s,s)=I$である。

(b) $U(t,s)U(s,r) = U(t,r)$,

$$0 \leq r \leq s \leq t \leq T.$$

(c) $U(t,s)Y \subset Y, \quad (t,s) \in \Delta$

かつ U は $\Delta \mapsto B(Y)$ の写像として強連続である。

(d) $(t,s) \in \Delta$ に対して、

$$\frac{dU(t,s)y}{dt} = -A(t)U(t,s)y, \quad y \in Y,$$

$$\frac{dU(t,s)y}{ds} = U(t,s)A(s)y, \quad y \in Y.$$

<u>*Remark*</u> 3.3. (i) *Theorem* 3.1は、[A.3]の仮定のA(t)が強連続より強い作用素ノルムで連続という条件の下で、最初にT. Katoによって証明された。([2], [3]を参照) その後、色々な人によって、[A.3]のような条件の下で証明されたが、ここでは *K. Kobayasi* による[6]を挙げておく。

(ii) 適当な条件の下で、$\mathcal{S}$はtに依存しても

良い。 ([3], [6], [9], [11]を参照)

このノートではTheorem 3.1の証明は省略する。その代わり、簡単な例により、仮定[A.1], [A.2], [A.3]の意味するところを検討してみる。

例 3.1. 次のような1階単独双曲型方程式を考える。

$$\frac{\partial u}{\partial t} + a(t,x)\frac{\partial u}{\partial x} = 0 , \quad 0 \leq t \leq T, \quad x \in \mathbb{R},$$
$$u(0,x) = u_0(x)$$

ここで、

$$a(t,x) \in W^{1,\infty}((0,T)\times\mathbb{R})$$

とする。次のようにおく。

$$X = L^2(\mathbb{R}),$$
$$A(t) = a(t,x)\frac{\partial}{\partial x},$$
$$D(A(t)) = \{C_0^1(\mathbb{R}) \text{のA(t)のグラフノルムの意味での完備化}\} \ (\supset H^1(\mathbb{R}))$$

以下、順次[A.1]〜[A.3]を確かめる。

([A.1]について) $(\cdot\,,\cdot)$ を X における内積

とすると、簡単な計算より

$$
\begin{aligned}
&\|A(t)u+\lambda u\|_X\ \|u\|_X \\
&\quad \geq (A(t)u+\lambda u\,,\,u) \\
&\quad = (a(t,x)\,u_x+\lambda u\,,\,u) \\
&\quad = -\tfrac{1}{2}(a_x(t,x)u\,,u)+\lambda\|u\|_X^2 \\
&\quad \geq (\lambda-\beta)\|u\|_X^2\ ,\quad \lambda\in\mathbb{R}\,,\ u\in C_0^1(\mathbb{R}).
\end{aligned}
$$

但し、

$$\beta=\frac{1}{2}\sup_{\substack{x\in\mathbb{R}\\ t\in[0,T]}}\left|\frac{\partial}{\partial x}a(t,x)\right| .$$

これと、$C_0^1(\mathbb{R})$ が $D(A(t))$ で稠密であることより、

$$(3.4)\quad \|(A(t)+\lambda)u\|_X\geq(\lambda-\beta)\|u\|_X\ ,$$

$$\lambda>\beta\ ,\quad u\in D(A(t)).$$

$\lambda>\beta$ に対しては、(3.4)より $A(t)+\lambda$ は有界な

逆を持つことが分かる。 $A(t)+\lambda$ $(\lambda>\beta)$ が

レゾルベント作用素であることを示すには、後は

$R(A(t)+\lambda)$ が $\lambda>\beta$ に対しては X で稠密であることを示せば良い。そのため、$A(t)$ の共役作用素 $A^*(t)$ を考える。形式的には、

$$A^*(t)u = -\frac{\partial}{\partial x}(a(t,x)u)$$

であるので、前と同様にして、

$$\|(A^*(t)+\lambda)u\|_X \geq (\lambda-\beta)\|u\|_X, \quad \lambda>\beta.$$

よって、$N(A^*(t)+\lambda)=R(A(t)+\lambda)^{\perp}=\{0\}$ となり、$R(A(t)+\lambda)$ $(\lambda>\beta)$ は X で稠密となる。

従って、$A(t)+\lambda$ $(\lambda>\beta)$ は $A(t)$ のレゾルベント作用素となる。($A(t)$ の共役作用素をきちんと決めるのは、少しやっかいである。例えば、[12] を参照せよ。) さらに、(3.4) より、

$$\|(A(t)+\lambda)^{-k}\|_{B(X)} \leq \|(A(t)+\lambda)^{-1}\|_{B(X)}^{k} \leq (\lambda-\beta)^{-k}, \quad \lambda>\beta,\ k\in \mathbb{N}.$$

Hille-Yosida の定理より、

$$A(t) \in G(X, 1, \beta), \quad t \in [0, T].$$

Remark 3.2 (ii) より、(3.1) は成立し、従って [A.1] は成立する。

<u>Remark</u> 3.4. 例 3.1 の方程式の L^2 エネルギーを計算すると、

$$\frac{1}{2}\frac{d}{dt}\|u\|_X^2 = +\frac{1}{2}(a_x(t,x)u, u)$$
$$\leq \beta \|u\|_X^2 .$$

両辺を t で積分し、式を整理すると、

$$\frac{\|u(t)\|_X}{\|u(0)\|_X} \leq e^{\beta t}, \quad 0 \leq t \leq T.$$

$u(0) \mapsto u(t)$ という写像を考えると、これはだいたい (3.3) を意味している。従って、[A.2] は、偏微分方程式の言葉で言い換えると、空間 X におけるエネルギー不等式が成立するという条件

となる。

（[A. 2]について）次のようにおく。

$Y = H^1(\mathbb{R})$,

$S = (1 - \frac{\partial}{\partial x})$

Sは$Y \mapsto X$の同型写像である。なぜなら、

部分積分により、

$$\|Su\|_X^2 = (u - u_x, u - u_x)$$

$$= \|u\|_X^2 + \|u_x\|_X^2 = \|u\|_Y^2, \quad u \in Y.$$

作用素A, Bに対して交換子（commutator）

[A, B]を次のように定義する。

$$[A, B] = AB - BA$$

すると

$$B(t) = [S, A(t)]S^{-1} = SA(t)S^{-1} - A(t)$$

$$= (1 - \frac{\partial}{\partial x})\left\{a(t,x)\frac{\partial}{\partial x}(1 - \frac{\partial}{\partial x})^{-1}\right\} - a(t,x)\frac{\partial}{\partial x}$$

$$= -\frac{\partial a(t,x)}{\partial x}\frac{\partial}{\partial x}(1 - \frac{\partial}{\partial x})^{-1}$$

よって、

$$\|B(t)\|_{B(X)} \le \sup_{\substack{x\in\mathbb{R}\\ t\in(0,T)}} \left|\frac{\partial a}{\partial x}(t,x)\right| .$$

従って、〔A.2〕は成立する。

<u>Remark</u> 3.5. (i) 具体的な問題に応用する際、Sの選び方は一意ではない。上の例でも $S=(1-\frac{\partial^2}{\partial x^2})^{-1/2}$ と取っても良い。しかし、Sの選び方が大きなpointとなることが多い。

(ii) 上の計算で分かるように、応用の際に[A.2]を確かめることは、SとAの交換子$[S, A(t)]S^{-1}$を計算するのとほぼ同じである。一方、例3.1の方程式の$H^1(\mathbb{R})$のエネルギーを計算すると、

$$\frac{\partial}{\partial t}(Su) + a(t,x)\frac{\partial}{\partial x}(Su) + [S, A(t)]u = 0$$

であるから、

$$\frac{1}{2}\frac{d}{dt}\|Su\|_X^2 \le \frac{1}{2}\beta\|Su\|_X^2 + ([S, A(t)]u, u)$$

$$= \frac{1}{2}\beta\|Su\|_X^2 + ([S, A(t)]S^{-1}(Su), u)$$

$$\le \frac{1}{2}\beta\|Su\|_X^2 + \|[S, A(t)]S^{-1}\|_{B(X)}\|Su\|_X\|u\|_X .$$

従って、もし

(3.5) $\|[S, A(t)]S^{-1}\|_{B(X)} \in L^1(0, T)$

ならば、Gronwall の不等式より、Y におけるエネルギー不等式が成立することとなる。(3.5) は [A.2] の条件とほぼ同値であり、結局 [A.2] は偏微分方程式の言葉で言い換えると、Y でのエネルギー不等式が ~~成立することを保証~~ する条件であるということになる。

([A.3] について) $Y \subset D(A(t))$ は明らか である。また、簡単な計算により、

$$A(t)u - A(s)u = a(t,x)\frac{\partial u}{\partial x} - a(s,x)\frac{\partial u}{\partial x}$$

$$= \int_0^1 \frac{\partial}{\partial \theta} a(\theta t + (1-\theta)s, x)\, d\theta \cdot \frac{\partial u}{\partial x}$$

$$= \int_0^1 \left(\frac{\partial}{\partial t} a\right)(\theta t + (1-\theta)s, x)\, d\theta\, (t-s) \frac{\partial u}{\partial x}$$

よって、

$$\| A(t)u - A(s)u \|_X$$

$$\leq |t-s| \sup \left|\frac{\partial}{\partial t} a(t,x)\right| \left\| \frac{\partial u}{\partial x} \right\|_X$$

従って、 $t \mapsto A(t) \in B(Y,X)$ は強連続となる。

これより、[A.3] が成立する。

<u>Remark</u> 3.6. $a(t,x) \in W^{1,\infty}((0,T)\times \mathbb{R})$

としたので、$t \mapsto A(t) \in B(Y,X)$ は実際は作用素ノルムの意味で連続となった。しかし、

$$a \in C([0,T]\times \mathbb{R}), \quad a, \frac{\partial a}{\partial x} \in L^\infty((0,T)\times \mathbb{R})$$

と弱めると、$t \mapsto A(t) \in B(Y,X)$ は強連続にしかならない。

以上 [A.1]〜[A.3] が確かめられたので、発展作用素 $U(t,s)$ が存在し、$u_0 \in Y$ に対して $U(t,0)u_0$ は解となる。

非線形問題に応用する際には、$U(t,s)$ の評価が必要であるので、*Theorem* 3.1 だけでは不十分である。記号を導入する。

$$\|U\|_{\infty,X} = \sup_{t,s\in\Delta} \|U(t,s)\|_X .$$

<u>*Theorem*</u> 3.2. *Theorem* 3.1 の仮定の下で

$$\|U\|_{\infty,X} \le M e^{\beta T},$$

$$\|U\|_{\infty,Y} \le \|S\|_{B(Y,X)} \|S^{-1}\|_{B(X,Y)} \times M e^{\beta T + M\|B\|^*_{1,X}}$$

但し、

$$\|B\|^*_{1,X} = \int_0^T (*) \|B(t)\|_{B(X)} dt$$

ここで、(*) は *upper integral* を表わす。

3.2. (L)の解

発展作用素 $U(t,s)$ が求まれば、(L)の解は形式的に次のように書ける。

$$(S)\quad u(t) = U(t,0)\phi + \int_0^t U(t,s) f(s)\,ds, \quad 0 \leq t \leq T.$$

(S)で与えられる $u(t)$ が実際に(L)の解となるためには、ϕ と $f(t)$ について条件が必要である。

<u>Theorem</u> 3.3. Theorem 3.1と同じ仮定が成立するものとし、$u(t)$ は(S)によって定義されたものとする。

(a) $\phi \in X$, $f \in L^1(0,T;X)$

$\Longrightarrow u \in C([0,T];X)$.

(b) $\phi \in Y$, $f \in L^1(0,T;Y)$

$\Longrightarrow u \in C([0,T];Y)$.

(c) $\phi \in Y$, $f \in C([0,T];X) \cap L^1(0,T;Y)$

$\Longrightarrow$ $u \in C([0,T];Y) \cap C^1([0,T];X)$で、

$u(t)$は(L)を満たす。

さらに、

(3.6) $\|u\|_{\infty,X} \leq \|V\|_{\infty,X} (\|\phi\|_X + \|f\|_{1,X})$,

(3.7) $\|u\|_{\infty,Y} \leq \|V\|_{\infty,Y} (\|\phi\|_Y + \|f\|_{1,Y})$,

(3.8) $\|\frac{du}{dt}\|_{\infty,X} \leq \|f\|_{\infty,X}$

$+ \|A\|_{\infty,B(Y,X)} (\|\phi\|_Y + \|f\|_{1,Y})$.

但しここで、

$$\|f\|_{\infty,X} = \sup_{0 \leq t \leq T} \|f(t)\|_X ,$$

$$\|f\|_{1,X} = \int_0^T \|f(t)\|_X \, dt ,$$

$$\|A\|_{\infty,B(Y,X)} = \sup_{0 \leq t \leq T} \|A(t)\|_{B(Y,X)} .$$

である。

§4. 非線形発展方程式

このセクションでは、次のような非線形発展方程式の時間局所解の存在定理を述べ、[4]に従って証明を概説する。

(Q) $\dfrac{du}{dt} + A(t,u)u = f(t,u),\ 0 \le t \le T,$

$u(0) = \phi$

まず、仮定を述べる。

[X] XとYは回帰的Banach空間で、YはXに連続かつ稠密に埋め込まれているものとする。さらに、YからXへの同型写像Sが存在し、$\|Sx\|_X = \|x\|_Y \ (x \in Y)$とする。

[B.1] あるYの開球Wと非負の定数βが存在して、次が成立するとする。

$\|e^{-sA(t,y)}\|_{B(X)} \le e^{\beta s},\ s \ge 0,\ 0 \le t \le T,\ y \in W.$

[B.2] ある正数λ_1が存在して、すべての$(t,y)\in[0,T]\times W$に対して、次が成立する。

$$SA(t,y)S^{-1}=A(t,y)+B(t,y),$$

$$B(t,y)\in B(X),\quad \|B(t,y)\|_{B(X)}\le\lambda_1.$$

[B.3] $A(t,y)\in B(Y,X)$, $(t,y)\in[0,T]\times W$ かつ各$y\in W$に対して、$t\mapsto A(t,y)\in B(Y,X)$は強連続であるとする。さらに、ある$\mu_1>0$に対して、次が成立するものとする。

$$\|A(t,y)-A(t,z)\|_{B(Y,X)}\le\mu_1\|y-z\|_X,$$

$$t\in[0,T],\quad y,z\in W.$$

[B.4] y_0をWの中心とすると、ある$\lambda_2>0$に対して、

$$\|A(t,y)y_0\|_Y\le\lambda_2,\quad (t,y)\in[0,T]\times W.$$

[f] ある $\lambda_3 > 0$ に対して次が成立するとする。

$$\|f(t,y)\|_Y \leq \lambda_3, \quad (t,y) \in [0,T] \times Y.$$

さらに、各 $y \in W$ に対して

$$f(t,y) \in C([0,T]; X)$$

で、ある $\mu_2 > 0$ に対して次が成立するとする。

$$\|f(t,y) - f(t,z)\|_X \leq \mu_2 \|y - z\|_X,$$

$$t \in [0,T], \quad y, z \in W.$$

<u>Remark</u> 4.1. (i) 応用上は、$A(t,y)$ はすべての $y \in Y$ に対して定義されることが多く、従って、たいていの場合、W を 0 を中心とした球と取れる。$y_0 = 0$ のときは、[B.4] は自動的に満たされる。

(ii) [B.2] が成立するための十分条件は、$A(t,y)$ の core (これは、t と y に依存して変化して

良い）に属するすべての w に対して、

$$\|[S,\ A(t,y)]\,S^{-1}w\|_X \le \lambda_1 \|w\|_X$$

が成立することである。（証明は[3]を参照せよ。）但し、集合 $F \subset D(A(t,y))$ が $A(t,y)$ の core であるとは、$\{(x,\ \ ) \in F \times X;\ \ = A(t,y)x\}$ が、グラフノルムの意味で、$A(t,y)$ のグラフが作る空間で稠密であることである。

[4] の主定理を述べる。

<u>Theorem</u> 4.1. [X], [B.1] ~ [B.4], [f] が成立するものとする。任意の $\phi \in W$ に対して、ある $T'>0$ $(0<T'\le T)$ が存在し、$[0,T']$ 上で (Q) は次のようなクラスの解 $u(t)$ を唯一つ持つ。

$$u(t) \in C([0,T'];W) \cap C^1([0,T'];X),$$

$$u(0)=\phi.$$

Theorem 4.1の証明をするために必要な補題を2つ述べる。

<u>Lemma</u> 4.1. [X]を仮定する。

AはYにおける有界閉凸集合とする　、AはXにおいても閉である。

<u>Lemma</u> 4.2. [X]を仮定する。関数 $g(t)$; $t \in [0,T] \mapsto X$ は有界で(強可測は仮定しない)かつ $g \in C([0,T];X)$とする。そのとき、$g \in C_w([0,T];Y)$である。 ここで、$C_w([0,T];Y)$はYの弱位相の意味で連続な関数の集合を表わす。

<u>Remark</u> 4.2. (i) Lemma 4.1と4.2は、回帰的Banach空間では有界集合は弱点列コンパクトであるということと、凸集合に対しては

強閉と弱閉が一致するということから示される。

(ii) [4] で与えられている Theorem 4.1 の証明では、Lemma 4.1 と 4.2 を本質的に用いるため、X と Y は回帰的でなければならない。最近、[10], [6] において、X と Y の回帰性を仮定しなくとも、全く同様な結果が得られることが示された。[10], [6] の証明は非常に巧妙であるが、[4] の証明は回帰性を仮定している分だけ非常に簡潔であり、その手法は偏微分方程式を個別に扱う際にも有効なものである。

Theorem 4.1 の証明を述べる。

<u>Proof of Theorem 4.1</u> $\phi \in W$ とすると、仮定から、

$$^{\exists}R>0;\ \|\phi-y_0\|_Y<R,$$

$$\{\|y-y_0\|_Y\le R\}\subset W.$$

ここで、

$$E=\{v;[0,T']\mapsto Y;$$

$$\|v(t)-y_0\|_Y\le R,\ v\in C([0,T'];X)\}$$

とおく。但し、T'は $0<T'\le T$で後で十分小さく選ぶ定数である。$v(t)\in E$なら、すべての$t\in[0,T']$に対して$v(t)\in W$であることを注意しておく。Eに距離を次のように入れる。

$$d(v,w)=\sup_{0\le t\le T'}\|v(t)-w(t)\|_X,\ v,w\in E.$$

Lemma 4.1より、Eは完備距離空間である。

$v\in E$に対して、次のようにおく。

$$A^v(t)=A(t,v(t)),$$

$$f^v(t)=f(t,v(t)).$$

このとき、$v\in E$に対して次のような(Q)の線形化方程式を考える。

$$(L^v)\quad \frac{du}{dt}+A^v(t)u=f^v(t),\quad 0\le t\le T',$$

$$u(0)=\phi\in W\subset Y.$$

もし各$v\in E$に対して(L^v)が解$u(t)$を持つならば、

$$\Phi;\ v\in E\mapsto u$$

という写像が定義できる。そこで、ΦがEからEへの縮少写像となっていれば、ΦはEの中に不動点を唯一つ持ち、(L^v)の定義より、その不動点が(Q)の解となる。以下、$T'>0$を十分小さく取れば、ΦがEからEへの縮少写像となることを示す。証明は2つのステップからなる。

（ステップ1）各$v \in E$に対して、(L^v)は解

$u(t) \in C([0,T'];Y) \cap C^1([0,T'];X)$を持つことを示す。

そこで、$A^v(t)$が発展作用素を生成することを示す。

§3の[A.1]〜[A.3]を確かめれば良い。

（[A.1]について）[B.1]より、各$t \in [0,T']$

に対して、$A^v(t) \in G(X,1,\beta)$であるから、

Remark 3.2 (ii)より、[A.1]は成立する。

（[A.3]について）[A.2]より先に[A.3]

を示す。[B.3]より、

$$Y \subset D(A^v(t)),$$

$$A^v(t) \in B(Y,X).$$

さらに、$y \in Y$に対して、[B.3]より、

$$\|A^v(t')y - A^v(t)y\|_X$$
$$\leq \|A(t',v(t'))y - A(t',v(t))y\|_X$$
$$+\|A(t',v(t))y - A(t,v(t))y\|_X$$

$$\leq \mu_1 \| v(t') - v(t) \|_X \| y \|_Y$$

$$+ \| A(t', v(t))y - A(t, v(t))y \|_X$$

$$\longrightarrow 0 \quad (t' \to t).$$

よって、$t \mapsto A^v(t) \in B(Y, X)$ は強連続である。従って、〔A.3〕が成立する。

（〔A.2〕について）〔B.2〕より、

$$S A^v(t) S^{-1} = A^v(t) + B^v(t), \; t \in [0, T'],$$

$$B^v(t) \in B(X), \quad \| B^v(t) \|_{B(X)} \leq \lambda_1 .$$

後は、$t \mapsto B^v(t) \in B(X)$ が弱連続であることを示せば十分である。$y \in Y$ に対して、

$$S^{-1} B^v(t) y = A^v(t) S^{-1} y - S^{-1} A^v(t) y$$

〔A.3〕をチェックする際に示したように、$A^v(t)$ は $t \mapsto A^v(t) \in B(Y, X)$ として強連続なので、右辺は X の位相で強連続である。一方、〔B.2〕

より、

$$\|S^{-1}B^{\nu}(t)\|_{B(X)} \leq \|S^{-1}\|_{B(X)}\lambda_1$$

かつ Yは Xに稠密に埋め込まれているので、

各$x \in X$に対して、$t \mapsto S^{-1}B^{\nu}(t)x$ は Xの

位相で強連続となる。ここで、

$$\|S^{-1}B^{\nu}(t)x\|_Y = \|B^{\nu}(t)x\|_X \leq \lambda_1\|x\|_X$$

なので、Lemma 4.2より、$t \mapsto S^{-1}B^{\nu}(t)x$

は Yの位相で弱連続となる。$S^{-1} \in B(X, Y)$

なので、$t \mapsto B^{\nu}(t)x$ は Xの位相で弱連続

である。従って、[A.2]は成立する。

以上、[A.1]〜[A.3]が確かめられた

ので、$A^{\nu}(t)$は発展作用素$U^{\nu}(t,s)$を生成する。

さらに、[f]より、

(4.1) $\|f^{\nu}(t)\|_Y \leq \lambda_3$, $\nu \in E$.

そして、やはり [f] より、

$$\|f^v(t') - f^v(t)\|_X$$

$$\leq \|f(t', v(t')) - f(t', v(t))\|_X$$

$$+ \|f(t', v(t)) - f(t, v(t))\|_X$$

$$\leq M_2 \|v(t') - v(t)\|_X$$

$$+ \|f(t', v(t)) - f(t, v(t))\|_X$$

$$\longrightarrow 0 \quad (t' \to t).$$

よって、$f^v(t) \in C([0,T'];X)$ となり、

Lemma 4.2 より、

$$f^v(t) \in C_w([0,T'];Y).$$

従って、$f^v(t)$ は Theorem 3.3(c) の条件を満たすので、Theorem 3.1, Theorem 3.3(c) より、(L^v) は次のような解 $u(t)$ を唯一つ持つ。

$$u \in C([0,T'];Y) \cap C^1([0,T'];X).$$

（ステップ2）　ステップ1より、$\Phi; v \in E \mapsto u$ という写像が定義できるが、$T' > 0$ を十分小さく取ると Φ は E から E への縮少写像となることを示す。　まず、Theorem 3.2より、

$$(4.2) \quad \| U^v \|_{B(X)} \leq e^{\beta T'},$$

$$(4.3) \quad \| U^v \|_{B(Y)} \leq e^{(\beta + \lambda_1) T'}.$$

$\tilde{u} = u - y_0$ とおくと、

$$\frac{d\tilde{u}}{dt} + A^v(t)\tilde{u} = f^v(t) - A^v(t) y_0,$$

$$\tilde{u}(0) = \phi - y_0.$$

積分方程式に書き直すと、

$$u(t) - y_0 = U^v(t,0)(\phi - y_0)$$

$$+ \int_0^t U^v(t,s)(f^v(s) - A^v(s) y_0)\, ds.$$

[B.4] と (4.1) より、

$$\| A^v(s) y_0 \|_Y \leq \lambda_3, \quad \| f^v(s) \|_Y \leq \lambda_3, \quad 0 \leq s \leq T'$$

なので、(4.3)より

$$\|u(t)-y_0\|_Y \le e^{(\beta+\lambda_1)T'} \times \left(\|\phi-y_0\|_Y + (\lambda_2+\lambda_3)T'\right).$$

右辺は $T'>0$ を十分小さく取ると、R より小さくできるので、

$$\Phi;\ v\in E \longmapsto u=\Phi v\in E$$

とできる。 類似の議論より、$T'>0$ を十分小さく取れば、

$$d(\Phi v, \Phi w) \le \frac{1}{2} d(v, w),\quad v, w\in E$$

となることが示せる。 よって、縮少写像の原理より、Theorem 4.1 が示された。

□

本来は、[4]に出ている豊富な応用例も解説すべきであると思われるが、応用の際の

計算の要点は§3の例3.1と同じなので割愛する。最後に、いくつかの注意を述べたい。

<u>Concluding remark.</u> (i) Theorem 4.1 の証明で、Eの距離 d は $C([0,T'];X)$ のノルムと同じものである。つまり、Yのノルムは含んでいない。このことは、実際に偏微分方程式に応用する際は重要なことで、非線形項の滑らかさをあまり必要しないということに対応している。

(ii) Theorem 4.1 を適用する際は、YとSをどう取るかが問題になる。双曲型の場合、微分作用素の定義域を明確な型で決定するのは難しい。その困難を回避するため、Yを導入し、さらにXとYを仲介するSを導入した

のは、非常にすばらしいアイディアであると言える。

しかし、初期境界値問題のように、境界条件が付くと、やはりYやSは複雑にならざるをえない。そのため、[4]の理論を境界条件付きの問題に応用するのは困難がともなう。T. Kato自身により、Theorem 4.1は境界条件付きの問題に適用できるように拡張された。([5]を参照) しかし、それでもなお、発展方程式の理論を境界条件付きの場合に適用するという問題は、多くの解決されなければならない点を含んでいるように思われる。

(iii) 実際にTheorem 4.1を適用する場合、仮定のうちきちんと確かめなければなら

ないのは[B.2]だけで、残りは簡単な場合が多い。[B.2]を確かめることは、SとA(t,y)の交換子を計算することに帰着される。このように、局所解の存在の問題を交換子の計算に帰着させたのは、[4]の理論の大きなメリットである。

REFERENCES

[1] M.G. Crandall and P.E. Souganidis, Convergence of difference approximations of quasilinear evolution equations, Nonlinear Analysis, TMA, 10 (1986), 425-445.

[2] T. Kato, Linear evolution equations of "hyperbolic" type, J. Fac. Sci. Univ. Tokyo, Sect. I, 17 (1970),241-258.

[3] T. Kato, Linear evolution equations of "hyperbolic" type II, J. Math. Soc. Japan, 25 (1973), 648-666.

[4] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, Lecture Notes in Math., 448, Springer, 1975, pp. 25-70.

[5] T. Kato, Abstract differential equations and nonlinear mixed problems, Fermi Lectures, Pisa, 1985.

[6] K. Kobayasi, On a theorem for linear evolution equations of hyperbolic type, J. Math. Soc. Japan, 31 (1979), 647-654.

[7] K. Kobayasi and N. Sanekata, A method of iterations for quasi-linear evolution equations in nonreflexive Banach spaces, Hiroshima Math. J., 19 (1989), 521-540.

[8] A. Majda, Compressible fluid flow and systems of conservation laws in several space variables, Applied Math. Sciences, Vol. 53, Springer, 1984.

[9] K. Masuda, Hattenhōteishiki (in Japanese), Kinokuniya-shoten, 1975.

[10] N. Sanekata, Abstract quasi-linear equations of evolution in nonreflexive Banach spaces, Hiroshima Math. J., 19 (1989), 109-139.

[11] H. Tanabe, Hattenhōteishiki (in Japanese), Iwanami-shoten, 1975.

[12] K.O. Friedrichs, The identity of weak and strong extensions of differential operators, Trans. Amer. Math. Soc., 55 (1944), 132-151.

[13] H. Federer, Geometric Measure Theory, Springer, 1969.

Stefan problems with dynamic boundary conditions

愛木　豊彦（長崎総合科学大学　工学研究所）

1 Introduction

In this work we deal with Stefan problems with dynamic boundary conditions.We refer to [2] for the physical background.

The problem is stated as follows. Let Ω be a bounded domain in $\mathbf{R}^{\mathbf{N}}$ ($\mathbf{N} \geq 1$) with smooth boundary $\Gamma = \partial\Omega$; and let T be a fixed positive number, $Q = (0,T) \times \Omega$, $\Sigma = (0,T) \times \Gamma$. The problem, denote by $P(u_0)$, is to find a function $u = u(t,x)$ on Q satisfying

$$\begin{aligned} &u_t - \Delta\beta(u) = f \quad \text{in } Q, \\ &u(0,\cdot) = u_0 \quad \text{in } \Omega, \\ &-\frac{\partial\beta(u)}{\partial\nu} = g(t,x,\beta(u)) + \beta(u)_t \quad \text{on } \Sigma. \end{aligned}$$

Here $\beta : \mathbf{R} \to \mathbf{R}$ is a given non-decreasing function; $f : Q \to \mathbf{R}$ is a given function; u_0 is a given initial datum; $g = g(t,x,\xi) : (0,T) \times \Gamma \times \mathbf{R} \to \mathbf{R}$ is a given function which is non-decreasing in $\xi \in \mathbf{R}$ for a.e. $(t,x) \in \Sigma$; $(\partial/\partial\nu)$ denotes the outward normal derivative on Γ . For the data we postulate that

(A1) β is non-decreasing and Lipschitz continuous on $\mathbf{R}$ with $\beta(0) = 0$ and bi-Lipschitz continuous both on $(-\infty, -r_0]$ and $[r_0, \infty)$ where r_0 is a positive constant; denote by C_β a Lipschitz constant of β;

(A2) $f \in L^\infty(Q)$;

(A3-1) $g(t,x,\xi)$ is non-decreasing in $\xi \in \mathbf{R}$ for a.e. $(t,x) \in \Sigma$ and $g(\cdot,\cdot,0) \in L^\infty(\Sigma)$;

(A3-2) $g(t,x,\xi)$ is Lipschitz continuous in ξ uniformly with respect to $(t,x) \in \Sigma$; denote by C_g a Lipschitz constant of $g(t,x,\cdot)$;

(A4) $u_0 \in L^\infty(\Omega)$ and $\beta(u_0) \in W^{1,2}(\Omega)$.

In particular, when $\beta'(r) > 0$ for any $r \in \mathbf{R}$, problem $P(u_0)$ was treated by [1]. Also, in case the flux condition is of the form $-\frac{\partial\beta(u)}{\partial\nu} = g(t,x,\beta(u))$, the problem was uniquely solved in variational sence by [3]. The purpose of the present paper is to establish a existence result for problem $P(u_0)$ and to show the uniqueness of the solution in variational sence.

2 Main results

We give a notion of solution to $P(u_0)$ in the variational sence.

Definition. A function $u : [0,T] \to L^2(\Omega)$ is a weak solution of $P(u_0)$, if it satisfies the following (S1) and (S2):

(S1) $u \in C_w(0,T;L^2(\Omega)) \cap L^\infty(Q)$, $\beta(u) \in L^2(0,T;W^{1,2}(\Omega))$ and $\beta(u)|_{(t,x)\in\Sigma} \in C_w(0,T;L^2(\Gamma))$;

$$(S2) \qquad -\int_Q u\eta_t dxdt - \int_\Omega u_0\eta(0)dx - \int_\Sigma \beta(u)\eta_t d\Gamma dt + \int_\Gamma \beta(u_0)\eta(0)d\Gamma +$$

$$\int_Q \nabla\beta(u)\cdot\nabla\eta dxdt + \int_\Sigma g(t,x,\beta(u))\eta d\Gamma dt = \int_Q f\eta dxdt \quad \text{for any } \eta \in W$$

where $d\Gamma$ denotes the usual surface element on Γ and $W = \{\eta \in W^{1,2}(0,T;W^{1,2}(\Omega)); \eta(T) = 0\}$.

THEOREM 1 *Suppose that (A1),(A2),(A3-1),(A3-2) and (A4) hold. Then $P(u_0)$ has at least one weak solution u.*

THEOREM 2 *Under the same assumptions as in Theorem 1, $P(u_0)$ has at most one weak solution u.*

Next we mention a comparison result for $P(u_0)$.

THEOREM 3 *Suppose that (A1),(A2),(A3-1),(A3-2) and (A4) hold and $\bar{u}_0$ satisfies (A4). Let u (resp. $\bar{u}$) be a weak solution of $P(u_0)$ (resp. $P(\bar{u}_0)$). Then for any $t \in [0,T]$,*

$$|[u(t) - \bar{u}(t)]^+|_{L^1(\Omega)} + |[\beta(u)(t) - \beta(\bar{u})(t)]^+|_{L^1(\Gamma)}$$

$$\leq |[u_0 - \bar{u}_0]^+|_{L^1(\Omega)} + |[\beta(u_0) - \beta(\bar{u}_0)]^+|_{L^1(\Gamma)}.$$

THEOREM 4 *If condition (A3-2) is replaced by the following condition (A3-3), then Theorems 1,2 and 3 remain valid:*

(A3-3) $g(t,x,\xi)$ is locally Lipschitz continuous in ξ uniformly with respect to $(t,x) \in \Sigma$, that is, for each $M > 0$ there is a constant $C_g(M) \geq 0$ such that

$$|g(t,x,\xi) - g(t,x,\xi')| \leq C_g(M)|\xi - \xi'|$$

$$\textit{for all } \xi,\ \xi' \textit{ with } |\xi| \leq M,\ |\xi'| \leq M \textit{ and for a.e. } (t,x) \in \Sigma;$$

there are constants m_1,m_2 with $m_1 \leq m_2$ such that

$$g(t,x,\beta(m_1)) \leq 0, g(t,x,\beta(m_2)) \geq 0 \textit{ for a.e. } (t,x) \in \Sigma.$$

We shall omit proofs of Theorems 1,3 and 4.

3 Sketch of the proof of Theorem 2

This section is almost paralleled to §.6 in [3], although the situation is slightly different.

Suppose that u_1, u_2 are solutions of $P(u_0)$. Observe first that the following integral identity

$$\begin{aligned} & -\int_Q (u_1 - u_2)\eta_t dxdt - \int_\Sigma (\beta(u_1) - \beta(u_2))\eta_t d\Gamma dt \\ = & \int_Q (\beta(u_1) - \beta(u_2))\Delta\eta dxdt - \int_\Sigma (\beta(u_1) - \beta(u_2))\frac{\partial \eta}{\partial \nu} d\Gamma dt \\ & - \int_\Sigma (g(t,x,\beta(u_1)) - g(t,x,\beta(u_2)))\eta d\Gamma dt \end{aligned} \tag{3.1}$$

for any $\eta \in \tilde{W}$

where $\tilde{W} = \{\eta \in C^{2,1}(\bar{Q}); \eta(T) = 0\}$.

In order to avoid some surplus notational complicacies we introduce the following functions:

$$u(t,x) = u_1(t,x) - u_2(t,x) \text{ in } Q,$$

$$U(t,x) = \beta(u_1)(t,x) - \beta(u_2)(t,x) \text{ in } Q,$$

$$a(t,x) = \begin{cases} \frac{U(t,x)}{u(t,x)} & \text{if } u(t,x) \neq 0, \\ 0 & \text{if } u(t,x) = 0, \end{cases}$$

$$V(t,x) = \begin{cases} \frac{g(t,x,\beta(u_1)) - g(t,x,\beta(u_2))}{U(t,x)} & \text{if } U(t,x) \neq 0, \\ 0 & \text{if } U(t,x) = 0. \end{cases}$$

By the definition of solutions to $P(u_0)$ and the assumptions (A1),(A3-2),

$$u \in L^\infty(Q), \tag{3.2}$$
$$0 \leq a \leq C_\beta \text{ in } Q, \tag{3.3}$$
$$0 \leq V \leq C_\beta C_g \text{ on } \Sigma \tag{3.4}$$

Using the above notations we can rewrite (3.1) in the form

$$\int_Q u(\eta_t + a\Delta\eta)dxdt + \int_\Sigma U(-\frac{\partial \eta}{\partial \nu} - V\eta + \eta_t)d\Gamma dt = 0 \text{ for any } \eta \in \tilde{W}. \tag{3.5}$$

By virtue of (3.3) we can choose the following sequence of functions:

(C) $\{a_n\} \subset C^\infty(\bar{Q})$ such that

(C1) $|a_n - a|_{L^2(Q)} \leq C_0 n^{-1}$, (C2) $a_n \geq n^{-1}$ in Q,

(C3) $a_n = n^{-1}$ on Σ. where C_0 is a positive constant independent of n.

Making use of the introduced approximation, we formulate the regularized parabolic problems:

$$z_t + a_n \Delta z = f_0 \quad \text{in } Q, \tag{3.6}$$

$$\frac{\partial z}{\partial \nu} = -Vz - z_t \quad \text{on } \Sigma, \tag{3.7}$$

$$z(T,x) = 0 \quad x \in \Omega \tag{3.8}$$

where $f_0 \in C^\infty(\bar{Q})$ with $f_0(0,x) = 0$ for $x \in \Omega$ and $f_0 = 0$ in Γ. (3.9)

LEMMA 3.1 *The regularized problems (3.6)-(3.8) have unique solutions* $z_n \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;W^{1,2}(\Omega))$ *with* $z_n|_{(t,x)\in\Sigma} \in W^{1,2}(0,T;L^2(\Gamma)))$.

We shall omit the proof of Lemma 3.1.

REMARK 3.1 *We can take the solutions* $z_n, n = 1,2,\cdots$ *of the problem (3.6)-(3.8) as the test functions in the integral identity (3.5).*

As a consequence of assumption (C) we can conclude the following a priori bounds:

LEMMA 3.2 *For* $n = 1,2,\cdots$, *let* z_n *be the solutions of problems (3.6) - (3.8). Then there exists positive constants* K_1, K_2 *independent of* n, *such that*

$$|z_n|_{L^\infty(Q)} \leq K_1, \tag{3.10}$$

$$|\Delta z_n|_{L^2(Q)} \leq K_2\sqrt{n}. \tag{3.11}$$

proof. For simplicity we write z for z_n. We put

$$m(t) = M(T-t) \text{ for } t \in [0,T]$$

where M is any positive number.
For a.e. $t \in [0,T]$,

$$\begin{aligned}
\int_\Omega z_t(z-m)^+ dx &= -\int_\Omega a_n \Delta z (z-m)^+ dx + \int_\Omega f_0(z-m)^+ dx \\
&= \int_\Omega a_n \nabla z \cdot \nabla (z-m)^+ dx + \int_\Omega (z-m)^+ \nabla z \cdot \nabla a_n dx \\
&\quad + \int_\Gamma a(V - z_t)(z-m)^+ d\Gamma + \int_\Omega f_0(z-m)^+ dx
\end{aligned}$$

$$
\begin{aligned}
&\ge \int_\Omega a_n(\nabla(z-m)^+)^2dx - \frac{1}{2n}\int_\Omega(\nabla(z-m)^+)^2dx \\
&\quad - \frac{n}{2}\int_\Omega(\nabla a_n)^2((z-m)^+)^2dx - \frac{1}{n}\int_\Gamma z_t(z-m)^+d\Gamma \\
&\quad + \int_\Omega f_0(z-m)^+dx \\
&\ge -\frac{n}{2}|(\nabla a_n)^2|_{L^\infty(Q)}\int_\Omega((z-m)^+)^2dx - \frac{1}{n}\int_\Gamma(z_t+M)(z-m)^+d\Gamma \\
&\quad + \int_\Omega f_0(z-m)^+dx \\
&= -\frac{n}{2}|(\nabla a_n)^2|_{L^\infty(Q)}\int_\Omega((z-m)^+)^2dx - \frac{1}{2n}\frac{d}{dt}\int_\Gamma((z-m)^+)^2d\Gamma \\
&\quad + \int_\Omega f_0(z-m)^+dx.
\end{aligned}
$$

On the other hand, for a.e. $t\in[0,T]$,

$$
\begin{aligned}
&\int_\Omega z_t(z-m)^+dx \\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega((z-m)^+)^2dx - M\int_\Omega(z-m)^+dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{d}{dt}\Big[\frac{1}{2}\int_\Omega((z-m)^+)^2dx + \frac{1}{2n}\int_\Gamma((z-m)^+)^2dx\Big] \\
&\ge -\frac{n}{2}|(\nabla a_n)^2|_{L^\infty(Q)}\int_\Omega((z-m)^+)^2dx + \int_\Omega(f_0+M)(z-m)^+dx
\end{aligned}
$$

for a.e. $t\in[0,T]$.

If $M\ge|f_0|_{L^\infty(Q)}$, then

$$
\begin{aligned}
&\frac{d}{dt}\Big[\frac{1}{2}\int_\Omega((z-m)^+)^2dx + \frac{1}{2n}\int_\Gamma((z-m)^+)^2dx\Big] \\
&\ge -\frac{n}{2}|(\nabla a_n)^2|_{L^\infty(Q)}\int_\Omega((z-m)^+)^2dx
\end{aligned}
$$

for a.e. $t\in[0,T]$.

Applying the Gronwall's inequality to this inequality, we obtain that

$$\int_\Omega((z-m)^+)^2)dx \le 0 \quad \text{for any } t\in[0,T].$$

Hence

$$z(t,x) \le m(t) \le MT \text{ for a.e.}(t,x) \in Q.$$

We obtain a similar estimate for $-z(t,x)$ to the above. Therefore we have an inequality of the form (3.10) with $K_1 = |f_0|_{L^\infty(Q)} \cdot T$.

Next we show (3.11)

$$\begin{aligned}
& \int_Q a_n (\Delta z)^2 dxdt \\
= & -\int_Q z_t \Delta z dxdt + \int_Q f_0 \Delta z dxdt \\
\le & \int_\Sigma V^2 z^2 d\Gamma dt + \int_Q \Delta f_0 z dxdt \\
\le & K_1^2 C_\beta^2 C_g^2 \int_\Sigma d\Gamma dt + K_1 |\Delta f_0|_{L^1(Q)} := K_3.
\end{aligned}$$

Hence,

$$|\Delta z|_{L^2(Q)} \le (K_3 n)^{\frac{1}{2}}.$$

Q.E.D.

Let us take the solutions $z_n, n = 1, 2, \cdots$ of the regularized problems (3.6)-(3.8) as the test functions in the integral identity (3.5). Therefore for $n = 1, 2 \cdots$,

$$\int_Q u(a - a_n) \Delta z_n dxdt = \int_Q f_0 u dxdt.$$

By Lemma 3.2, assumption (C) and the definition of solution to $P(u_0)$, that is, $u \in L^\infty(Q)$,

$$\begin{aligned}
& |\int_Q u(a - a_n) \Delta z_n dxdt| \\
\le & |u|_{L^\infty(Q)} |a - a_n|_{L^2(Q)} |\Delta z_n|_{L^2(Q)} \\
\le & |u|_{L^\infty(Q)} C_0 K_2 n^{-\frac{1}{2}} \\
\to & 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence for each functions f_0 satisfying (3.9)

$$\int_Q f_0 u dxdt = 0.$$

This implies that $u_1 = u_2$ a.e. in Q. Thus Theorem 2 holds.

References

[1] T. Hintermann. "Evolution equation with dynamic boundary conditions". *Proc. Roy. Soc. Edinburgh*, A 113:43–60, 1989.

[2] R. E. Langer. "A problem in diffusion or in the flow of heat for a solid in contact with fluid". *Tôhoku Math. J.*, (1)35:260–275, 1932.

[3] M. Niezgodka and I. Pawlow. "A generalized Stefan Problem in Sevral space variables". *Applied Math. Opt.*, 9:193–224, 1983.

Viscosity solutions of nonlinear elliptic PDEs with implicit obstacle

KATSUYUKI ISHII
(Kobe Univ. of Mercantile Marine)

§1. Introduction

In this article we consider the following nonlinear elliptic partial differential equation (PDE) with implicit obstacle:

$$(1.1)\qquad \begin{cases} \max\{-\Delta u + u - f, u - Mu\} = 0 & \text{in } \Omega, \\ \max\{u - g, u - Mu\} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \in \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and M is the nonlocal operator defined as follows:

$$Mu(x) = 1 + \inf\{u(x+\xi) \mid \xi \in (\mathbb{R}^+)^N, x + \xi \in \overline{\Omega}\}.$$

It is known that the equation (1.1) is associated with the impluse control problems (cf. [1]).

Concerning the existence and uniqueness of solutions of (1.1), see [1] and [5] etc. They obtained them by assuming that there exists a subsolution $\underline{u}$ satisfying $\underline{u} \leq g \leq M\underline{u}$ on $\partial\Omega$. Without this assumption B. Perthame showed the existence and uniqueness of viscosity solutions of (1.1) (cf. [6]).

Our main purpose is to get the comparison principle and existence of viscosity solutions of (1.1) by applying the results in [3] and [2]. By these methods we can treat the nonlinear PDEs of the type (1.1) whose principal parts are in some classes of general (possibly degenerate) elliptic operators.

§2. Assumptions and Definitions

We make the following assumptions.

(A.1) $\Omega \subset \mathbb{R}^N$ is a bounded and convex domain with smooth boundary $\partial\Omega$.

(A.2) $f, g \in C(\overline{\Omega})$ and $f, g \geqq 0$ on $\overline{\Omega}$.

Let $\mathcal{O}$ be a subset of $\mathbb{R}^N$. For any function $u : \mathcal{O} \to \mathbb{R} \cup \{-\infty, +\infty\}$, we define the function u^*, $u_* : \mathcal{O} \to \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$u^*(x) = \lim_{r\to 0} \sup\{u(y) \mid y \in \mathcal{O},\ |y-x| < r\}, \quad u_* = -(-u)^*$$

For each $x \in \mathcal{O}$, we set

$$J^{2,+}_{\mathcal{O}} u(x) = \left\{(p, X) \in \mathbb{R}^N \times \mathbb{S}^N \,\middle|\, u(y) \leqq u(x) + \langle p, y-x\rangle \right.$$
$$\left. + \frac{1}{2}\langle X(y-x), y-x\rangle + o(|y-x|^2) \quad \text{as } \mathcal{O} \ni y \to x \right\}$$

and $J^{2,-}_{\mathcal{O}} u(x) = -J^{2,+}_{\mathcal{O}}(-u(x))$. Here $\mathbb{S}^N$ denotes the set of all $N \times N$ real symmetric matrices and $\langle \cdot, \cdot \rangle$ is the Euclidian inner product in $\mathbb{R}^N$. We denote by $\bar{J}^{2,+}_{\mathcal{O}} u(x)$ and $\bar{J}^{2,-}_{\mathcal{O}} u(x)$ the following sets:

$$\bar{J}^{2,+}_{\mathcal{O}} u(x) = \{(p, X) \in \mathbb{R}^N \times \mathbb{S}^N \mid {}^\exists(x_n, p_n, X_n) \in \mathcal{O} \times \mathbb{R}^N \times \mathbb{S}^N$$
$$\text{such that } (p_n, X_n) \in J^{2,+}_{\mathcal{O}} u(x_n) \text{ and}$$
$$(x_n, u(x_n), p_n, X_n) \to (x, u(x), p, X) \text{ as } n \to +\infty\},$$

and $$\bar{J}^{2,-}_{\mathcal{O}} u(x) = -\bar{J}^{2,+}_{\mathcal{O}}(-u(x)).$$

We give the definition of viscosity solutions of the following nonlinear elliptic PDEs:

$$\max\{u + F(x, Du, D^2u), u - Mu\} = 0 \quad \text{in } \Omega, \tag{2.1}$$

where F is a continuous function on $\Omega \times \mathbb{R}^N \times \mathbb{S}^N$ satisfying the degenerate ellipticity condition:

$$F(x, p, Y) \leqq F(x, p, X) \qquad \text{for all } x \in \Omega,\ p \in \mathbb{R}^N,\ X, Y \in \mathbb{S}^N \text{ and } Y \geqq X.$$

Definition 2.1. *Let u be a function defined on $\overline{\Omega}$.*

(1) *u is a viscosity subsolution (resp., supersolution) of (2.1) if $u^*(x) < \infty$ (resp., $u_*(x) > -\infty$) on $\overline{\Omega}$ and*

$$\max\{u^*(x) + F(x, p, X), u^*(x) - Mu^*(x)\} \leqq 0 \tag{2.2}$$

$$(\text{resp., } \max\{u_*(x) + F(x, p, X), u_*(x) - Mu_*(x)\} \geqq 0\,) \tag{2.3}$$

for all $x \in \Omega$, $(p, X) \in \bar{J}_\Omega^{2,+}u^*(x)$ (*resp.*, $(p, X) \in \bar{J}_\Omega^{2,-}u_*(x)$).

(2) *u is a viscosity solution of* (2.1) *if u is a viscosity subsolution and supersolution of* (2.1).

§3. Main results

Our main results are stated as follows. See [4] for the details.

Theorem 3.1. *Assume* (A.1) *and* (A.2). *Let u and v be, respectively, a viscosity subsolution and a supersolution of* (1.1). *If u and v satisfy*

$$\max\{u^* - g, u^* - Mu^*\} \leqq 0 \text{ and } \max\{v_* - g, v_* - Mv_*\} \geqq 0 \quad \text{on } \partial\Omega, \tag{3.1}$$

then $u^* \leqq v_*$ on $\overline{\Omega}$.

Theorem 3.2. *Assume* (A.1) *and* (A.2). *Then there exist a unique viscosity solution $u \in C(\overline{\Omega})$ of* (1.1) *satisfying* $\max\{u - g, u - Mu\} = 0$ *on* $\partial\Omega$.

In what follows we mention the sketch of the proofs of Theorems.

Proof of Theorem 3.1. We may assume u (resp., v) is upper (resp., lower) semicontinuous on $\overline{\Omega}$. We use some perturbation of viscosity subsolution. For each $m \in \mathbf{N}$, $u_m = (1 - 1/m)u - 1/m$ is a viscosity subsolution of the following PDE:

$$\begin{cases} \max\{-\Delta u_m + u_m - f, u_m - Mu_m\} + \dfrac{1}{m} = 0 & \text{in } \Omega, \\ \max\{u_m - g, u_m - Mu_m\} + \dfrac{1}{m} \leqq 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

In order to prove $u_m \leqq v$ on $\overline{\Omega}$ for all $m \in \mathbf{N}$, we suppose the contrary, i.e., $\sup_{\overline{\Omega}}(u_{m_0} - v) = \theta > 0$ for some $m_0 \in \mathbf{N}$. We take $z \in \overline{\Omega}$ such that $\theta = u_{m_0}(z) - v(z)$. Then by (3.1) and (3.2), we may consider $z \in \Omega$.

Let $\Phi(x, y)$ be a function defined by

$$\Phi(x, y) = u_{m_0}(x) - |x - z|^4 - v(y) - \frac{1}{2\varepsilon}|x - y|^2 \quad \text{on } \overline{\Omega \times \Omega}$$

and let $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega \times \Omega}$ be a maximum point of $\Phi(x, y)$. From the inequality $\theta \leqq \Phi(x_\varepsilon, y_\varepsilon)$ and the semicontinuity of u_{m_0}, v, we have the behaviors of x_ε, y_ε, $u_{m_0}(x_\varepsilon)$, and $v(y_\varepsilon)$ as $\varepsilon \to 0$:

$$x_\varepsilon, y_\varepsilon \to z, \quad u_{m_0}(x_\varepsilon) \to u_{m_0}(z), \quad v(y_\varepsilon) \to v(z), \quad \frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2 \to 0.$$

Thus we can consider x_ε, $y_\varepsilon \in \Omega$. Moreover there exist X_ε, $Y_\varepsilon \in \mathbb{S}^N$ such that

$$\left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), X_\varepsilon\right) \in \bar{J}_\Omega^{2,+}(u_{m_0}(x_\varepsilon) - |x_\varepsilon - z|^4), \ \left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), Y_\varepsilon\right) \in \bar{J}_\Omega^{2,-} v(y_\varepsilon),$$
$$-\frac{3}{\varepsilon}\begin{pmatrix} I & O \\ O & I \end{pmatrix} \leqq \begin{pmatrix} X_\varepsilon & O \\ O & -Y_\varepsilon \end{pmatrix} \leqq \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \quad (I = \text{identity matrix}).$$

We remark that $((x_\varepsilon - y_\varepsilon)/\varepsilon + 4|x_\varepsilon - z|^2(x_\varepsilon - z), X_\varepsilon + Z_\varepsilon) \in \bar{J}_\Omega^{2,+} u_{m_0}(x_\varepsilon)$ $(Z_\varepsilon = 4|x_\varepsilon - z|^2 I + 8(x_\varepsilon - z) \otimes (x_\varepsilon - z))$. Hence using the facts that u_{m_0} and v are viscosity subsolution of (3.2) and supersolution of (1.1), respectively, we obtain the following inequalities:

$$\max\{-tr(X_\varepsilon + Z_\varepsilon) + u_{m_0}(x_\varepsilon) - f(x_\varepsilon), u_{m_0}(x_\varepsilon) - Mu_{m_0}(x_\varepsilon)\} + \frac{1}{m_0} \leqq 0,$$
$$\max\{-trY_\varepsilon + v(y_\varepsilon) - f(y_\varepsilon), v(y_\varepsilon) - Mv(y_\varepsilon)\} \geqq 0.$$

From these inequalities, we get a contradiction. Therefore we have $u_m \leqq v$ on $\overline{\Omega}$ for all $m \in \mathbb{N}$. Letting $m \to \infty$, we obtain the result. ∎

Proof of Theorem 3.2. It is easily seen that there exist a viscosity subsolution $\underline{u}$ and a supersolution $\overline{u}$ of (1.1) such that $\max\{\underline{u}^* - g, \underline{u}^* - M\underline{u}^*\} \leqq 0$ and $\max\{\overline{u}_* - g, \overline{u}_* - M\overline{u}_*\} \geqq 0$, respectively on $\partial\Omega$.

We define the set $\mathcal{S}$ and the function u as follows:

$$\mathcal{S} = \{u : \text{viscosity subsolution of (1.1)} \mid \max\{u^* - g, u^* - Mu^*\} \leqq 0 \text{ on } \partial\Omega\},$$
$$u(x) = \sup\{v(x) \mid v \in \mathcal{S}\} \quad (x \in \overline{\Omega}).$$

We observe that Perron's method can be used (cf. [2]). Therefore we obtain that u is a viscosity solution of (1.1) satisfying

$$\max\{u^* - g, u^* - Mu^*\} \leqq 0 \qquad \text{on} \quad \partial\Omega. \tag{3.3}$$

On the other hand, using the barrier argument we get

$$\max\{u_* - g, u_* - Mu_*\} \geqq 0 \quad \text{on} \quad \partial\Omega. \tag{3.4}$$

Hence it follows from Theorem 3.1 that $u^* = u = u_*$ on $\overline{\Omega}$ and thus $u \in C(\overline{\Omega})$. Then (3.3) and (3.4) yield $\max\{u - g, u - Mu\} = 0$ on $\partial\Omega$. Theorem 3.1 also implies the uniqueness of viscosity solutions of (1.1) satisfying the boundary condition. ∎

§4. Some remarks

In this section we shall give some remarks for Theorem 3.1. First we consider the boundary value problem of Dirichlet type, whose boundary value is interpreted in the viscosity sense:

$$\begin{cases} \max\{u + F(x, Du, D^2u), u - Mu\} = 0 & \text{in} \quad \Omega, \\ \max\{u - g, u - Mu\} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.1}$$

where F is a continuous function on $\overline{\Omega} \times \mathbb{R}^N \times \mathbb{S}^N \times \mathbb{R}$ satisfying the degenerate ellipticity condition.

Definition 4.1. *Let u be a function defined on $\overline{\Omega}$.*

(1) *u is a viscosity subsolution (resp., supersolution) of (4.1) provided $u^*(x) < \infty$ (resp., $u_*(x) > -\infty$) on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$, $(p, X) \in J^{2,+}_{\overline{\Omega}} u^*(x)$ (resp., $(p, X) \in J^{2,-}_{\overline{\Omega}} u_*(x)$), if $x \in \Omega$, then u^* (resp., u_*) satisfies (2.2) (resp., (2.3)) and if $x \in \partial\Omega$, then*

$$\max\{u^*(x) + F(x, p, X), u^*(x) - Mu^*(x)\} \leqq 0$$
$$\text{or } \max\{u^*(x) - g(x), u^*(x) - Mu^*(x)\} \leqq 0$$
$$(\text{resp., } \max\{u_*(x) + F(x, p, X), u_*(x) - Mu_*(x)\} \geqq 0$$
$$\text{or } \max\{u_*(x) - g(x), u_*(x) - Mu_*(x)\} \geqq 0).$$

(2) *u is a viscosity solution of (4.1) if u is a viscosity subsolution and supersolution of (4.1).*

We assume a kind of continuity for F:

(F.1) There exists a modulus of continuity ω such that

$$F(y,\alpha(x-y),Y)-F(x,\alpha(x-y),X) \leqq \omega(\alpha|x-y|^2+|x-y|)$$

for $\alpha > 1$, x, $y \in \overline{\Omega}$ and $X, Y \in \mathbb{S}^N$ satisfying

$$-3\alpha\begin{pmatrix} I & O \\ O & I \end{pmatrix} \leqq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leqq 3\alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Theorem 4.1. *Suppose* (A.1), (F.1) and $g \in C(\overline{\Omega})$. *Moreover suppose that F is uniformly continuous with respect to $p \in \mathbb{R}^N$. Let u and v be, respectively, a viscosity subsolution and a supersolution of* (4.1). *If u and v satisfy $u^* = u_*$ and $v^* = v_*$ on $\partial\Omega$, then $u^* \leqq v_*$ on $\overline{\Omega}$.*

Next we consider the boundary value problem of Neumann type:

$$(4.2)\qquad \begin{cases} \max\{u+F(x,Du,D^2u), u-Mu\}=0 & \text{in } \Omega, \\ \max\left\{\dfrac{\partial u}{\partial n}, u-Mu\right\}=0 & \text{on } \partial\Omega, \end{cases}$$

where $n(x)$ denotes the outward unit normal to Ω at $x \in \partial\Omega$.

Definition 4.2. *Let u be a function defined on $\overline{\Omega}$.*

(1) *u is a viscosity subsolution (resp., supersolution) of* (4.2) *provided $u^*(x) < \infty$ (resp., $u_*(x) > -\infty$) on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$, $(p,X) \in \overline{J}^{2,+}_{\overline{\Omega}}u^*(x)$ (resp., $(p,X) \in \overline{J}^{2,-}_{\overline{\Omega}}u_*(x)$), if $x \in \Omega$, then u^* (resp., u_*) satisfies* (2.2) (resp., (2.3)) *and if $x \in \partial\Omega$, then*

$$\max\{u^*(x)+F(x,p,X), u^*(x)-Mu^*(x)\} \leqq 0$$
$$\text{or } \max\{\langle n(x),p\rangle, u^*(x)-Mu^*(x)\} \leqq 0$$
$$(\text{resp., } \max\{u_*(x)+F(x,p,X), u_*(x)-Mu_*(x)\} \geqq 0$$
$$\text{or } \max\{\langle n(x),p\rangle, u_*(x)-Mu_*(x)\} \geqq 0).$$

(2) *u is a viscosity solution of* (4.2) *if u is a viscosity subsolution and supersolution of* (4.2).

Theorem 4.2. *Suppose* (A.1), (F.1) *and the uniform continuity of* F *with respect to* $(p, X) \in \mathbf{R}^N \times \mathbf{S}^N$. *Let* u *and* v *be, respectively, a viscosity subsolution and a supersolution of* (4.2). *Then* $u^* \leq v_*$ *on* $\overline{\Omega}$.

We omit the proofs of Theorems 4.1 and 4.2 because the methods are similar to that of Theorem 3.1.

References

[1] Bensoussan, A. and Lions, J. L., *Contrôlle impulsionnel et Inéquations Quasi-Variationnelles*, Dunod, Paris, 1982.

[2] Crandall, M. G., Ishii, H. and Lions, P. L., User's guide to viscosity solutions of second order partial differential equations, preprint.

[3] Ishii, H. and Lions, P. L., Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, J. Differential Equations, 83 (1990), 26–78.

[4] Ishii, K., Viscosity solutions of nonlinear second order elliptic PDEs associated with impulse control problems, to appear in Funkcial. Ekvac..

[5] Perthame, B., Quasi-variational inequalities and Hamilton-Jacobi-Bellman equations in a bounded region, Comm. Partial Differential Equations, 9 (1984), 561–595.

[6] ——————, Some remarks on quasi-variational inequalities and the associated impulsive control problems, Ann. Inst. Henri. Poincaré, 2 (1985), 237–260.

Limit Shape of the Section of Shrinking Doughnuts

NAOYUKI ISHIMURA*

Department of Mathematics
Faculty of Science
University of Tokyo
Tokyo, 113, JAPAN

Abstract. We discuss the limit shape of the generating curve of symmetric tori which are shrinking to a circle by the mean curvature flow. The problem naturally arises from a joint work with K.Ahara. Employing the backward heat kernel analysis introduced by G.Huisken we prove that it is a circle even under a little more general hypothesis than our previous work.

§1 INTRODUCTION

In this article we solve the question raised in a joint work [1] with K.Ahara; how is the limit shape of the section of symmetric 2-tori which are shrinking to a circle by the mean curvature flow.

The mean curvature flow problem, in its typical form, is to find the family of hypersurfaces $F_t : M_t \hookrightarrow \mathbf{R}^{n+1} (n \geq 2)$ satisfying

$$(1) \qquad \begin{cases} \dfrac{\partial F}{\partial t}(x,t) = -H(x,t) \cdot N(x,t) \\ F(x,0) = F_0(x) : M_0 \hookrightarrow \mathbf{R}^{n+1}, \end{cases}$$

where N denotes the outward unit normal and H is the mean curvature with respect to N. Notice that in terms of the induced metric on M_t the right hand side of (1) is the Laplace-Beltrami operator Δ_{M_t} on M_t.

We briefly recall some known facts about this problem. When the initial surface M_0 is strictly convex, G.Huisken [10], employing the method of R.Hamilton [9], showed that (1) shrinks M_0 to a round point within finite time, and also proved that for the area preserving rescaled flow M_0 really converges to a sphere in the C^∞-topology. Later M.Grayson [7] gave the counterexample which shows the convexity assumption in Huisken's theorem cannot be omitted; not all compact hypersurfaces with genus zero shrink to a point. Our previous work [1], on the other hand, dealt with the symmetric 2-torus and proved that under a rather

*Partially supported by Grant-in-Aid for Scientific Research, Ministry of Education, Science and Culture
(No. 03740075)

restrictive hypothesis the torus might be shrunk to a circle by the mean curvature flow (see Theorem 2.1). Our idea is based on applying the method of M.Gage and R.Hamilton [6], which discuss the curve shortening problem, to the equation for the generating curve.

The aim of the present article is to discuss the shape of the generating curve of symmetric tori which are shrinking to a circle by the mean curvature flow and to show that in many cases it is a circle. As to our previous work [1] it is so (see Corollary 2.3).

This limit shape problem is related to the rescaled flow analysis in Huisken's work [10] and to the problem of the formation of singularities in curve shortening (see [3]). It also corresponds to the self-similar or homothetic solutions. See [2] [11]. Indeed, in the limit we do arrive at such solutions.

The method of our proof is to utilize the backward heat kernel, which is first introduced by M.Struwe [12] for the study of heat flow for harmonic mappings and later used cleverly by Huisken [11] for the mean curvature flow. We mainly follow the idea of Huisken. In the limit the effect of the rotation around the axis is dropped and the problem becomes the "plane" situation.

The author is greatly indebted to Prof. Hiroshi Matano for helpful suggestions and to Dr. Kazushi Ahara for stimulating discussions.

§2 Notation and results

We use the same notation as in [1]. But we present it for completeness.

Let M_t be a family of an embedding of a 2-torus $F_t : T^2 \hookrightarrow \mathbf{R}^3$ such that they are rotationally symmetric about the z-axis. We represent them by

$$F_t(u,\varphi) = (f(t,u)\cos\varphi, f(t,u)\sin\varphi, g(t,u)),$$

where $u \in S^1$ is a parameter independent of t and $0 \leq \varphi < 2\pi$. We call M_t doughnuts hereafter. Let C_t be their generating curves, i.e., C_t are the intersection of M_t with the half xz-plane $\{(x,0,z)|x > 0\}$. C_t are represented by

$$C_t(u) = (f(t,u), 0, g(t,u)).$$

We define the speed $v(t,u)$ of C_t by

$$v(t,u)^2 \equiv f'(t,u)^2 + g'(t,u)^2,$$

where $' = \partial/\partial u$. The mean curvature $H(t,u)$ of M_t is then given by

$$H(t,u) = \frac{f'g'' - f''g'}{v^3} + \frac{g'}{fv} \equiv k_m + k_l.$$

Here

$$k_m = \frac{f'g'' - f''g'}{v^3} : \text{ the meridional sectional curvature.}$$

$$k_l = \frac{g'}{fv} \qquad : \text{ the latitudinal sectional curvature.}$$

Notice that k_m is a planar curvature of the generating curve C_t and k_l is a curvature of rotation. Since the outer unit normal N on M_t is given by

$$N = \left(\frac{g'}{v}\cos\varphi, \frac{g'}{v}\sin\varphi, -\frac{f'}{v}\right),$$

the equation (1) is described as the one for the generating curve:

$$\frac{\partial}{\partial t}\begin{pmatrix} f \\ g \end{pmatrix} = -(k_m + k_l)\cdot\frac{1}{v}\begin{pmatrix} g' \\ -f' \end{pmatrix}, \tag{2}$$

or explicitly

$$\begin{cases} \dfrac{\partial f}{\partial t} = \dfrac{1}{v}\left(\dfrac{f'}{v}\right)' - \dfrac{g'^2}{fv^2} \\ \dfrac{\partial g}{\partial t} = \dfrac{1}{v}\left(\dfrac{g'}{v}\right)' + \dfrac{f'g'}{fv^2}, \end{cases}$$

with the periodic condition

$$\begin{cases} f(t, u + 2\pi) = f(t,u) \\ g(t, u + 2\pi) = g(t,u) \end{cases}$$

and the initial condition.

We regard (2) as the perturbed plane curve shortening equation and hence, dropping the y-coordinate, we take a coordinate (x, z) only in the sequel.

For later use we denote the length of C_t and the area enclosed by C_t by L and A, respectively:

$$L = \int_{C_t} ds, \qquad A = \frac{1}{2}\int_{C_t} \langle F, N \rangle \, ds,$$

where $ds = v du$ is the arc-length parameter.

Our previous result is now stated as follows.

THEOREM 2.1 ([1]). *Suppose M_0 satisfies the following assumption (A). Then the mean curvature flow shrinks M_0 to a circle within finite time.*

(A) There exists a positive constant ε such that

$$f > \varepsilon \text{ and } k_m > \frac{1}{\varepsilon}\frac{1+\sqrt{5}}{2}.$$

The next question naturally arises; how is the shape of the generating curve becomes? Is it becoming circular as in the case of plane curve shortening [4][5]? The answer is positive even in a little more general situation. This is the focus of this article.

Now let (f, g) be the solution of (2). We *assume* (f, g) converges to $(1, 0)$ smoothly as $t \to T$. Let $\rho(X, t)$ be the backward heat kernel at $((1, 0), T)$, namely (see [11][12]),

$$\rho(X,t) = \frac{1}{\sqrt{4\pi(T-t)}} \exp\left\{-\frac{|X|^2}{4(T-t)}\right\} \qquad t < T. \tag{3}$$

Here we put $X = (f - 1, g)$.

We next define the rescaled immersions $\widetilde{X} \equiv (\widetilde{f} - 1, \widetilde{g})$ by

$$\widetilde{X}(\cdot, \widetilde{t}) = \frac{1}{\sqrt{2(T-t)}} X(\cdot, t), \qquad \widetilde{t}(t) = -\frac{1}{2}\log(T-t). \tag{4}$$

Similarly we denote the rescaled quantities by $\widetilde{\cdot}$ (for example, $\widetilde{A}, \widetilde{L}, \ldots$).

Our main result is then stated as follows:

THEOREM 2.2. *Suppose the solution (f, g) of (2) converges smoothly to $(1, 0)$ as $t \to T$. Suppose also that the isoperimetric ratio L^2/A of C_t is bounded as it converges. Then for each sequence $\widetilde{t}_j \to \infty$ there is a subsequence $\widetilde{t}_{jk}$ such that the generating curve $\widetilde{C}_{\widetilde{t}_{jk}}$ of $\widetilde{M}_{\widetilde{t}_{jk}} \equiv \widetilde{F}(\cdot, \widetilde{t}_{jk})$ converges smoothly to a unit circle centered at $(1, 0)$.*

In particular when C_t stays convex as it converges the corresponding isoperimetric ratio is bounded and so the result holds.

COROLLARY 2.3. *In the situation of [1] the limit shape of its generating curve is a circle.*

We remark here that the boundedness of the isoperimetric ratio seems to be an unpleasant assumption. But in [8] Grayson showed that in a figure-eight curve shortening the unboundedness of the isoperimetric ratio is equivalent to that the loops bound regions of equal area. We also notice that in a convex plane curve shortening Gage [4] proved that the isoperimetric ratio is monotone decreasing and so it is bounded.

REFERENCES

[1] K.Ahara and N.Ishimura, *On the mean curvature flow of "thin" doughnuts*, to appear in Lect.Notes Num.Appl.Anal. see also Preprint series UTYO-MATH 91-14 (1991).

[2] S.B.Angenent, *Shrinking doughnuts*, Proc.Conf.Ellip. Parabolic Eqs.,Gregynog, Wales (1989).

[3] S.B.Angenent, *On the formation of singularities in the curve shortening flow*, J.Diff.Geom. **33** (1991), 601-633.

[4] M.E.Gage, *An isoperimetric inequality with applications to curve shortening*, Duke Math.J. **50** (1983), 1225-1229.

[5] M.E.Gage, *Curve shortening makes convex curves circular*, Invent.Math. **76** (1984), 357-364.

[6] M.E.Gage and R.S.Hamilton, *The shrinking of convex plane curves by the heat equation*, J.Diff.Geom. **23** (1986), 69-96.

[7] M.A.Grayson, *A short note on the evolution of a surface by its mean curvature*, Duke Math.J. **58** (1989), 555-558.

[8] M.A.Grayson, *The shape of a figure-eight under the curve shortening flow*, Invent.Math. **96** (1989), 177-180.

[9] R.S.Hamilton, *Three-manifolds with positive Ricci curvature*, J.Diff.Geom. **17** (1982), 255-306.

[10] G.Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Diff. Geom. **20** (1984), 237-266.

[11] G.Huisken, *Asymptotic behaviour for singularities of the mean curvature flow*, J.Diff.Geom. (1990), 285-299.

[12] M.Struwe, *On the evolution of harmonic maps in higher dimensions*, J. Diff. Geom. (1988), 485-502.

On Burgers' type equation with nonlocal term

Kazuo Ito

1. Introduction

We consider the initial value problem:

$$u_t + a(\frac{u^2}{2})_x + b(\int_0^\infty u(x+\beta s)u_x(x+s)ds)_x = u_{xx}, t > 0, x \in \mathbf{R},$$

(1)

$$u(0,x) = u_0(x),\ x \in \mathbf{R},$$

where $u = u(t,x)$ is a unknown real − valued function with the constraint

$$\lim_{x\to\pm\infty} u = 0,$$

and a, b and β are constants such that $a \neq b$ and $\beta > 1$. We remark that if $\beta = 1$, then (1) turns to Burgers equation.

Here we state the motivation to consider (1). Majda and Rosales [2] proposed the following equation:

$$u_t + a(\frac{u^2}{2})_x + b(\int_0^\infty u(x+\beta s)u_x(x+s)ds)_x = 0, t > 0, x \in \mathbf{R},$$

(2)

$$u(0,x) = u_0(x),\ x \in \mathbf{R}.$$

Equation (2) arises as an asymptotic approximation which governs the growth of multidimensional perturbations in planar detonation front solutions of the equations of reactive gas dynamics in two space variables. In particular, if $\varphi_x = u$, then φ describes the evolution of a 2 − D perturbation in the primary planar front. In (2), Gardner [1] has proved the local existence theorem for smooth solutions, and also proved that smooth solutions develop shock in finite time.

We will study the solutions of (1) with εu_{xx} in the right − hand side, and plan to construct the solutions of (2) by putting ε close to 0.

In this paper, to (1), we show the local existence and uniqueness theorem and the global existence theorem with small initial data and the large − time behaviour of solutions.

Notations $L^p, 1 \le p \le \infty$, denotes the usual Lebesgue space on **R** with the norm $|\ |_p$. $W^{1,1}$ denotes the usual Sobolev space on **R** with the norm $|\cdot|_{1,1}$. For $0 < T \le \infty, X_T$ denotes the space of bounded and continuous functions from $[\ 0, T)$ to $W^{1,1}$. X denotes the subspace of X_∞ such that

$$\sup_{t \ge 0} |u(t)|_1 + \sup_{t \ge 0} (t+1)^{1/2} |u_x(t)|_1 < \infty.$$

$S(t)$ is the operator defined by

$$(S(t)u)(x) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi t}} \exp(-\frac{(x-y)^2}{4t}) u(y) dy,$$

$i.e. S(t)u$ is the solution of the linear heat equation

$$w_t - w_{xx} = 0, t > 0, x \in \mathbf{R},$$

$$w(0, x) = u(x), x \in \mathbf{R}.$$

We alter (1) to the following form

$$(3) \qquad u(t) = S(t)u - \int_0^t S(t-\tau)(\frac{a}{2}u^2 + b\int_0^\infty u(x+\beta s)u_x(x+s)ds)_x(\tau)d\tau,$$

and study the solutions of (3).

2. Results

The existence and uniqueness theorem is the following one.

Theorem 1 (i) (*uniqueness*) *For* $0 < T \le \infty$, *if* (3) *has two solutions* u *and* v *in* X_T , *then* $u = v$.
(ii) (*local existence*) *For any* $u_0 \in W^{1,1}$, *there exists* $T > 0$ *such that* (3) *has a solution in* X_T.
(iii) (*global existence with small initial data*) *Suppose that* $u_0 \in W^{1,1}$ *and* $|u_0|_{1,1}$ *is sufficiently small* , *then* (3) *has a solution in* X.

Now we observe the large – time behaviour of solutions of Theorem 1 (iii). To this end, we consider the self – similar solution of (1) of the form

$$\frac{1}{\sqrt{t+1}}\phi\ (\frac{x}{\sqrt{t+1}}).$$

Equation that ϕ should satisfy is the following one :

(4) $$\phi'(\xi)+\frac{\xi}{2}\phi\ (\xi)=\frac{a}{2}\phi\ (\xi)^2+b\int_0^\infty \phi\ (\xi+\beta\eta)\ \phi'(\xi+\eta)d\eta,$$

where

$$\xi=\frac{x}{\sqrt{t+1}}.$$

In addition, we impose the following condition on (4):

(5) $$\int_{-\infty}^{+\infty}\phi\ (\xi)d\xi=m,$$

where m is a given number.

We need the existence theorem of the solution of (4) and (5).

Theorem 2 (*existence of the self – similar solution*)
Suppose that $|m|$ is sufficiently small, where m is in (5). Then (4) and (5) have a unique C^1 solution.

From Theorem 1 (iii) and Theorem 2, we have the following result.

Theorem 3 (*large – time behaviour of solutions*)
Let $\varepsilon>0$ is an arbitrary and sufficiently small parameter. Suppose that $u_0\in W^{1,1}$, $|u_0|_{1,1}<K\varepsilon$ and $\int_{-\infty}^{+\infty}|u_0(x)||x|dx<\infty$, where K is a constant defined in the proof. Let u be the solution of (3) and ϕ be the solution of (4) and (5) with $m=\int_{-\infty}^{+\infty}u_0(x)dx$. Then, we have

$$|\partial_x^k\{u(t,\cdot)-\frac{1}{\sqrt{t+1}}\phi\ (\frac{\cdot}{\sqrt{t+1}})\}|_1$$

$$\leq C\frac{|u_0|_*}{1-(K\varepsilon)^{-1}|u_0|_{1,1}}(t+1)^{-(k+1)/2+\epsilon}$$

for $t\geq 0$ and $k=0,1$, where C is a constant independent of t,k and ε, and $|u_0|_*=|u_0|_{1,1}+\int_{-\infty}^{+\infty}|u_0(x)||x|dx$.

3. Outline of the proofs

In this section, we show the proof of Theorem 1 (iii), Theorem 2 and Theorem 3 shortly. Here and below, C denotes a generic constant.

Outline of the proof of Theorem 1 (iii)

First we note that a simple computation shows

$$|\int_0^\infty u(x+\beta s)v(x+s)ds|_1 \le (\beta-1)^{-1}|u|_1|v|_1 \tag{6}$$

for $u, v \in L^1$.

We define the mapping $\Phi : X \to X$ by

$$(\Phi u)(t) = S(t)u_0 - \int_0^t S(t-\tau)(\frac{a}{2}u^2 + b\int_0^\infty u(x+\beta s)u_x(x+s)ds)_x(\tau)d\tau.$$

Then,applying (6), we obtain the following basic estimates for u and v in some ball centered at 0 in X :

$$|(\Phi u)(t)|_1 \le |u_0|_1 + C\|u\|_X^2,$$

$$|(\Phi u)_x(t)|_1 \le C(t+1)^{-1/2}|u_0|_{1,1} + C(t+1)^{-1/2}\|u\|_X^2,$$

where

$$\|u\|_X = \sup_{t\ge0}|u(t)|_1 + \sup_{t\ge0}(t+1)^{1/2}|u_x(t)|_1.$$

The above estimates lead to the following inequality

$$\|\Phi u - \Phi v\|_X \le C|u_0|_{1,1}\|u-v\|_X$$

for u and v in some ball centered at 0 in X. Applying the contraction mapping theorem, we conclude that Φ has a fixed point. This completes the proof.

Outline of the proof of Theorem 2

First we alter (4) to the following integral equation:

$$(7) \qquad \phi(\xi) = \exp(-\frac{\xi^2}{4})\,\phi_0$$

$$+\int_0^\xi \exp(-\frac{\xi^2-\eta^2}{4})(\frac{a}{2}\phi(\eta)^2 + b\int_0^\infty \phi(\eta+\beta\zeta)\,\phi'(\eta+\zeta)d\zeta)d\eta,$$

where $\phi_0 = \phi(0)$. We assume that $|\phi_0|$ is sufficiently small.
Put $Y = \{\phi \in C^1(\mathbf{R})\,;\, \|\phi\|_Y < \infty\}$ where

$$\|\phi\|_Y = \sup_{\xi<0}(1+|\xi|)^\alpha|\phi(\xi)| + \sup_{\xi\geq 0}\exp(\frac{\xi^2}{4})|\phi(\xi)|$$

$$+\sup_{\xi<0}(1+|\xi|)^{\alpha-1}|\phi'(\xi)| + \sup_{\xi\geq 0}(1+|\xi|)^{-1}\exp(\frac{\xi^2}{4})|\phi'(\xi)|,$$

and $\alpha > 2$ is a parameter.

We define $\Psi : Y \to Y$ by $\Psi(\phi)$ = the right − hand side of (7), and estimate $\Psi(\phi)$ by using $Y-$ norm to get the inequality

$$\|\Psi(\phi) - \Psi(\psi)\|_Y \leq C|\phi_0|\|\phi - \psi\|_Y$$

for ϕ, ψ in some ball centered at 0. Applying the contraction mapping theorem, we conclude that (7) has a C^1 solution.

On the other hand, we can show that the mapping

$$\phi_0 \to \int_{-\infty}^{+\infty} \phi(\xi)d\xi$$

is one − to − one provided that $|\phi_0|$ is sufficientry small. Thus, for a sufficientry small m, there is a initial data ϕ_0 such that $\int_{-\infty}^{+\infty}\phi(\xi)d\xi = m$. This ϕ is the solution we want to look for.

Outline of the proof of Theorem 3

For any $0 < T < \infty$, we put

$$\|u\|_T = \sup_{0\leq t\leq T}(t+1)^{1/2-\epsilon}|u(t)|_1 + \sup_{0\leq t\leq T}(t+1)^{1-\epsilon}|u_x(t)|_1.$$

We remark the following fact:

(∗) If

$$\int_{-\infty}^{+\infty} w(x)dx = 0,$$

then

$$|\partial_x^k S(t)w|_1 \leq Ct^{-(k+1)/2} \int_{-\infty}^{+\infty} |w(x)||x|dx,$$

for $k = 0, 1, 2, \cdots$.

In our case, $w = u_0 - \phi$.
we estimate

$$U(t,x) = u(t,x) - \frac{1}{\sqrt{t+1}} \phi \left(\frac{x}{\sqrt{t+1}}\right)$$

by using L^1- norm and applying ($*$), then we have

$$\|U\|_T \leq C|u_0|_* + (K\varepsilon)^{-1}|u_0|_{1,1}\|U\|_T.$$

By the asumption, $(K\varepsilon)^{-1}|u_0|_{1,1} < 1$. Therefore we obtain

$$\|U\|_T \leq \frac{C|u_0|_*}{1-(K\varepsilon)^{-1}|u_0|_{1,1}}$$

Since T is arbitrary, we complete the proof.

References

[1] Gardner.R, Solutions of a nonlocal conservation law arising in combustion theory, SIAM J. Math. Anal, 18 (1987), 172 – 183.

[2] Majda.A and Rosales.R, A theory for spontaneous Mach stem formation in reacting shock fronts. I, the basic perturbation analysis, SIAM J. Appl. Math, 43 (1983), 1310 – 1334.

Mean ergodic theorems for integrated semigroups and integrated cosine families

HIROKAZU OKA

Department of Mathematics, Waseda University

Let $(X, \|\cdot\|)$ be a Banach space. We denote by $B(X)$ the set of all bounded linear operators from X into itself.

Let n be a positive integer, which is fixed in this paper.
A family $\{U(t) : t \geq 0\}$ in $B(X)$ is called an $n-$times integrated semigroup, if the following $(a), (b)$, and (c) are satisfied :

(a) $U(\cdot)x : [0, \infty) \to X$ is continuous for $x \in X$,

(b) $U(t)U(s)x = \frac{1}{(n-1)!}(\int_t^{s+t}(s+t-r)^{n-1}U(r)xdr - \int_0^s(s+t-r)^{n-1}U(r)xdr)$ for $x \in X$ and $s, t \geq 0$, and $U(0) = 0$,

(c) It implies $x = 0$ that $U(t)x = 0$ for all $t > 0$.

Let $\{U(t) : t \geq 0\}$ be an $n-$times integrated semigroup. If we assume the condition ;

(d) There is a constant $M \geq 0$ such that $\|U(t) - U(s)\| \leq M|t^n - s^n|$ for $s, t \geq 0$,

then there exists a unique closed linear operator A such that $(0, \infty) \subset \rho(A)$ (the resolvent set of A) and

$$R(\lambda; A)x(\equiv (\lambda - A)^{-1}x) = \int_0^\infty \lambda^n e^{-\lambda t}U(t)xdt \quad \text{for } x \in X \text{ and } \lambda > 0.$$

The operator A is called the generator of $\{U(t) : t \geq 0\}$.

In this talk, we give a mean ergodic theorem for $n-$times integrated semigroups and show that the ergodic theorem extends the mean ergodic theorem which has been proved by Shaw [5] recently. Also, we establish a mean ergodic theorem for $n-$times integrated cosine families.

The domain, the null space, and the range of an operator B in X will be denoted by $D(B), N(B)$, and $R(B)$ respectively.

THEOREM 1. *Let A be the generator of an $n-$times integrated semigroup $\{U(t) : t \geq 0\}$ satisfying the condition (d). We define an operator P by*

$$\begin{cases} D(P) = \{x \in X : \lim_{t\to\infty} n!U(t)x/t^n \text{ exists}\} \\ Px = \lim_{t\to\infty} n!U(t)x/t^n \text{ for } x \in D(P). \end{cases}$$

Then P is a bounded linear projection with $\|P\| \leq M, R(P) = N(A), N(P) = \overline{R(A)}$, and

$D(P) = N(A) \oplus \overline{R(A)} = \{x \in X : \{n!U(t)x/t^n : t > 0\}$ contains a weakly convergent subsequence as $t \to \infty\}$.

As a direct consequence of Theorem 1, we have the following corollary which has been given by Shaw [5] recently.

COROLLARY 2. *Let A be the generator of an $n-$times integrated semigroup $\{U(t) : t \geq 0\}$ satisfying the condition that $\|U(t)\| = O(t^n)$ as $t \to \infty$. We define an operator P' by*

$$\begin{cases} D(P') = \{x \in X : \lim_{t\to\infty} \frac{(n+1)!}{t^{n+1}} \int_0^t U(s)x ds \text{ exists}\} \\ P'x = \lim_{t\to\infty} \frac{(n+1)!}{t^{n+1}} \int_0^t U(s)x ds \text{ for } x \in D(P'). \end{cases}$$

Then P' is a bounded linear projection with $R(P') = N(A), N(P') = \overline{R(A)}$, and $D(P') = N(A) \oplus \overline{R(A)} = \{x \in X : \{\frac{(n+1)!}{t^{n+1}} \int_0^t U(s)x ds : t > 0\}$ contains a weakly convergent subsequence as $t \to \infty\}$.

By Theorem 1, we get the next result which was shown by Hashimoto [2] in the case where $n = 1$.

COROLLARY 3. *Under the assumption of Theorem 1, the following conditions are mutually equivalent :*

(i) $y \in A(D(A) \cap \overline{R(A)})$;

(ii) $s - \lim_{\lambda\downarrow 0} R(\lambda; A)y$ *exists* ;

(iii) $x = s - \lim_{t\to\infty} \frac{n!}{t^n} \int_0^t U(s)yds$ *exists ;*

(iv) *There is a sequence* $t_k \to \infty$ *as* $k \to \infty$ *such that*

$x = w - \lim_{k\to\infty} \frac{n!}{t_k^n} \int_0^{t_k} U(s)yds$ *exists.*

Moreover, the limit x *is the unique solution of* $Ax = y$ *in* $\overline{R(A)}$.

Similarly, we can prove a mean ergodic theorem for $n-$times integrated cosine families $\{C(t) : t \in \mathbf{R}\}$ which was introduced by Kato [4]. See also Arendt and Kellermann [1].

A family $\{C(t) : t \in \mathbf{R}\}$ in $B(X)$ is called an $n-$times integrated cosine family, if

(1) $C(\cdot)x : \mathbf{R} \to X$ is continuous for $x \in X$,

(2) $C(t) = (-1)^n C(-t)$ for $t \geq 0$ and $C(0) = 0$,

(3) $\int_0^s C(r)(C(t) - \frac{t^n}{n!})xdr + (C(s) - \frac{s^n}{n!}) \int_0^t C(r)xdr$
$= \frac{1}{n!}[\int_t^{s+t} (s+t-r)^n C(r)xdr - \int_0^s (s+t-r)^n C(r)xdr]$ for $x \in X$ and $s, t \geq 0$,

(4) It implies $x = 0$ that $C(t)x = 0$ for all $t > 0$.

Let $\{C(t) : t \in \mathbf{R}\}$ be an $n-$times integrated cosine family. If we assume the condition ;

(5) There is a constant $M \geq 0$ such that $\|C(t) - C(s)\| \leq M|t^n - s^n|$ for $s, t \geq 0$,

then there exists a unique closed linear operator A such that $(0, \infty) \subset \rho(A)$ and

$$(\lambda^2 - A)^{-1}x = \int_0^\infty \lambda^{n-1} e^{-\lambda t} C(t)xdt \quad \text{for } x \in X \text{ and } \lambda > 0.$$

The operator A is called the generator of $\{C(t) : t \in \mathbf{R}\}$. Theorem 4, Corollaries 5 and 6 are the corresponding results to Theorem 1, Corollaries 2 and 3 respectively.

THEOREM 4. *Let A be the generator of an $n-$times integrated cosine family $\{C(t) : t \in \mathbf{R}\}$ satisfying the condition (5). We define an operator P by*

$$\begin{cases} D(P) = \{x \in X : \lim_{t\to\infty} \frac{(n+1)!}{t^{n+1}} \int_0^t C(s)xds \ \text{ exists}\} \\ Px = \lim_{t\to\infty} \frac{(n+1)!}{t^{n+1}} \int_0^t C(s)xds \ \text{ for } x \in D(P). \end{cases}$$

Then P is a bounded linear projection with $\|P\| \le M, R(P) = N(A), N(P) = \overline{R(A)}$, and

$D(P) = N(A) \oplus \overline{R(A)} = \{x \in X : \{\frac{(n+1)!}{t^{n+1}} \int_0^t C(s)xds : t > 0\}$ contains a weakly convergent subsequence as $t \to \infty\}$.

COROLLARY 5. *Let A be the generator of an $n-$times integrated cosine family $\{C(t) : t \in \mathbf{R}\}$ satisfying $\|C(t)\| = O(t^n)$ as $t \to \infty$. We define an operator P' by*

$$\begin{cases} D(P') = \{x \in X : \lim_{t\to\infty} \frac{(n+2)!}{t^{n+2}} \int_0^t \int_0^s C(r)xdrds \ \text{ exists}\} \\ P'x = \lim_{t\to\infty} \frac{(n+2)!}{t^{n+2}} \int_0^t \int_0^s C(r)xdrds \ \text{ for } x \in D(P'). \end{cases}$$

Then P' is a bounded linear projection with $R(P') = N(A), N(P') = \overline{R(A)}$, and

$D(P') = N(A) \oplus \overline{R(A)} = \{x \in X : \{\frac{(n+2)!}{t^{n+2}} \int_0^t \int_0^s C(r)xdrds : t > 0\}$ contains a weakly convergent subsequence as $t \to \infty\}$.

COROLLARY 6. *Under the assumption of Theorem 4, the following conditions are mutually equivalent :*

(i) $y \in A(D(A) \cap \overline{R(A)})$;

(ii) $s - \lim_{\lambda \downarrow 0} R(\lambda^2; A)y$ *exists* ;

(iii) $x = s - \lim_{t\to\infty} \frac{(n+1)!}{t^{n+1}} \int_0^t \int_0^s \int_0^r C(w)ydwdrds$ *exists* ;

(iv) There is a sequence $t_k \to \infty$ as $k \to \infty$ such that

$x = w - \lim_{k\to\infty} \frac{(n+1)!}{t_k^{n+1}} \int_0^{t_k} \int_0^s \int_0^r C(w)ydwdrds$ *exists.*

Moreover, the limit x is the unique solution of $Ax = y$ in $\overline{R(A)}$.

EXAMPLE. *(Arendt and Kellermann [1], Hieber [3])*

We consider an elliptic differential operator $A = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ *with some constant coefficients on* $L^p(\mathbf{R}^n), 1 < p < \infty$. *Such an operator A is called elliptic if the polynomial* $\sum_{|\alpha|\leq m} a_\alpha i^{|\alpha|} x^\alpha$ *is elliptic (i.e.* $\sum_{|\alpha|=m} a_\alpha x_1^\alpha \cdots x_n^\alpha = 0$ *implies* $x_1 = \cdots = x_n = 0$). *If* $Re \sum_{|\alpha|\leq m} a_\alpha i^{|\alpha|} x^\alpha = 0$ *for* $x \in \mathbf{R}^n$ *and* $m \geq 2, A$ *(with a suitable domain) generates a* $k-$*times integrated semigroup* $\{S(t) : t \geq 0\}$ *on* $L^p(\mathbf{R}^n), 1 < p < \infty$, *with* $\|S(t)\| \leq ct^k$ *for* $t \geq 0$ *and with* $k > \frac{n}{2}$, *where c is some constant. For details, see Hieber [3].*

REFERENCES

1. W.Arendt and H.Kellermann, *Integrated solutions of Volterra integro-differential equations and applications*, in "Volterra integro-differential equations in Banach spaces and applications ", Pitmann Res. Notes in Math. **190** (1987), 21–51.
2. K.Hashimoto, *The range of generators and mean ergodicity of integrated semigroups*, preprint.
3. M.Hieber, *Integrated semigroups and differential operators on L^p spaces*, to appear.
4. N.Kato, *Integrated cosine family and integrated semigroups*, Proceedings of the seminar of evolution equations **13** (1987).
5. S.-Y.Shaw, *Uniform convergence of ergodic limits and approximate solutions*, Proc. Amer. Math. Soc., to appear.

2次元非有界領域における Navier-Stokes 流の強解の減衰について

小薗英雄　九州大・教養
小川卓克　名古屋大・理

§1 導入と結果.

Ω ($\subset \mathbf{R}^2$) は非有界領域でその境界 $\partial\Omega$ は一様に C^m 級であるとする。$Q_T = \Omega \times (0,T)$ において次の初期値境界値問題を考える

$$
\text{(N.S)} \quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0, \text{ in } Q_T, \\ \text{div } u = 0, \quad \text{in } Q_T, \\ u = 0, \quad \text{on } \partial\Omega \times (0,T), \\ u|_{t=0} = a, \end{cases}
$$

ここに速度ベクトル $u = (u_1(x,t), u_2(x,t))$ および圧力 $p = p(x,t)$ は未知函数、$a = (a_1(x,t), a_2(x,t))$ は与えられた初期値である。

ここでは $a \in L^2_\sigma(\Omega)$ に対する (N.S) の時間大域的強解の存在とその $t \to \infty$ での漸近挙動を調べたい。Ωが $\mathbf{R}^n$ ($n \geq 3$) の外部領域の場合は弱解の L^2-norm および 強解の L^p-norm の代数巾による減衰が得られている（ Borchers-Miyakawa [1], [2]、Iwashita [8])。　$n = 2$ のときは $\|u(t)\|_{L^2} \to 0$ のみが知られている（ Masuda [11])。

定義. $a \in L^2_\sigma(\Omega)$ とする。uが $(0,T)$ 上の *(N.S)* の強解であるとは次の *(1),(2),(3)* の条件を満たすことである

(1) $u \in C([0,T); L^2_\sigma(\Omega)) \cap C^1((0,T); L^2_\sigma(\Omega))$
(2) $u(t) \in D(A)$ *for* $t > 0$, $Au \in C((0,T); L^2_\sigma(\Omega))$
(3) u は次の式を満たす。

$$
\text{(A-N.S)} \quad \begin{cases} \dfrac{du}{dt} + Au + P(u \cdot \nabla u) = 0, \quad 0 < t < T, \\ u(0) = a. \end{cases}
$$

ここに Pは $L^2(\Omega)$ から $L^2_\sigma(\Omega)$ への直交射影, $A\equiv -P\Delta$, $(D(A)=\{u\in H^2(\Omega); u|_{\partial\Omega}=0\}\cap L^2_\sigma)$ は Stokes 作用素を表わす。

(A-N.S) の解の存在と減衰について次のような結果を得た.

定理. $a\in L^2_\sigma(\Omega)$ とする。このとき $(0,\infty)$ 上の $(N.S)$ の強解 u が一意的に存在する。更に u は次の性質を満たす。

(1) *(smoothness)* $u(t)\in C^1((0,\infty); D(A^\alpha))$ ただし $0\le\alpha<1$。

(2) *(decay)*

$$\|u(t)\|_p=\begin{cases} o(t^{1/p-1/2}), & \text{for } 2\le p<\infty,\\ o(t^{-1/2}\sqrt{\log t}), & \text{for } p=\infty;\end{cases} \tag{1.1}$$

$$\|A^\alpha u(t)\|_2=\begin{cases} o(t^{-\alpha}), & 0<\alpha<1\\ o(t^{-1}\sqrt{\log t}), & \alpha=1;\end{cases} \tag{1.2}$$

$$\|\dot u(t)\|_p=\begin{cases} o(t^{1/p-3/2}), & 2\le p<\infty,\\ o(t^{-3/2}\sqrt{\log t}), & p=\infty;\end{cases} \tag{1.3}$$

$$\|A^\alpha \dot u(t)\|_2=o(t^{-\alpha-1}),\qquad 0<\alpha<1, \tag{1.4}$$

as $t\to\infty$.

§2 準備.

定理の証明には以下の補題が重要である。

補題1. $\varepsilon>0$ $0<\delta<1/2$. $u,v\in D(A^{1/2})\cap L^\infty$とする。

$$\Longrightarrow\quad \|(A+\varepsilon)^{-\delta}P(u\cdot\nabla v)\|_2\le C_\delta\|A^{1/2-\delta}u\|_2\|A^{1/2}v\|_2$$

ただし C_δは ε, u, vによらない定数。

注意

Ωが外部のときは A が有界な逆を持たないことに注意する。

補題1により次のような双線型作用素 $F_\delta(\cdot,\cdot)$ が定義できる

$$F_\delta(u,v)=\text{w-}\lim_{\varepsilon\to 0}(A+\varepsilon)^{-\delta}P(u\cdot\nabla v)\quad u,v\in D(A^{1/2})\cap L^\infty$$

この F_δを density を用いて $D(A^{1/2})$ 上に拡張したものに対して、補題 1 より以下が得られる。

補題 2．

(1) $\|F_\delta(u,v)\|_2 \le C_\delta\|A^{1/2-\delta}u\|_2\|A^{1/2}v\|_2, \quad u,v \in D(A^{1/2})$
(2) $(F_\delta(u,v), A^\delta\phi) = (P(u\cdot\nabla v),\phi)$ for $u,v \in D(A^{1/2})$, $\phi \in D(A^\delta)$
(3) $A^\delta F_\delta(u,v) = P(u\cdot\nabla v)$ for $u,v\in D(A^{1/2})\cap L^\infty$

補題 1 の証明は正値自己共役作用素の分数巾に対する Heinz の不等式に注意すると、

$$\|(-\Delta+\varepsilon)^{-\delta}P(u\cdot\nabla v)\|_2 \le C_\delta\|(-\Delta)^{1/2-\delta}u\|_2\|(-\Delta)^{1/2}v\|_2 \tag{2.1}$$

を得れば十分である (Kato-Fujita [9]参照)。(2.1) は$-\Delta+\lambda$の $\mathbf{R}^2$における 基本解の積分表示を用いて示される次の不等式によって得られる。

$$G_\alpha(x,y,\varepsilon) \le \frac{\Gamma(1-\alpha)}{4^\alpha\pi\Gamma(\alpha)}|x-y|^{2\alpha-2} \qquad (0<\alpha<1)$$

ただし

$$G_\alpha(x,y,\varepsilon) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^\infty \lambda^{-\alpha}G(x,y,\varepsilon+\lambda)d\lambda$$

ここに $G(x,y,\varepsilon)$ は $(-\Delta+\varepsilon)^{-1}$の$\Omega$における Green kernel である.

一方、次の補題は u と$\dot{u}$の L^∞評価を得るのに用いられる。

補題 3．$u\in D(A^{s/2})(1<s\le 2)$ とする。このとき $2<p<\infty$に対して

$$\|u\|_\infty \le C_s p^{1/2-\beta/2s}\|A^{1/2}u\|_2^{1-\beta}(\|u\|_2+\|A^{s/2}u\|_2)^\beta$$

(ここで $u\in D(A^{s/2})$, $\beta=2s/(2+p(s-1))$) ただし C_sは s にのみよる定数。

補題 3 は $n=2$ における Gagliardo-Nirenberg の不等式

$$\|u\|_p \le Cp^{1/2}\|u\|_2^{2/p}\|\nabla u\|_2^{1-2/p} \quad u\in H_0^1(\Omega), \quad 2\le p<\infty$$

と

$$\|u\|_\infty \le C_s\|u\|_2^{1-\alpha}\|u\|_{H^s}^{\alpha}. \qquad u\in H^s(\Omega)$$

(ただし$\alpha = 2/(2+p(s-1))$) および

$$\|\nabla u\|_2 = \|A^{1/2}u\|_2 \qquad u \in D(A^{1/2})$$

により得られる。

3 定理の証明の概略

強解の存在を示すにはつぎの iteration scheme

$$\begin{cases} u_0(t) = e^{-tA}a, \\ u_{j+1}(t) = e^{-tA}a - \int_0^t A^{1-\gamma}e^{-(t-s)A}F_{1-\gamma}(u_j, u_j)(s)ds, \quad 1/2 < \gamma < 1 \end{cases}$$

に対して、A の分数巾、A^α $(0 < \alpha < 1)$ を作用させ、非線型項を補題2により評価する。

$$K_{j,\alpha} \equiv \sup_{0<t\leq T} t^\alpha \|A^\alpha u_j(t)\|_2$$

とおけば、次を得る。

$$K_{j+1,\alpha} \leq K_{0,\alpha} + C_{1-\gamma}B(\gamma-\alpha, 1-\gamma)K_{j,\gamma-1/2}K_{j,1/2}.$$

したがって

$$k_j(T) = \max\{K_{j,\gamma-1/2}(T), K_{j,1/2}(T)\} \quad (j = 0, 1, \cdots),$$
$$\beta_\gamma = C_{1-\gamma}\max\{B(1/2, 1-\gamma), B(\gamma-1/2, 1-\gamma)\}$$

とおけば

$$k_{j+1} \leq k_0 + \beta_\gamma(k_j)^2,$$

を得て、k_0が小さければ k_jが有界列であることがわかる。ほぼ同様にして $u_{j+1}-u_j$を評価して u_j が収束列であることがわかり極限 u が解になることが示される。

この時更に

(1) $\|a\| < (4\beta_\gamma)^{-1}$ならば $u(t)$ は大域解となり、$\|A^\alpha u(t)\| \leq Ct^{-\alpha}$ $0 < \alpha < 1$ を得る.

(2) 初期値が滑らか、すなわち $a \in D(A^\epsilon)$ $(\varepsilon > 0)$ ならば局所解 $u(t)$ の存在時間 T は $T = (4\beta_\gamma\|A^\epsilon a\|)^{-1/\epsilon}$ と取れる.

さらに方程式に $u(t)$ と $A^{2\gamma-1}u(t)$ をかけて部分積分することにより、エネルギー等式

$$\|u(t)\|_2^2+2\int_0^t\|\nabla u(\tau)\|_2^2d\tau=\|a\|_2^2$$

と a priori 評価

$$\|A^\epsilon u(t)\|_2^2\le\|A^\epsilon a\|_2^2\exp(C_\epsilon\|a\|_2^2)\qquad 0<\varepsilon<1/2$$

を得る．これらの評価とはじめの局所解が $t>0$ で regularity があがることを用いれば解が時間大域的に接続できることがわかる。

解の減衰はまず Masuda [11]の結果より

$$\|u(t)\|_2\to 0\quad t\to 0$$

が得られることに注意する。それにより (2) から $\|A^\alpha u(t)\|_2=o(t^{-\alpha})$ が得られ Gagliardo-Nirenberg の不等式より L^p 減衰が得られる。

次に解の L^∞ 評価は補題 3 に

$$\|A^{1/2}u(t)\|_2=o(t^{-1/2})$$

を用い $p=\log t$ と選ぶことにより得られる。

REFERENCES

[1] Borchers,W. and Miyakawa, T., *Algebraic L^2 decay for Navier-Stokes flows in exerior domains*, Acta Math. 165 (1990), 189–227.

[2] Borchers,W. and Miyakawa, T., *L^2 decay for the Navier-Stokes flows in unbounded domains with application to exterior stationary flows*, preprint.

[3] Borchers,W. and Sohr, H., *On the semigroup of the Stokes operator for exterior domains in L^q-spaces*, Math. Z. 196 (1987), 415–425.

[4] Fujita, H. and Kato, T., *On the Navier-Stokes initial value problem 1*, Arch. Rat. Mech. Anal. 46 (1964), 269–315.

[5] Giga, Y., *Domains of fractional powers of the Stokes operator in L_r spaces*, Arch. Rat. Mech. Anal. 89 (1985), 251–265.

[6] Giga, Y. and Miyakawa, T., *Solution in L_r of the Navier-Stokes initial value problem*, Arch. Rat. Mech. Anal. 89 (1985), 267–281.

[7] Giga,Y. and Sohr, H., *On the Stokes operator in exterior domains*, J. Fac. Sci. Univ. Tokyo, Ser. IA Math. 36 (1986), 103–130.

[8] Iwashita, H., *$L_q - L_r$ estimates for solutions of nonstationary in an exterior domain and the Navier-Stokes initial value problem in L_q spaces*, Math. Ann. 285 (1989), 265–288.

[9] Kato, T. and Fujita, H., *On the nonstationary Navier-Stokes system*, Rend. Sem. Math. Univ. Padova 32 (1962), 243–260.

[10] Masuda, K., *On the stability of incompressible viscous fluid motions past object*, J. Math. Soc. Japan 27 (1975), 294–327.

[11] Masuda, K., *Weak solutions of the Navier-Stokes equations*, Tohoku Math. J. 36 (1984), 623–646.

[12] Miyakawa, T., *On stationary solutions of the Navier-Stokes equations in an exterior domain*, Hiroshima Math. J. 12 (1982), 115–140.

[13] Miyakawa, T. and Sohr, H., *On energy inequality, smoothness and large time behavior in L^2 for weak solutions of the Navier-Stokes equations in exterior domains*, Math. Z. 199 (1988), 455–478.

[14] Yao Jing-Qi, *Comportment à l'infini des solution d'une équation de Schrödinger nonlinéaires dans un domaine extérier*, C. R. Acad. Sci. Paris, 15 (1982), 163–166.

Nonlinear Scattering for Long Range Interaction

Tohru OZAWA (RIMS, Kyoto University)

In the scattering theory for nonlinear waves, the basic idea is that for large times the solutions of nonlinear wave equations behave like the solutions of the corresponding free equations. This is possible only when we can take the point of view that the nonlinear interaction has no effect for large times, which in turn imposes restrictive conditions on the degree of nonlinearities and on the space dimensions in connection with decay rate in time of the free solutions. Even in the small data setting the conditions often exclude the possibility of scattering theory for many famous equations especially in lower space dimensions, such as the (modified) K-dV equation, the (derivative) nonlinear Schrödinger equation in 1+1 dimensions, the Klein-Gordon equation with cubic (resp. quadratic) nonlinearity in 1+1 (resp. 1+2) dimensions, and systems of quantum fields with Yukawa coupling (Maxwell-Dirac, Klein-Gordon-Dirac, Klein-Gordon-Schrödinger, e.t.c.). In fact, most of these equations have no nontrivial solutions with the asymptotic form of the free solutions. In this talk, I present a new framework for the nonlinear scattering in the case where the degree of nonlinearities is not high enough to ensure asymptotically free solutions. The results given here, together with the recent papers [1][2], give an answer to the third problem of M. Reed.

We consider scattering for the nonlinear Schrödinger equation

$$i\partial_t u + (1/2)\Delta u = f(u)u. \qquad (1)$$

Here u is a $\mathbb{C}$-valued function of $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, Δ is the Laplacian in $\mathbb{R}^n$, and f is an $\mathbb{R}$-valued function on $\mathbb{C}$. We treat the following two cases.

(I) The single power interaction in one space dimension:

$$f(u) = \lambda|u|^2, \ \lambda \in \mathbb{R}\setminus\{0\}, \ (t,x) \in \mathbb{R}\times\mathbb{R}. \qquad (2)$$

In this case, (1) is derived from the electromagnetic wave equation for the propagation of a laser beam in a nonlinear medium, from the Zakharov system for the propagation of the Langumuir waves in a plasma, from the Davey-Stewartson system for the propagation of surfaces of water waves, from isotropic Heisenberg equation for the evolution of classical spins, from the Ginsburg-Landau model for superconductivity, and so on.

(II) The Hartree type interaction in more than one space dimension:

$$f(u) = V*|u|^2, \ V(x) = \lambda|x|^{-1}, \ (t,x) \in \mathbb{R}\times\mathbb{R}^n, \ n \geq 2, \qquad (3)$$

where $*$ denotes the convolution in $\mathbb{R}^n$. In this case (1) is derived from a multibody Schrödinger equation in the self-consistent field approximation for a quantum system of bosons interacting through two body potential V. The associated time-independent version also arises in the quantum field theory, especially in the Hartree-Fock theory.

The above examples (2)-(3) have the following properties in common.

(a) Gauge invariance: $f(e^{i\theta}u) = f(u)$, $\theta \in \mathbb{R}$.

(b) Homogeneity: $D(t)^{-1}f(D(t)u) = t^{-1}f(u)$, $t > 0$,

where $D(t)$ is the dilation operator given by $(D(t)\psi)(x) = t^{-n/2}\psi(t^{-1}x)$. Property (a) leads to the conservation of the prpbability density which enables us to establish the well-posedness of the Cauchy problem for (1). More precisely, in both cases (I)-(II) it is proved that there is a unique group of nonlinear operators $\{S(t);\ t \in \mathbb{R}\}$ such that for any $k \in \mathbb{N}\cup\{0\}$

(1) $S(t)$ is a homeomorphism in the usual Sobolev space H^k and is an isometry in the L^2 norm for any $t \in \mathbb{R}$.

(2) $S(t+s) = S(t)S(s)$ for any $t, s \in \mathbb{R}$, $S(0) = 1$.

(3) For any $\phi \in H^k$, the map $t \longmapsto S(t)\phi$ is continuous from $\mathbb{R}$ to H^k.

(4) For any $t_0 \in \mathbb{R}$ and $\phi \in H^k$, $u(t) = S(t-t_0)\phi$ is a unique solution satisfying $u \in C(\mathbb{R};\ H^k) \cap \bigcap_{0\le\delta(q)<1} L^{2/\delta(q)}_{loc}(\mathbb{R};\ W^{k,q})$ and

$$u(t) = U(t-t_0)\phi - i\int_{t_0}^{t} U(t-\tau)f(u(\tau))u(\tau)\,d\tau,\quad t \in \mathbb{R}, \tag{4}$$

where $U(t) = \exp(i(t/2)\Delta)$ and $\delta(q) = n/2-n/q$.

In scattering for (1) a cucial effect is given by the degree of nonlinear term $f(u)$ at $u = 0$, which is measured by the decay rate in t of the dilated potential $D(t)^{-1}f(D(t)u)$. In the ordinary scattering we compare solutions u to free solutions $U(t)\phi_\pm$ on the basis of the asymptotics

$$\|u(t) - U(t)\phi_\pm\|_2 \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \pm\infty. \tag{5}$$

We would say that scattering theory for (1) had been possible only in the case where $D(t)^{-1}f(D(t)u) \sim t^{-\gamma}g(u)$ as $t \longrightarrow \infty$ for some function g and $\gamma > 1$. In n space dimensions, this corresponds $p > 1+2/n$ for $f(u) = \lambda|u|^{p-1}$ and $\gamma > 1$ for $f(u) = V*|u|^2$ with $V(x) = \lambda|x|^{-\gamma}$. On the other hand J. Ginibre (private communication) proved that (5) is impossible for any nontrivial solution when $D(t)^{-1}f(D(t)u) \sim t^{-\gamma}g(u)$ as $t \longrightarrow \infty$ for some g and $\gamma \le 1$. Property (b) therefore shows that the usual setting of scattering just fails for (2)-(3).

In order to state the main results we use the following notations. $W^{m,p}$ denotes the Sobolev space given by

$$W^{m,p} = \{\psi \in L^p;\ \|\psi\|_{W^{m,p}} = \sum_{|\alpha|\le m} \|\partial^\alpha\psi\|_p < \infty\},\ m \in \mathbb{N}\cup\{0\},\ p \in [1,\infty].$$

Here $\|\cdot\|_p$ denotes the norm in $L^p = L^p(\mathbb{R}^n)$ and $\partial^\alpha = \Pi_{j=1}^n \partial_j^{\alpha_j}$, $\partial_j = \partial/\partial x_j$, for a multi-index α. $H^{m,s}$ denotes the weighted Sobolev space given by

$$H^{m,s} = \{\psi \in \mathcal{S}';\ \|\psi\|_{m,s} = \|(1+|x|^2)^{s/2}(1-\Delta)^{m/2}\psi\|_2 < \infty\},\ m,\ s \in \mathbb{R}.$$

For $\phi_\pm \in H^{0,1}$ we define the phase functions $S^\pm$ by

$$S^\pm(t,x) = \mp \log|t|\cdot f(\hat{\phi}_\pm)(t^{-1}x),\ t \in \mathbb{R}\setminus\{0\},$$

where $\hat{}$ denotes the Fourier transform given by

$$\hat{\psi}(\xi) = (\mathcal{F}\psi)(\xi) = (2\pi)^{-n/2} \int \exp(-ix\cdot\xi)\psi(\xi)\, d\xi.$$

We define the unitary operators $\exp(iS_\pm(t))$ by

$$\exp(iS_\pm(t)) = \exp(iS^\pm(t,-it\nabla)) = \mathcal{F}^{-1}\exp(\mp i\log|t|\cdot f(\hat{\phi}_\pm))\mathcal{F}.$$

<u>Theorem 1.</u> Let f be as in (I) and let $k \in \mathbb{N}\cup\{0\}$. Then there is a constant $\varepsilon > 0$ with the following properties.

(1) For any $\phi_+ \in H^{k,2}\cap H^{0,k+2}$ with $\|\hat{\phi}_+\|_\infty \le \varepsilon$ there exists a unique $\phi \in H^{k,0}$ such that for any $\theta \in (1/2,1)$

$$\|S(t)\phi - \exp(iS_+(t))U(t)\phi_+\|_{k,0} = O(t^{-\theta}), \tag{6}_+$$

$$\left(\int_t^{+\infty} \|S(\tau)\phi - \exp(iS_+(\tau))U(\tau)\phi_+\|^4_{W^{4,\infty}}\, d\tau\right)^{1/4} = O(t^{-\theta/2}) \quad \text{as} \quad t \longrightarrow +\infty. \tag{7}_+$$

(2) For any $\phi_- \in H^{k,2}\cap H^{0,k+2}$ with $\|\hat{\phi}_-\|_\infty \le \varepsilon$ there exists a unique $\phi \in H^{k,0}$ such that for any $\theta \in (1/2,1)$

$$\|S(t)\phi - \exp(iS_-(t))U(t)\phi_-\|_{k,0} = O(|t|^{-\theta}), \tag{6}_-$$

$$\left(\int_{-\infty}^{t} \|S(\tau)\phi - \exp(iS_-(\tau))U(\tau)\phi_-\|^4_{W^{4,\infty}}\, d\tau\right)^{1/4} = O(|t|^{-\theta/2}) \quad \text{as} \quad t \longrightarrow -\infty. \tag{7}_-$$

<u>Theorem 2.</u> Let f be as in (II) and let $k \in \mathbb{N}\cup\{0\}$. Then there is a constant $\varepsilon > 0$ with the following properties.

(1) Let $\phi_+ \in H^{k,2}\cap H^{0,k+2}$ for $n \ge 3$ and $\phi_+ \in H^{k,2}\cap H^{0,k+3}$ for $n = 2$. Suppose that there is $\sigma \in (0,1/(n-1))$ such that $\|\hat{\phi}_+\|_{p(\sigma)}\|\hat{\phi}_+\|_{p(-\sigma)} \le \varepsilon$, where $p(\pm\sigma)$ is given by $p(\pm\sigma) = 2n/((1\pm\sigma)(n-1))$. Then there exists a unique $\phi \in H^{k,0}$ such that for any $\theta \in (1/2,1)$

$$\|S(t)\phi-\exp(iS_+(t))U(t)\phi_+\|_{k,0} = O(t^{-\theta}), \qquad (8)_+$$

$$\left(\int_t^{+\infty} \|S(\tau)\phi-\exp(iS_+(\tau))U(\tau)\phi_+\|^4_{W^{4,\infty}}\, d\tau\right)^{1/4} = O(t^{-\theta/2}) \quad \text{as} \quad t \longrightarrow +\infty. \qquad (9)_+$$

(2) Let $\phi_- \in H^{k,2}\cap H^{0,k+2}$ for $n \geq 3$ and $\phi_- \in H^{k,2}\cap H^{0,k+3}$ for $n = 2$. Assume that there is $\sigma \in (0,1/(n-1))$ such that $\|\hat{\phi}_-\|_{p(\sigma)}\|\hat{\phi}_-\|_{p(-\sigma)} \leq \varepsilon_2$, then there exists a unique $\phi \in H^{k,0}$ such that for any $\theta \in (1/2,1)$

$$\|S(t)\phi-\exp(iS_-(t))U(t)\phi_-\|_{k,0} = O(|t|^{-\theta}), \qquad (8)_-$$

$$\left(\int_{-\infty}^{t} \|S(\tau)\phi-\exp(iS_-(\tau))U(\tau)\phi_-\|^4_{W^{4,\infty}}\, d\tau\right)^{1/4} = O(|t|^{-\theta/2}) \quad \text{as} \quad t \longrightarrow -\infty. \qquad (9)_-$$

By Theorems 1-2, the modified wave operators $W_\pm$ is defined as maps $\phi_\pm \longmapsto \phi$ from B^k to $H^{k,0}$, where B^k is the domain of $W_\pm$ given by $\{\psi \in H^{k,2}\cap H^{0,k+2};\ \|\hat{\psi}\|_\infty \leq \varepsilon\}$ in the case (I), for example. The Cauchy problem for (1) is solved so that the asymptotic behavior of solutions is described as $(6)_\pm$ or $(8)_\pm$ when the initial data are in the ranges of $W_\pm$. Moreover, we see: (A) $W_\pm$ are injective and isometries in the L^2 norm. (B) $W_\pm$ are continuous from B^k to $H^{k,0}$, with B^k topologized from the associated weighted Sobolev space. (C) Under the evolution $S(t)$, Range($W_\pm$) are asymptotically orthogonal to every bound state for (1). (D) $W_\pm$ have the intertwining properties: $S(t)W_\pm = W_\pm U(t)$ on $B^0\cap H^{2,0}$.

Our modified wave operators $W_\pm$ have some properties analogous to the modified wave operators of Dollard type for the Coulomb scattering. First, $W_\pm$ intertwine the interacting dynamics and the usual free dynamics as described in (d). Secondly, the modification of the wave operators has no contribution to the asymptotic behavior of the probability density both in the position and momentum space. Lastly, the asymptotic motion of solution of (1) is closely approximated by the solutions $w_\pm$ of

$$i\partial_t w_\pm + (1/2)\Delta w_\pm = f(\hat{\phi}_\pm)(-it\nabla)w_\pm.$$

In the scattering with long range potentials V, with the interacting dynamics given by the unitary operator $\exp(-it(-(1/2)\Delta+V))$ we often associate the modified free evolution given by the solution w of

$$i\partial_t w + (1/2)\Delta w = V(-it\nabla)w.$$

The substitution x by $-it\nabla$ in the potential term is common both to the linear and nonlinear case. Unfortunately, this is not enough for the present nonlinear case and it is our claim that the nonlinear potential $f(u)$ must be modified as $f(\hat{\phi}_{\pm})(-it\nabla)$ through the introduction of $\hat{\phi}_{\pm}$.

It is a simple matter to see how the standard method breaks down in (I) and (II). The standard theory is carried out by solving the equations

$$u(t) = U(t)\phi_{\pm} + i\int_{t}^{\pm\infty} U(t-\tau)f(u(\tau))u(\tau)\,d\tau \qquad (10)$$

for given $\phi_{\pm}$. If the procedure is to work, the integral in (10) should converge in L^2. But this is impossible since every nontrivial solution of (1) does not decay faster than the free solutions and the integrand decays like $O(|t|^{-1})$ at best. The very same situation happens in the Coulomb scattering, where Cook's integral diverges logarithmically.

Our method depends on solving another integral equations around modified free evolutions $v_{\pm}$ instead of $U(t)\phi_{\pm}$ in order that the equations could have convergent integrals. Rather than (10), we consider

$$u(t) = v_{\pm}(t) + i\int_{t}^{\pm\infty} U(t-\tau)(f(u(\tau))u(\tau) - (i\partial_{\tau}+(1/2)\Delta)v_{\pm}(\tau))\,d\tau \qquad (11)$$

for suitable $v_{\pm}$ which give a nice cancellation of the divergent part of $f(u)u$. To this end we introduce the following approximate solutions $v_{\pm}$.

$$v_{\pm}(t) = \exp(iS^{\pm}(t))U(t)M(-t)\phi_{\pm} = i^{-n/2}\exp(iS^{\pm}(t))M(t)D(t)\hat{\phi}_{\pm}, \qquad (12)$$

where $M(t) = \exp(i|x|^2/2t)$. $v_{\pm}$ turn out to satisfy (1) up to the rate $O(|t|^{-2}(\log|t|)^2)$ in L^2 as $t \longrightarrow \pm\infty$, because of the exact cancellation of the divergent terms $f(v_{\pm})v_{\pm}$ and $|t|^{-1}f(\hat{\phi}_{\pm})(t^{-1}x)v_{\pm}$ from $i\partial_t v_{\pm}$. Then, (11) are solvable near $t = \pm\infty$ by a contraction argument on the space defined as a closed ball centered at $v_{\pm}$. The space-time estimates of Strichartz type are essential in this step. The solution u, defined for large times, behaves like $v_{\pm}(t)$ as $t \longrightarrow \pm\infty$, and extends to all times by means of $S(t)$, and then ϕ in the theorems is given by $u(0) = \phi$. The rest of the statements of the theorems follow by proving $v_{\pm}(t) \sim \exp(iS_{\pm}(t))U(t)\phi_{\pm}$ as $t \longrightarrow \pm\infty$. Details will be given elsewhere.

References.

[1] N. Hayashi, T. Ozawa, Modified wave operators for the derivative nonlinear Schrödinger equation, preprint RIMS-746, 1991.

[2] T. Ozawa, Long range scattering for nonlinear Schrödinger equations in one space dimension, RIMS-731, 1990, Comm. Math. Phys. to appear.

Shape Optimization for Periodic Solutions to Multi-Phase Stefan Problems

Atsushi KADOYA

Department of Mathematics
Graduate School of Science and Technology
Chiba University

1.Formulation of an Optimization Problem

Let us consider periodic solutions for a multi-phase Stefan problem described as follows:

$$SP(\Omega)\begin{cases} u_t - \Delta\beta(u) = f & \text{in } Q(\Omega) := R \times \Omega, \\ \beta(u) = g & \text{on } \Sigma(\Omega) := R \times \partial\Omega, \end{cases}$$

where $\hat{\Omega}$ is a fixed bounded domain in $R^N (N \geq 2)$ with smooth boundary $\partial\hat{\Omega}$; Ω is a smooth subdomain of $\hat{\Omega}$; $\hat{Q} := R \times \hat{\Omega}$ and $\hat{\Sigma} := R \times \partial\hat{\Omega}$; $\beta : R \to R$ is a non-decreasing function on R such that

$$(1.1) \qquad \begin{cases} \beta(0) = 0, \ |\beta(r)| \geq C_0|r| - C_0' & \text{for all } r \in R\,, \\ |\beta(r) - \beta(r')| \leq L_0|r - r'| & \text{for all } r, r' \in R\,, \end{cases}$$

where $C_0 > 0$, $C_0' \geq 0$, $L_0 > 0$ are constants. Let T be a given positive constant. Here we suppose that $f \in L^2_{loc}(R; L^2(\hat{\Omega}))$ and $g \in W^{2,2}_{loc}(R; L^2(\hat{\Omega})) \cap L^2_{loc}(R; H^2(\hat{\Omega}))$ is given T-periodic functions.

We use the following function spaces and notations:

(1) We define a bilinear form $a_\Omega(\cdot,\cdot)$ on $H^1(\Omega)$ by

$$a_\Omega(u,v) = \int_\Omega \nabla u \cdot \nabla v dx \quad \text{for } u, v \in H^1(\Omega).$$

We denote by $\langle\cdot,\cdot\rangle_\Omega$ the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$, and by F_Ω the duality mapping from $H^1_0(\Omega)$ onto $H^{-1}(\Omega)$ which is given by the formula

$$\langle F_\Omega v, z\rangle = a_\Omega(v,z) \text{ for all } v, z \in H^1_0(\Omega)$$

Moreover, $(\cdot,\cdot)_\Omega$ denotes the inner product in $L^2(\Omega)$.

(2) We denote by $O := \{\ \Omega \subset \hat{\Omega}; \Omega \text{ is a smooth subdomain of } \hat{\Omega}\ \}$ and by $V(\Omega)$ the set

$$\{z \in H^1_0(\hat{\Omega}); z = 0 \ \text{ a.e. on } \hat{\Omega} - \Omega\} \quad \text{for each } \Omega \in O.$$

Clearly, $V(\Omega)$ is a closed linear subspace of $H^1_0(\hat{\Omega})$. This space is a Hilbert space with inner product $a(\cdot,\cdot) := a_{\hat{\Omega}}(\cdot,\cdot)$ and with norm

$$|v|_{\hat{\Omega}} := a(v,v)^{1/2} (= |\nabla v|_{L^2(\hat{\Omega})}) \quad \text{for } v \in V(\Omega).$$

(3) Now, we introduce a notion of convergence of closed convex sets in a Banach space Y, which is due to Mosco [7]. Let $\{K_n\}$ be a sequence of closed convex sets in Y, and K be a closed convex set in Y. Then we mean by "$K_n \to K$ in Y as $n \to \infty$ (in the sense of Mosco)" that the following two conditions (M1) and (M2) are satisfied:

(M1) If $\{n_k\}$ is a subsequence of $\{n\}$, $z_k \in K_{n_k}$, and $z_k \to z$ weakly in Y as $k \to \infty$, then $z \in K$.

(M2) For any $z \in K$ there is a sequence $\{z_n\} \subset Y$ such that $z_n \in K_n, n = 1, 2, \ldots$, and $z_n \to z$ in Y as $n \to \infty$.

(4) We denote by χ_Ω the characteristic function of Ω on $\hat{\Omega}$ for any subset Ω of $\hat{\Omega}$.

Our shape optimization problem is considered for any non-empty subset O_c of O which is compact in the following sense:

$$(C)\begin{cases} \text{For any sequence } \{\Omega_n\} \subset O_c \text{ there are a subsequence } \{\Omega_{n_k}\} \text{ of } \{\Omega_n\} \text{ and } \Omega \in O_c \\ \text{such that } \chi_{\Omega_{n_k}} \to \chi_\Omega \text{ in } L^1(\hat{\Omega}) \text{ as } k \to \infty \text{ and } V(\Omega_{n_k}) \to V(\Omega) \text{ in } H_0^1(\hat{\Omega}) \\ \text{as } k \to \infty \text{ (in the sense of Mosco).} \end{cases}$$

EXAMPLE 1.1. (c.f. [4]) (1) Let Θ be the class of all C^1-diffeomorphisms from $\overline{\hat{\Omega}}$ onto itself. Now, let Ω' be a subdomain of $\hat{\Omega}$ with smooth boundary $\partial\Omega'$ and $\overline{\Omega'} \subset \hat{\Omega}$. For a given non-empty compact subset Θ_c of Θ, put $O_c = \{\theta(\Omega'); \theta \in \Theta_c\}$. Then this O_c is compact in the sense of (C).

(2) Let $\hat{\Omega} := \{x; |x| < 2\} \subset R^3$, $\Omega_a := \{x; a < |x| < 1\}$ for any $0 < a \le \frac{1}{2}$ and $\Omega := \{x; |x| < 1\}$. Put $O_c := \{\Omega_a; 0 < a \le \frac{1}{2}\} \cup \{\Omega\}$. Then, we see that this subset O_c of O satisfies compactness. ◇

Now, we give the weak formulation of $SP(\Omega)$.

DEFINITION 1.1. Denote by I a compact interval $[t_0, t_1]$ in R. A function $u : I \to L^2(\Omega)$ is called a weak solution of $SP(\Omega)$ on I, if the following two conditions are satisfied:

(w1) $u \in C_w(I; L^2(\Omega))$, $\beta(u) - g \in L^2(I; H_0^1(\Omega))$;

(w2) $-\int_{I\times\Omega} u\eta_t dxdt + \int_I a_\Omega(\beta(u), \eta)dt = \int_{I\times\Omega} f\eta dxdt + \int_\Omega u(t_0, x)\eta(t_0, x)dx$ for all $\eta \in W(I, \Omega)$.

where $C_w(I; L^2(\Omega))$ is the space of all weakly continuous functions from I to $L^2(\Omega)$ and

$$W(I, \Omega) := \{\eta \in H^1((t_0, t_1) \times \Omega); \eta = 0 \text{ on } (t_0, t_1) \times \partial\Omega, \eta(t_1, \cdot) = 0 \text{ on } \Omega\}.$$

DEFINITION 1.2. For a general interval J in R, a function $u : J \to L^2(\Omega)$ is called a weak solution of $SP(\Omega)$ on J if u is a weak solution of $SP(\Omega)$ on I for every compact subinterval I of J in the above sense. In particular, if $J = R$, we call that u is a weak solution of $SP(\Omega)$.

For any $t \in R$ and $\Omega \in O$, let $\{\varphi_\Omega^t\}$ be a family of proper lower-semicontinuous functions on $H^{-1}(\Omega)$ which is defined as follows:

$$\varphi_\Omega^t(z) = \begin{cases} \int_\Omega \hat{\beta}(z(x))dx - (g(t), z)_\Omega & \text{for } z \in L^2(\Omega), \\ +\infty & \text{for } z \in H^{-1}(\Omega)\backslash L^2(\Omega), \end{cases} \tag{1.2}$$

where $\widehat{\beta}$ is the primitive of β with $\widehat{\beta}(0) = 0$, i.e.

(1.3) $$\widehat{\beta}(r) = \int_0^r \beta(s)ds \quad \text{for any } r \in R.$$

Then, concerning the subdifferential $\partial\varphi_\Omega^t$ in $H^{-1}(\Omega)$ it is easy to see that $\partial\varphi_\Omega^t$ is single-valued in $H^{-1}(\Omega)$ and

(1.4) $$\begin{aligned}&\partial\varphi_\Omega^t(z) = F_\Omega(\beta(z) - g(t))\\ &\text{for any } z \in D(\partial\varphi_\Omega^t) = \{z \in L^2(\Omega); \beta(z) - g(t) \in H_0^1(\Omega)\}.\end{aligned}$$

For any interval I of R, a weak solution u of $SP(\Omega)$ on I is obtained as a solution of the following evolution problem in $H^{-1}(\Omega)$:

(1.5) $$u'(t) + F_\Omega(\beta(u(t)) - g(t)) = f(t) + \Delta g(t) \quad \text{for a.e. } t \in I.$$

According to [2; Theorem 2.4], we see that problem $SP(\Omega)$ has a T-periodic solution u that $\beta(u)$ is uniquely determined by Ω.

Now, we consider a shape optimization problem. For a given non-empty subset O_c of O, our optimization problem, denoted by $P(O_c)$, is formulated as follows:

$$P(O_c) \qquad \Omega_* \in O_c; J(\Omega_*) = \inf_{\Omega \in O_c} J(\Omega),$$

where

(1.6) $$J(\Omega) = \frac{1}{2}\int_0^T |\beta(u_\Omega(t)) - \beta_d(t)|^2_{L^2(\Omega)}dt + \frac{1}{2}\int_0^T |g(t)|^2_{L^2(\widehat{\Omega}-\Omega)}dt \quad \text{for } \Omega \in O,$$

u_Ω is a T-periodic weak solution of $SP(\Omega)$, and β_d is a given T-periodic function in $L^2_{loc}(R; L^2(\widehat{\Omega}))$ with period T.

The main results are stated in the following theorems.

THEOREM 1.1. *Let $\{\Omega_n\} \subset O$ and $\Omega \in O$ such that $V(\Omega_n) \to V(\Omega)$ in $H_0^1(\widehat{\Omega})$ as $n \to \infty$ (in the sense of Mosco) and $\chi_{\Omega_n} \to \chi_\Omega$ in $L^1(\widehat{\Omega})$ as $n \to \infty$. Also, denote by u_n and u T-periodic weak solutions of $SP(\Omega_n)$ and $SP(\Omega)$, respectively. Then, as $n \to \infty$,*

(1.7) $$(u_n(t), z)_{\Omega_n} \to (u(t), z)_\Omega \quad \textit{for any } z \in L^2(\widehat{\Omega}),\ t \in R$$

and

(1.8) $$\widetilde{\beta}(u_n) \to \widetilde{\beta}(u) \quad \textit{in } L^2_{loc}(R; L^2(\widehat{\Omega})),$$

where

$$\widetilde{\beta}(u_{\Omega'}) = \begin{cases} \beta(u_{\Omega'}) & \textit{in } Q(\Omega') \\ g & \textit{in } \widehat{Q} - Q(\Omega') \quad \textit{for any } \Omega' \in O. \end{cases}$$

THEOREM 1.2. *Problem $P(O_c)$ has at least one solution Ω_*.*

2. Uniform Estimates for $SP(\Omega)$

In this section, we prove the uniform estimates for T-periodic weak solutions of $SP(\Omega)$ with respect to $\Omega \in O$.

LEMMA 2.1 *There exists a positive constant $M_1 > 0$ such that*

$$\sup_{t\in R} |u_\Omega(t)|_{L^2(\Omega)} \le M_1, \ \sup_{t\in R} |\beta(u_\Omega)|_{L^2(t,t+T;H^1(\Omega))} \le M_1 \tag{2.1}$$

for all $\Omega \in O$, where u_Ω is a T-periodic weak solution of $SP(\Omega)$.

Proof. Multiply (1.5) by $u(\tau)$ in $H^{-1}(\Omega)$ to obtain

$$\frac{1}{2}\frac{d}{d\tau}|u(\tau)|^2_{H^{-1}(\Omega)} = (u'(\tau), u(\tau))_{H^{-1}(\Omega)} = (f(\tau) - \partial\varphi^\tau_\Omega(u(\tau)), u(\tau))_{H^{-1}(\Omega)}.$$

According to [3],

$$\begin{cases} a_2|z|^2_{L^2(\Omega)} + b_1 \ge \varphi^t_\Omega(z) \ge a_1|z|^2_{L^2(\Omega)} - b_1 & \text{for any } z \in L^2(\Omega) \text{ and } t \in R, \\ k_2|z|_{L^2(\Omega)} \ge |z|_{H^{-1}(\Omega)} & \text{for all } z \in L^2(\Omega) \end{cases} \tag{2.2}$$

where a_1, a_2, b_1, k_2 is positive constants independent of $\Omega \in O$.
By (2.2), we have

$$\begin{aligned}(\partial\varphi^\tau_\Omega(u(\tau)), u(\tau))_{H^{-1}(\Omega)} \ge \varphi^\tau_\Omega(u(\tau)) &\ge a_1|u(\tau)|^2_{L^2(\Omega)} - b_1 \\ &\ge a_1k_2^{-2}|u(\tau)|^2_{H^{-1}(\Omega)} - b_1.\end{aligned}$$

Then, we obtain

$$\frac{d}{d\tau}|u(\tau)|^2_{H^{-1}(\Omega)} + a_1k_2^{-2}|u(\tau)|^2_{H^{-1}(\Omega)} \le 2b_1 + \frac{k_2^2}{2a_1}|f(\tau)|^2_{H^{-1}(\Omega)}.$$

After some calculations, we get that

$$\begin{cases} \sup_{t\in R}|u(t)|_{H^{-1}(\Omega)} \le M_1', \ \sup_{t\in R}|u|_{L^2(t,t+T;L^2(\Omega))} \le M_1', \\ \sup_{t\in R}\int_t^{t+T} |\varphi^\tau_\Omega(u(\tau))| d\tau \le M_1'. \end{cases} \tag{2.3}$$

Multiply (1.5) by $u'(\tau)$ in $H^{-1}(\Omega)$ to obtain

$$(\partial\varphi^\tau_\Omega(u(\tau)), u'(\tau))_{H^{-1}(\Omega)} = (f(\tau) - u'(\tau), u'(\tau))_{H^{-1}(\Omega)}.$$

According to [3], we have

$$\frac{d}{d\tau}\varphi^\tau_\Omega(u(\tau)) - (\partial\varphi^\tau_\Omega(u(\tau)), u'(\tau))_{H^{-1}(\Omega)} \le |g'(\tau)|_{L^2(\Omega)}(a_2\varphi^\tau_\Omega(u(\tau)) + b_1).$$

Then, we have

$$\begin{aligned}&\frac{d}{d\tau}\{(\tau - s)\varphi^\tau_\Omega(u(\tau))\} + \frac{1}{2}(\tau - s)|u'(\tau)|^2_{H^{-1}(\Omega)} \\ &\le a_2|g'(\tau)|_{L^2(\Omega)}(\tau - s)\varphi^\tau_\Omega(u(\tau)) + b_1(\tau - s)|g'(\tau)|_{L^2(\Omega)} \\ &\qquad + \frac{1}{2}(\tau - s)|f(\tau)|^2_{H^{-1}(\Omega)} + \varphi^\tau_\Omega(u(\tau)).\end{aligned}$$

After some calculations, we get

$$
\begin{aligned}
&(t-s)|\varphi_\Omega^t(u(t))| + \frac{1}{2}\int_s^t (\tau - s)|u'(\tau)|^2_{H^{-1}(\Omega)} d\tau \\
(2.4)\qquad &\le \{b_1(t-s)\int_s^t |g'(\tau)|_{L^2(\hat{\Omega})} d\tau + \frac{1}{2}(t-s)\int_s^t |f(\tau)|^2_{H^{-1}(\Omega)} d\tau \\
&\qquad + \int_s^t |\varphi_\Omega^\tau(u(\tau))| d\tau\} \exp(W(t) - W(s)).
\end{aligned}
$$

where $t > s$ and

$$W(\tau) = a_2 \int_s^\tau |g'(\sigma)|_{L^2(\Omega)} d\sigma.$$

By (2.3) and (2.4), we derive (2.1). ◇

LEMMA 2.2 *There exists a positive constant $M_2 > 0$ such that*

$$(2.5)\qquad \sup_{s\in R} |\frac{d}{dt}\beta(u_\Omega)|_{L^2(s,s+T;L^2(\Omega))} \le M_2,\ \sup_{t\in R} |\beta(u_\Omega(t))|_{H^1(\Omega))} \le M_2$$

for all $\Omega \in O$, where u_Ω is a T-periodic weak solution of $SP(\Omega)$.

Proof. As was seen in [3], problem $SP(\Omega)$ is able to be approximated by non-degenerate problem $SP(\Omega)^\varepsilon$, $\varepsilon \in (0,1]$:

$$SP(\Omega)^\varepsilon \begin{cases} u_t - \Delta\beta^\varepsilon(u) = f & \text{in } Q(\Omega), \\ \beta^\varepsilon(u) = g & \text{on } \Sigma(\Omega), \end{cases}$$

where $\beta^\varepsilon(r) = \beta(r) + \varepsilon r$, for $r \in R$.
In fact, this problem has a unique T-periodic weak solution $u^\varepsilon \in C_{loc}(R; L^2(\Omega))$ such that $\frac{d}{dt}\beta^\varepsilon(u^\varepsilon) \in L^2_{loc}(R; L^2(\Omega))$ and $\beta^\varepsilon(u^\varepsilon) \in L^2_{loc}(R; H^1(\Omega))$, and besides $u^\varepsilon \to u_\Omega$ in $C_{w\ loc}(R; L^2(\Omega))$ and $\beta^\varepsilon(u^\varepsilon) \to \beta(u_\Omega)$ weakly in $L^2_{loc}(R; H^1(\Omega))$, as $\varepsilon \to 0$. There exists a positive constant C' independent of ε and Ω such that

$$(2.6)\qquad \sup_{t\in R} |u^\varepsilon(t)|^2_{L^2(\Omega)} + \sup_{t\in R} \int_t^{t+T} |\nabla(\beta^\varepsilon(u^\varepsilon))|^2_{L^2(\Omega)} d\tau \le C'.$$

In fact, (2.6) is obtained in a similar way to the proof of Lemma 2.1. Moreover, multiply both sides of $u_t - \Delta(\beta^\varepsilon(u^\varepsilon) - g) = f + \Delta g$ by $\frac{d}{dt}(\beta^\varepsilon(u^\varepsilon) - g)$ and integrate over $(s,t) \times \Omega$ $(s < t)$. Then, by (2.6), we have

$$(2.7)\qquad \sup_{t\in R} |\beta^\varepsilon(u^\varepsilon(t))|_{H^1(\Omega)} \le C'',\ \sup_{s\in R} |\frac{d}{dt}\beta^\varepsilon(u^\varepsilon)|_{L^2(s,s+T;L^2(\Omega))} \le C'',$$
$$\text{for any } \varepsilon \in (0,1] \text{ and } \Omega \in O,$$

where C'' is a constant independent of $\varepsilon \in (0,1]$ and $\Omega \in O$. Therefore, letting $\varepsilon \to 0$, we see that (2.5) holds. ◇

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let $I_t := [t, t+T]$, $Q(\Omega)_t := I_t \times \Omega$ and $Q_t := I_t \times \hat{\Omega}$ for all $t \in R$. Put

$$v_n = \begin{cases} \beta(u_n) & \text{in } Q_n := Q(\Omega_n), \\ g & \text{in } \hat{Q} - Q_n. \end{cases}$$

Consider a function $u_g \in L^\infty(R; L^2(\hat{\Omega}))$ such that $g(t,x) = \beta(u_g(t,x))$ on $\hat{Q}$. Here, we put

$$\tilde{u}_n = \begin{cases} u_n & \text{in } Q_n, \\ u_g & \text{in } \hat{Q} - Q_n. \end{cases}$$

Then, we see that $\tilde{u}_n \in L^\infty(R; L^2(\hat{\Omega}))$. By Lemmas 2.1 and 2.2, there exist a subsequence $\{n_k\}$ of $\{n\}$ and $\tilde{u} \in L^\infty(R; L^2(\hat{\Omega}))$ such that

$$\tilde{u}_{n_k} \to \tilde{u} \text{ weakly* in } L^\infty(R; L^2(\hat{\Omega})). \tag{3.1}$$

Moreover, for any $t \in R$

$$\begin{cases} \tilde{u}_{n_k} \to \tilde{u} & \text{weakly in } W^{1,2}(I_t; H^{-1}(\hat{\Omega})) \text{ and weakly in } L^2(I_t; L^2(\hat{\Omega})), \\ v_{n_k} \to v & \text{weakly in } L^2(I_t; H^1(\hat{\Omega})), \\ v_{n_k}(t) \to v(t) & \text{weakly in } H^1(\hat{\Omega}), \end{cases} \tag{3.2}$$

By Ascoli-Arzela's theorem, we see that

$$v_{n_k} \to v \quad \text{in } C(I_t; L^2(\hat{\Omega})) \text{ and } L^2(I_t; L^2(\hat{\Omega})).$$

By the periodicity of $\tilde{u}_n$, $\tilde{u}$ is also a T-periodic function. Since $v_{n_k} = \beta(\tilde{u}_{n_k})$ in $\hat{Q}$, (3.1) and (3.2), we see that $v = \beta(\tilde{u})$ and that $\beta(\tilde{u}(t)) - g(t) \in V(\Omega)$ for any $t \in R$.

Next, let z be any function in $V(\Omega)$ and ρ be any function in $D(I_t)$. By the assumption, there exists a sequence $\{z_n\}$ such that $z_n \in V(\Omega_n)$ and $z_n \to z$ in $H^1(\hat{\Omega})$. Then, by $z_{n_k} = 0$ a.e. on $\hat{\Omega} - \Omega_{n_k}$, we obtain

$$-\int_t^{t+T} (\tilde{u}_{n_k}(\tau), z_{n_k})_{\hat{\Omega}} \rho'(\tau) d\tau + \int_t^{t+T} a(v_{n_k}(\tau), z_{n_k}) \rho(\tau) d\tau = \int_t^{t+T} (f(\tau), z_{n_k})_{\hat{\Omega}} \rho(\tau) d\tau.$$

Since $z = 0$ on $\hat{\Omega} - \Omega$, we see

$$-\int_t^{t+T} (\tilde{u}(\tau), z)_\Omega \rho'(\tau) d\tau + \int_t^{t+T} a_\Omega(v(\tau), z) \rho(\tau) d\tau = \int_t^{t+T} (f(\tau), z)_\Omega \rho(\tau) d\tau,$$

as $k \to \infty$. This shows that $u = \tilde{u}|_{Q(\Omega)}$ is a periodic solution of $SP(\Omega)$. Then we obtain (1.8). ◇

Proof of Theorem 1.2. Choose a sequence $\{\Omega_n\}$ in O_c such that

$$J(\Omega_n) \to J_* := \inf\{J(\Omega); \Omega \in O_c\}.$$

Then, by assumption, we may assume that $V(\Omega_n) \to V(\Omega_*)$ in $H_0^1(\hat{\Omega})$ (in the sense of Mosco) for some $\Omega_* \in O_c$ and $\chi_{\Omega_n} \to \chi_{\Omega_*}$ in $L^1(\hat{\Omega})$ as $n \to \infty$. Now, denote by u_n a T-periodic weak solution of $SP(\Omega_n)$ and by u_* a T-periodic weak solution of $SP(\Omega_*)$. Then put

$$v_n = \begin{cases} \beta(u_n) & \text{in } Q_n = Q(\Omega_n), \\ g & \text{in } \hat{Q} - Q_n, \end{cases}$$

and

$$v = \begin{cases} \beta(u_*) & \text{in } Q = Q(\Omega_*), \\ g & \text{in } \hat{Q} - Q. \end{cases}$$

From Theorem 1.1, it follows that $v_n \to v$ in $L^2_{loc}(R; L^2(\hat{\Omega}))$ and hence

$$J(\Omega_n) \to J(\Omega_*).$$

Therefore $J(\Omega_*) = J_*$ and Ω_* is a solution of $P(O_c)$. ⋄

References

[1] A.Damlamian, Some results on the multi-phase Stefan problem, Comm. P.D.E. 2. (1977), 1017 - 1044.

[2] A.Damlamian and N.Kenmochi, Periodicity and almost periodicity of solutions to a multi-phase Stefan problem in several space variables, Nonlinear Anal. 12(1988), 921 - 934.

[3] A.Damlamian and N.Kenmochi, Asymptotic Behavior of Solutions to a Multi-phase Stefan Problem, Japan J. Appl. Math., 3(1986), 15 - 36.

[4] A.Kadoya and N.Kenmochi, Optimal Shape Design in Multi-phase Stefan Problems, Advances in Math. Sci. Appl., To appear.

[5] N.Kenmochi, Solvability of nonlinear evolution equations with time-dependent constraints and applicationa,Bull. Fac. Education,Chiba Univ. 30(1981),1 - 87.

[6] M.Kubo, Periodic and almost periodic stability of solutions to degenerate parabolic equations,Hiroshima Math. J. 19(1989),499 - 514.

[7] U.Mosco, Convergence of convex sets and of solutions of variational inequarities, Advances Math. 3(1969), 515 - 585.

Existence Theorems for Quasilinear Elliptic Problems on $\mathbf{R}^n$

By

Yoshitsugu Kabeya

Graduate School of Science and Technology
Kobe University

§1. Introduction

In this paper we consider the following quasilinear elliptic problem :

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = q(x)|u|^{\sigma}u \quad \text{on } \mathbf{R}^n \tag{1}$$

where p and σ are constants which satisfy certain conditions stated later, and λ is a positive constant. We seek a nontrivial solution of (1) as a critical point of the functional

$$\Phi_\lambda(u) = \frac{1}{p}\int_{\mathbf{R}^n}(|\nabla u|^p + \lambda|u|^p)dx - \frac{1}{\sigma+2}\int_{\mathbf{R}^n} q(x)|u|^{\sigma+2}dx. \tag{2}$$

in the Banach space $W^{1,p}(\mathbf{R}^n)$.

From the homogeniety of the first term of $\Phi_\lambda(u)$, under appropriate assumptions on the potential $q(x)$, we can get a nontrivial solution of (1) by solving the constrained minimization problem:

$$\inf_{u\in W^{1,p}(\mathbf{R}^n),\|u\|_\lambda=1}\left(-\int_{\mathbf{R}^n} q(x)|u|^{\sigma+2}dx\right)$$

where $\|u\|_\lambda = \left\{\int_{\mathbf{R}^n}(|\nabla u|^p+\lambda|u|^p)dx\right\}^{1/p}$. Unlike the case $p=2$, it seems that few papers have treated the case of p-Laplace equations with a potential $q(x)$ which may change its sign.

For the sake of simplicity, we consider only the radial csae. But we can get similar result in the non-radial case (see Kabeya [3]). We don't mention the regularity of solutions of our problem here, however, there are several results in the regularity of $p-$Laplace equations including DiBenedetto [1] and Uhlenbeck [7].

For the case $p=2$, many authors including Ding and Ni [2] and Rother [4], [5] considered equations of this type. The former authors studied the case of positive potentials and the latter potentials which may change its sign. In both papers, they used " *à la* uniform integrability" so that the treatment of the problem on $\mathbf{R}^n$ could be similar to that in a bounded domain. Following the idea of them, we consider a more general case, i.e. the case of p-Laplace equations (1).

§2. The radial case

In this section we will study the radial case,i.e., the cae when the potential $q(x)$ in (1) is a function of the variable $r = |x|$.

We define

$$C_{0,r}^{\infty} = \{u \in C_0^{\infty}(\mathbf{R}^n) \mid u \text{ is radial}\}.$$

and denote by $W_r^{1,p}$ the completion of $C_{0,r}^{\infty}$ with respect to the norm $W^{1,p}$. We also denote the area of $\partial B_1(0)$ by ω_n. We use the same letter C for expressing various constants in this section.

We can now prove the following radial lemma which helps us to weaken the assumptions on q.

Lemma 1 (the radial lemma). *For $u \in W_r^{1,p}$ and $1 \leq p < n$, if $x \neq 0$, then*

$$|u(x)|^p \leq C|x|^{p-n}\|u\|_{\lambda}^p.$$

Remark. For the case $p = 2$, the radial lemma will be found in Struwe[6].

Proof. It suffices to show the lemma for $u \in C_{0,r}^{\infty}$. For such u, we have

$$|u(x)|^p = -\int_{|x|}^{\infty} \frac{d}{dr}\{|u(r)|^p\}dr.$$

The right-hand side is estimated as follows:

$$\left|\int_{|x|}^{\infty} \frac{d}{dr}\{|u(r)|^p\}dr\right| \leq p\int_{|x|}^{\infty} |u(r)|^{p-1}\left|\frac{d}{dr}u(r)\right|dr.$$

Now we decompose the last integrand in such a way that identity

$$|u(r)|^{p-1}\left|\frac{d}{dr}u(r)\right| = r^{-(n-1)(n+1-p)/n}\{|u(r)|r^{(n-1)/p^*}\}^{p-1}\left|\frac{d}{dr}u(r)\right|r^{(n-1)/p}$$

holds. The total sum of the exponents of r is equal to 0. In fact,

$$\begin{aligned}
&-\frac{(n-1)(n+1-p)}{n} + \frac{(n-p)(n-1)(p-1)}{pn} + \frac{(n-1)}{p} \\
=&\frac{n-1}{pn}\{-p(n+1-p) + (n-p)(p-1)\} + \frac{n(n-1)}{pn} \\
=&\frac{-n(n-1) + n(n-1)}{pn} \\
=&0.
\end{aligned}$$

We will estimate the integral using the Hölder inequality. First we observe that the Hölder inequality can be applied, because we can raise the power of the decomposed parts to $\alpha,\ \beta,\ \gamma$, respectively, where $\alpha = n/(p-1),\ \beta = p^*/(p-1),\ \gamma = p$, since

$$\begin{aligned}\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} &= \frac{p-1}{n}+\frac{(n-p)(p-1)}{pn}+\frac{1}{p}\\ &= \frac{p(p-1)+(n-p)(p-1)+n}{pn} = 1.\end{aligned}$$

Hence

$$\begin{aligned}&\Big|\int_{|x|}^{\infty}\frac{d}{dx}\{|u(r)|^p\}dr\Big| \le p\int_{|x|}^{\infty}|u(r)|^{p-1}\Big|\frac{d}{dr}u(r)\Big|dr\\ &\le p\Big(\int_{|x|}^{\infty}r^{-(n-1)(n+1-p)/(p-1)}dr\Big)^{(p-1)/n}\\ &\quad\times\Big(\int_{|x|}^{\infty}|u(r)|^{pn/(n-p)}r^{n-1}dr\Big)^{(n-p)(p-1)/pn}\Big(\int_{|x|}^{\infty}\Big|\frac{d}{dr}u(r)\Big|^p r^{n-1}dr\Big)^{1/p}.\end{aligned}$$

If we observe

$$\begin{aligned}&\int_{|x|}^{\infty}r^{-(n-1)(n+1-p)/(p-1)}dr\\ &=\Big\{1-\frac{(n-1)(n+1-p)}{p-1}\Big\}^{-1}\Big[r^{1-(n-1)(n+1-p)/(p-1)}\Big]_{|x|}^{\infty}\\ &=\frac{p-1}{n(n-p)}|x|^{-n(n-p)/(p-1)},\end{aligned}$$

we get

$$\begin{aligned}|u(x)|^p &\le p\{\frac{p-1}{n(n-p)}\}^{(p-1)/n}\omega_n^{-(n-p)(p-1)/pn}\omega_n^{-1/p}|x|^{p-n}\|u\|_{L^{p^*}}^{p-1}\|\nabla u\|_{L^p}\\ &\le p\{\frac{p-1}{n(n-p)}\}^{(p-1)/n}\omega_n^{-(n-p)(p-1)/pn}\omega_n^{-1/p}|x|^{p-n}\Big(\|u\|_{L^{p^*}}+\|\nabla u\|_{L^p}\Big)^p\\ &\le C|x|^{p-n}\|\nabla u\|_{L^p}^p \quad \text{(by the Sobolev embedding theorem)}\\ &\le C|x|^{p-n}\|u\|_{\lambda}^p,\end{aligned}$$

where C is a constant independent of $u \in C_{0,r}^{\infty}$, but depending on p and n.

The proof is complete.

We are now in a position to state our main theorem. We assume that $q(x)$ is a radially symmetric function which is allowed to satisfy some growth condition at infinity.

Theorem 2. *Let $1 < p < n$, and $p^* - 2 < \sigma$. We assume $q : \mathbf{R}^n \to \mathbf{R}$ is measurable, radially symmetric, and satisfies the following assumptions:*

(*A* 4) $$q = q_+ - q_-,\ q_- \in L^1_{loc}.$$

(A 5) $$0 \leq q_+(|x|) \leq f(|x|)|x|^{k(\sigma)}$$

where $f \in L^\infty$ and $k(\sigma) = \frac{n-p}{p}\{(\sigma+2) - p^*\} - \delta$, where δ is a positive constant. Furthermore

(A 6) $$0 \leq f(|x|) \leq C|x|^{2\delta} \text{ on } B_\eta(0)$$

where $\eta > 0$ is a small constant.

(A 7) There exists $u_0 \in W_r^{1,p}$ such that $\int_{\mathbf{R}^n} q|u_0|^{\sigma+2}dx > 0$

Then for all positive λ, there exists a nontrivial weak solution u of (1) in $W_r^{1,p}$.

Remark. Theorem 2 is valid for all $\delta > 0$, not only for a suitable δ. But, according to δ in (A 5), $f(|x|)$ must vanish at the origin as stated in (A 6). *Proof.* we define $D_r = \{u \in W_r^{1,p} \mid \int_{\mathbf{R}^n} q_-|u|^{\sigma+2}dx < \infty,\ \|u\|_\lambda = 1\}$.

Then, by the radial lemma, we have

$$\begin{aligned}\int_{\mathbf{R}^n} q_+|u|^{\sigma+2}dx =&\omega_n \int_0^\infty q_+|u|^{\sigma+2}r^{n-1}dr\\ \leq&C\omega_n\Big(\int_0^\eta q_+\|u\|_\lambda^{\sigma+2}r^{(p-n)(\sigma+2)/p+n-1}dr\\ &+\int_\eta^\infty q_+\|u\|_\lambda^{\sigma+2}r^{(p-n)(\sigma+2)/p+n-1}dr\Big).\end{aligned}$$

We take u in $W_r^{1,p}$ and, from Assumptions (A 5) and (A 6), we get

$$\int_{\mathbf{R}^n} q_+|u|^{\sigma+2}dx \leq C\omega_n\Big\{\int_0^\eta r^\mu dr + \int_\eta^\infty r^\nu dr\Big\},$$

where

$$\begin{aligned}\mu =&2\delta + \frac{n-p}{p}\{(\sigma+2) - \frac{np}{n-p}\} - \delta - \frac{n-p}{p}(\sigma+2) + n - 1,\\ \nu =&\frac{n-p}{p}\{(\sigma+2) - \frac{np}{n-p}\} - \delta - \frac{n-p}{p}(\sigma+2) + n - 1.\end{aligned}$$

Then these values yield $\mu = \delta - 1,\ \nu = -\delta - 1$. Hence, finally, we have

$$\begin{aligned}\int_{\mathbf{R}^n} q_+|u|^{\sigma+2}dx \leq&C\omega_n\Big\{\big[\frac{1}{\delta}r^\delta\big]_0^\eta + \|f\|_{L^\infty}\big[-\frac{1}{\delta}r^{-\delta}\big]_\eta^\infty\Big\}\\ =&C\omega_n\frac{1}{\delta}\Big\{\eta^\delta + \|f\|_{L^\infty}\eta^{-\delta}\Big\}.\end{aligned} \tag{5}$$

This value is independent of $u \in D_r$. Let $\{u_j\}$ be a minimizing sequence for S_λ in D_r. By the Assumption (A 7) we have

$$-\infty < S_\lambda \leq I(u_0) < 0.$$

Since $\{u_j\}$ is a minimizing sequence for $S_\lambda (< 0)$, we may further assume $I(u_j) < 0$. From the fact that

$$\int_{\mathbf{R}^n} q_+ |u_j|^{\sigma+2} dx \leq C$$

and

$$S_\lambda \leq -\int_{\mathbf{R}^n} q_+ |u_j|^{\sigma+2} dx + \int_{\mathbf{R}^n} q_- |u_j|^{\sigma+2} dx < 0,$$

we get

$$\int_{\mathbf{R}^n} q_- |u_j|^{\sigma+2} dx \leq C \ .$$

So we have $\int_{\mathbf{R}^n} q_- |u_j|^{\sigma+2} dx \leq C$ for all j. Moreover, we may assume

$$u_j \rightharpoonup v \text{ weakly in } W^{1,p}, \text{ and } u_j \to v \text{ a.e. in } \mathbf{R}^n.$$

Then we have

$$\|v\|_\lambda \leq \liminf_{j \to \infty} \|u_j\|_\lambda \leq 1$$

and

$$\int_{\mathbf{R}^n} q_- |v|^{\sigma+2} dx \leq \liminf_{j \to \infty} \int_{\mathbf{R}^n} q_- |u_j|^{\sigma+2} dx \leq C.$$

By the fact that $\|u_j\|_\lambda = 1$ and the above estimate (5), for every $\varepsilon > 0$ there exist positive R_ε and r_ε such that

$$\int_{|x| \geq R_\varepsilon} q_+ |u|^{\sigma+2} dx \leq \varepsilon, \quad \int_{|x| \leq r_\varepsilon} q_+ |u|^{\sigma+2} dx \leq \varepsilon$$

for $u \in D_r$.

We now set $T_\varepsilon = \{x \in \mathbf{R}^n \mid r_\varepsilon \leq |x| \leq R_\varepsilon \}$ and apply the Lebesgue dominant convergence theorem (from Lemma 1 and (A 5), we can take a summable dominant function; see the above estimate on $q_+ |u|^{\sigma+2}$) to obtain

$$\int_{T_\varepsilon} q_+ |u_j|^{\sigma+2} dx \longrightarrow \int_{T_\varepsilon} q_+ |v|^{\sigma+2} dx \quad \text{as } j \to \infty.$$

Since

$$I(v) \leq \int_{\mathbf{R}^n} q_- |v|^{\sigma+2} dx - \int_{T_\varepsilon} q_+ |v|^{\sigma+2} dx,$$

we get in view of the above estimates,

$$
\begin{aligned}
I(v) =& -\int_{\mathbf{R}^n} (q_+ - q_-)|v|^{\sigma+2}dx \\
=& \int_{\mathbf{R}^n} q_-|v|^{\sigma+2}dx - \int_{\mathbf{R}^n} q_+|v|^{\sigma+2}dx \\
\leq& \int_{\mathbf{R}^n} q_-|v|^{\sigma+2}dx - \int_{T_\varepsilon} q_+|v|^{\sigma+2}dx \\
\leq& \liminf_{j\to\infty}\Big(\int_{\mathbf{R}^n} q_-|u_j|^{\sigma+2}dx - \int_{T_\varepsilon} q_+|u_j|^{\sigma+2}dx\Big) \\
\leq& \liminf_{j\to\infty}\Big(\int_{\mathbf{R}^n} q_-|u_j|^{\sigma+2}dx - \int_{\mathbf{R}^n} q_+|u_j|^{\sigma+2}dx + 2\varepsilon\Big) \\
=& S_\lambda + 2\varepsilon.
\end{aligned}
$$

So we have

$$I(v) \leq \liminf_{j\to\infty}(I(u_j) + 2\varepsilon) = S_\lambda + 2\varepsilon.$$

Hence, we obtain $I(v) \leq S_\lambda$. Finally we must show $v \in D_r$. We set $\alpha = \|v\|_\lambda$, then $\alpha \in (0,1]$ and $\frac{1}{\alpha}v \in D_r$. Thus

$$S_\lambda \leq I(\frac{1}{\alpha}v) = \alpha^{-(\sigma+2)}I(v) \leq \alpha^{-(\sigma+2)}S_\lambda < 0$$

Since $S_\lambda < 0$, we get $\alpha = 1$. Hence $v \in D_r$ and $I(v) = S_\lambda$. We note that $|q||v|^{\sigma+1}$ is locally integrable. This is because

$$
\begin{aligned}
\int_B |q||v|^{\sigma+1}dx =& \int_B |q|^{1/(\sigma+2)}|q|^{(\sigma+1)/(\sigma+2)}|v|^{\sigma+1}dx \\
&\leq \Big(\int_B |q|dx\Big)^{1/(\sigma+2)} \cdot \Big(\int_B |q||v|^{\sigma+2}\Big)^{(\sigma+1)/(\sigma+2)} < +\infty
\end{aligned}
$$

holds for all bounded domains $B \subset \mathbf{R}^n$ in view of the Hölder inequality. By the Gateaux derivative at v in D_r, we have

$$\int_{\mathbf{R}^n}\Big\{|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi + \lambda|u|^{p-2}u\varphi\Big\}dx = |S_\lambda|^{-1}\int_{\mathbf{R}^n} q|u|^\sigma u\varphi dx$$

for every $\varphi \in C_0^\infty(\mathbf{R}^n)$.

Thus in view of the Lagrange multiplier rule(see Struwe [6]), we find that $u = |S_\lambda|^{-1/(\sigma-p+2)}v$ is a nontrivial weak solution of (1).

The proof is complete.

References

[1] DiBenedetto, E. , $C^{1,\alpha}$ local reguralarity of weak solutions of degenerate elliptic equations, Nonlinear Analysis 7(1983), 827–850.

[2] Ding W.-Y.and Ni W.-M., On the existence of positive entire solutions of a semilinear elliptic equation, Archs. Ration. Mech. Analysis 91(1986), 283–308.

[3] Kabeya, Y. , Existence theorems for quasilinear elliptic problems on $\mathbf{R}^n$, *to appear.*

[4] Rother, W. , Existence theorems for a nonlinear elliptic eigenvalue problem on $\mathbf{R}^n$, Nonlinear Analysis 15(1990), 381–386.

[5] Rother, W. , Some existence results for the equations $\Delta U + K(x)U^p = 0$, Commun.in P.D.E. 15(1990), 1461–1473.

[6] Struwe, M. , *Variational methods*, Springer-Verlag (1990).

[7] Uhlenbeck, K. , Regularity for a class of non-linear elliptic systems, Acta Math. 138(1977), 219–240.

非線型発展方程式の局所解の存在について

大阪大学 理学部　川口謙一

§1. 序

次の発展方程式の初期値問題について考える。

$$(P)\begin{cases} u'(t) \in A(t)u(t) + G(u)(t) & s \le t \le T \\ u(s) = u_0 \end{cases}$$

ここで、$A(t)$は実Banach空間Xの部分集合$D(A(t))$で定義されたm-dissipative operatorである。この初期値問題の局所解の存在について、$A(t)=A$がtに依存しない場合がI.I. Vrabie [V1], [V2], N. Hirano [H]によって研究されている。

ここでは、$A(t)$がtに依存する場合を考える。

§2 [H]を拡張した結果 ── 局所解の存在 (I)

(仮定)

各$t \in [0,T]$に対し、$A(t) \subset X \times X$はm-dissipativeであるとする。更に$\{A(t); 0 \le t \le T\}$は次の条件を満たすとする。

(A1) $^\exists f : [0,T] \longrightarrow X$; 連続かつ有界変動

$^\exists L : [0,\infty[\longrightarrow [0,\infty[$; 非減少かつ連続

$^\exists \lambda_0 > 0$ s.t. $0<\lambda<\lambda_0$, $t, \tau \in [0,T]$, $x \in X$に対し、

$$\|A_\lambda(t)x - A_\lambda(\tau)x\| \le \|f(t) - f(\tau)\| L(\|x\|)(1 + \|A_\lambda(\tau)x\|)$$

(A 2) $^\forall\lambda>0, {}^\forall t\in[0,T]$ に対し. $J_\lambda(t)$ は compact

ここで、$A_\lambda(t)$, $J_\lambda(t)$ は次で定義されている。
$J_\lambda(t)=(I-\lambda A(t))^{-1}$, $A_\lambda(t)=\lambda^{-1}(J_\lambda(t)-I)$

$s\geq 0$ とし、U を X の空でない開部分集合として作用素 G が次の条件を満たすとする。

(C 1) $s<a$ なる a に対し.
$G: C([s,a];U)\longrightarrow C([s,a];X)$ は連続

(C 2) $^\exists k:(0,\infty)\longrightarrow(0,\infty)$
$^\exists H:[0,\infty)\longrightarrow[0,\infty)$: 非減少 s.t.
$var(G(u);[s,t])\leq k(d)H(var(u;[s,t]))$
whenever $u\in C([s,a];U)$ は有界変動で.
$\|u(t)\|\leq d$ for $s\leq t\leq a$

(C 3) $u\in C([s,T];U)$ に対し.
$G(u|_{[s,T_1]})=G(u)|_{[s,T_1]}$ for $s<T_1\leq T$

このような G の例:

$$G(u)(t)=\int_s^t a(t-\tau)g(\tau,u(\tau))d\tau$$

$a:[0,\infty)\longrightarrow\mathbb{R}$: 連続. $\alpha\in L^1_{loc}(0,\infty)$
$g:[0,\infty)\times X\longrightarrow X$: 連続
$\|g(t,x)\|\leq b(t)\|x\|+c$
$b\in L^1_{loc}(0,\infty)$, $c>0$

$\hat{D}(A(t)) = \{x \in X ; \lim_{\lambda \downarrow 0} \|A_\lambda(t)x\| < +\infty\}$

とおく。(A1)を仮定すると、$\hat{D}(A(t))$はtに依存しない。

$h \in C([s,T];X)$, $var(h;[s,T]) < +\infty$

なるhに対して、

$$B(t;h) = A(t) + h(t)$$

すなわち、

$$B(t;h)x = A(t)x + h(t) \quad \text{for } x \in D(A(t))$$

$$D(B(t;h)) = D(A(t))$$

とおくと、$\{B(t;h) ; s \leq t \leq T\}$ は evolution operator $\{U^h(t,\tau) ; s \leq \tau \leq t \leq T\}$ を生成する。

このとき、次の定理が成り立つ。

定理1 (A1), (A2), (C1)〜(C3)を仮定すると、

$\hat{D}(A(s)) \cap U$ の任意の元 u_0 に対して、適当な $T_1 \in (s,T]$、$u \in C([s,T_1];U)$ が存在して、次を満たす。;

$$U^{G(u)}(t,s)u_0 = u(t) \quad \text{for } t \in [s,T_1]$$

さらに、Xがreflexiveならば、uは(P)の strong solution になる。

証明には次の補題を用いる。

補題1

Kが$C([s,T];X)$の有界部分集合で、各元が有界変動ならば、正の数C_1, C_2が存在して、

$$\|U^h(t,s)u_0 - U^h(\tau,s)u_0\| \leq (t-\tau)(C_1 + C_2 var(h;[s,\tau]))$$

for $s \leq \tau < t \leq T$, $h \in K$

補題1は $U^{\lambda}(t,s)u_0$ が $(I-\lambda B(t;u))^{-1}$ を用いて積公式で表わされることと、(A1)を用いて証明できる。

定理1の証明の概略

正定数 M, r, T をとって、

$T > s$, $N(u_0, r) \equiv \{x \in X ; \|x - u_0\| \leq r\} \subset U$,

$\|G(u)(t)\| \leq M$

for $u \in C([s,T]; U)$, $u(t) \in N(u_0, r)$ on $[s,T]$

$(T-s)M + \|U(t,s)u_0 - u_0\| \leq r$ for $s \leq t \leq T$

をみたすようにする。ここで $\{U(t,s)\}$ は $\{A(t)\}$ から生成される evolution operator とする。

$$K^{T_1}(V) = \{u \in C([s,T_1]; U) ; var(u; [s,T_1]) \leq V, \; u(t) \in N(u_0, r) \text{ on } [s,T_1]\}$$

とおくと、補題1と(C2)を用いて、次が示される。

補題2

$V > 0$ に対し、適当な $T_1 \in (s, T]$ が存在して、次を満たす。

$u \in K^{T_1}(V)$ ならば、$U^{G(u)}(\cdot, s)u_0 \in K^{T_1}(V)$

そこで、写像 $Q : K^{T_1}(V) \longrightarrow C([s,T_1]; X)$ を

$(Qu)(t) = U^{G(u)}(t,s)u_0$ on $[s,T_1]$ for $u \in K^{T_1}(V)$

で定めると、$QK^{T_1}(V) \subset K^{T_1}(V)$ となる。

また、(C1)より、Q は連続である。

更に、(A2)を用いて、

補題3

$QK^{T_1}(V)$ は $C([s,T_1]; X)$ で relatively compact

$K^{T_1}(V)$は閉凸集合なので、Schauderの不動点定理より、Qの不動点が存在する。Qの不動点uは、

$u \in C([s,T_1];U)$

$U^{G(u)}(t,s)u_0 = u(t) \qquad \text{for } t \in [s,T_1]$

を満たす。 □

§3 大域解の存在

ここでは $U = X$, $u_0 \in \hat{D}(A(s))$ とする。

(仮定)

(C2)′ $H(\rho) = \rho + 1$ として、(C2)が成り立つ。

(C4) Gは有界集合を有界集合にうつす。

(C5) ${}^\forall T > s$ に対し、${}^\exists \alpha = \alpha(T) > 0$ s.t.

${}^\forall t \in [s,T]$, $w \in C([s,t];X)$ に対し、

$\|G(w)\|_{C([s,t];X)} \leq \alpha(\|w\|_{C([s,t];X)} + 1)$

定理2 任意の正数Tに対し、(A1),(A2),(C1),(C2)′,(C3)が成り立つと仮定する。更に(C4),(C5)を仮定すると、

${}^\exists u \in C([0,\infty);X)$ s.t.

$U^{G(u)}(t,s)u_0 = u(t) \qquad \text{for } t \in [0,\infty)$

さらに、Xがreflexiveならば、uは初期値問題：

$$\begin{cases} u'(t) \in A(t)u(t) + G(u)(t) & s \leq t \\ u(s) = u_0 \end{cases}$$

のstrong solutionである。

証明には、Gronwallの補題を用いればよい。

§4 [V2]を拡張した結果 — 局所解の存在(II)

ここでは、(A1)の仮定のfが絶対連続であるとする。
また(A1)を仮定すると、$\hat{D}(A(t))$はtに依存しないので、
$\mathcal{U}_A = \hat{D}(A(t)) \cap \mathcal{U}$とおく。

$g : [s, T] \times \mathcal{U}_A \longrightarrow X$

$k : [0, T-s] \longrightarrow L(X)$

をともに連続関数とする。また、kは有界変動とする。
ただし、$L(X)$はXからXへの有界線型作用素で、定義域がXに一致するものの全体である。

~~$$G(u)(t) = \int_s^t k(t-\tau)\, g(\tau, u(\tau))\, d\tau$$~~

とおく。

定理3 (A1), (A2)を仮定する。このとき$\forall u_0 \in \mathcal{U}_A$に対して、
$T_1 \in (s, T]$, $u \in C([s, T_1]; \mathcal{U})$が存在して、次をみたす。;
$G(u) \in C([s, T_1]; X)$は有界変動で、

$$U^{G(u)}(t, s)u_0 = u(t) \qquad \text{for } t \in [s, T_1]$$

さらに、Xがreflexiveならば、uは(P)のstrong solutionになる。

§5 evolution operatorのcompact性

仮定(A2)と$\{A(t)\}$から生成されるevolution operator $\{U(t, s)\}$のcompact性との関係について、次の結果が成り立つ。

定理4 次の(I), (II)は同値である。;
(I) $0 \le s < t \le T$に対し、$U(t, s)$は$\hat{D}$上compact
(II) 1°) $\lambda > 0$, $0 \le s \le T$に対し、$J_\lambda(s)$はcompact
2°) $U(t, s)$は$\hat{D}$の有界部分集合上で$t_0 > s$で同程度連続

References

[H] : N. Hirano : Local existence theorems for nonlinear differential equations, SIAM J. Math. anal. 14 (1983) 117−125

[V1] ; I. I. Vrabie : The nonlinear version of Pazy's local existence theorem, Israel J. Math. 32 (1979) 221−235

[V 2] : I. I. Vrabie : Compactness Methods for Nonlinear Evolutions, Pitmann Monogr. Surv. Pur. Appl. Math. 32, 1987 264−269

ON EXACT SOLUTION OF SOME QUASILINEAR HYPERBOLIC EQUATION

黒木場 正城　（福岡大学　理学部）

次のような初期値問題について考える。

$$(P)\quad \begin{cases} u_{tt} - f(\xi)\Delta u = g(x,t,u), & x \in R^n,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in R^n, \end{cases}$$

但し、 $\xi = \int_{R^n} |\nabla u|^2 dx$ 、 $f(\xi) \geq 0$ で $f(\xi) \in C^1[0,\infty)$ 、
$g(x,t,u)$は、与えられた関数とする。$g(x,t,u)=0$ であれば "Kirchhoff quasi-linear hyperbolic equation" と呼ばれる方程式で初期値問題、初期値境界値問題について多くの研究がなされている。(c.f. [1]-[16])

ここでは、$f=\sqrt{\xi}$ 、$g(x,t,u)=u^2$として１次元の場合について考える。

そこでまず変数分離をする。

$u(x,t) = v(x)\varphi(t)$ として（P）に代入すると

$$(P)'\begin{cases} v(x)\varphi_{tt}(t) - \left(\int_{-\infty}^{\infty} |v_x(x)|^2 dx\right)^{1/2} |\varphi(t)|\, v_{xx}(x) = v^2(x)\varphi^2(t) \quad \cdots\cdots(0) \\ u(x,0) = \varphi(0)v(x) = \varphi_0 v(x) \\ u_t(x,0) = \varphi_t(0)v(x) = \varphi_1 v(x), \end{cases}$$

$\varphi(t) \geq 0$と仮定すれば　（０）より

$$\frac{\beta v_{xx}(x) + v^2(x)}{v(x)} = \frac{\varphi_{tt}(t)}{\varphi^2(t)} = \lambda$$

但し、$\beta^2 = \int_{-\infty}^{\infty} |v_x(x)|^2 dx$ であり　λは正の定数とする。

これより、次の問題に分けられる。

$$
(*)\begin{cases} \beta v_{xx}(x) - \lambda v(x) = -v^2(x) \cdots\cdots\cdots (1) \\ \beta^2 = \int_{-\infty}^{\infty} |v_x(x)|^2 dx < +\infty \end{cases}
$$

,

$$
(**)\begin{cases} \varphi_{tt}(t) = \lambda\varphi^2(t) \cdots\cdots (2) \\ \varphi(0) = \varphi_0, \quad \varphi_t(0) = \varphi_1 \\ \varphi(t) \geq 0 \end{cases}
$$

,

そこでまず（＊）について考える。なめらかな偶関数で $\lim_{|x|\to+\infty} v(x) = 0$ となる様な解を構成するために次の x ≧ 0 に於ける問題を扱う。

$$
(*)'\begin{cases} \beta v_{xx}(x) - \lambda v(x) = -v^2(x) \cdots\cdots\cdots\cdots (1) \\ v_x(0) = 0, \quad \lim_{x\to+\infty} v(x) = 0, \\ \frac{\beta^2}{2} = \int_0^{\infty} |v_x(x)|^2 dx < +\infty \end{cases}
$$

,

（１）式にｖ（ｘ）をかけて０からｘまで積分すると

$$
\beta v_x^2(x) = \lambda v^2(x) - \frac{2}{3} v^3(x) + \frac{2}{3} v^3(0) - \lambda v^2(0)
$$

ここで $v(0) = \frac{3}{2}\lambda$ とすれば

$$
\sqrt{\beta}\, v_x(x) = \pm \sqrt{\lambda v^2(x) - (2/3)v^3(x)} \qquad (3)
$$

もし　$v(x) > 0$ で $v_x(x) < 0$ であれば（３）より、

$$\frac{v_x(x)}{v(x)\sqrt{\lambda - (2/3)v(x)}} = -\frac{1}{\sqrt{\beta}} \qquad (4)$$

（４）を$\frac{3}{2}\lambda$からｖ（ｘ）まで積分すれば

$$\int_{v}^{(3/2)\lambda} \frac{dw}{w\sqrt{\lambda - (2/3)w}} = \frac{1}{\sqrt{\beta}}x \qquad (5)$$

が得られる。（５）より　$\lim_{x\to\infty} v(x) = 0$　がわかる。（５）の左辺を計算して

$$\frac{1}{\sqrt{\lambda}}\log\frac{\sqrt{\lambda}+\sqrt{\lambda-(2/3)v(x)}}{\sqrt{\lambda}-\sqrt{\lambda-(2/3)v(x)}}\left(=\frac{1}{\sqrt{\beta}}x\right)$$

したがって

$$\begin{aligned} v(x) &= \frac{3}{2}\lambda\left\{1-\left(\frac{\sinh\mu x}{\cosh\mu x}\right)^2\right\} \\ &= \frac{3\lambda}{2\cosh^2\mu x} \end{aligned} \qquad (6)$$

但し、$\mu = \frac{1}{2}\sqrt{\frac{\lambda}{\beta}}$　、また（６）を微分すると

$$v_x(x) = \frac{-3\lambda\mu\sinh\mu x}{\cosh^3\mu x}$$

であるから

$$\int_0^\infty |v_x(x)|^2 dx = \frac{6}{5}\lambda^2\mu = \frac{3\lambda^2}{5}\sqrt{\frac{\lambda}{\beta}}$$

である。

これらの事より次の定理が成り立つ。

定理　1

$\beta = \sqrt[5]{\frac{36}{25}}\lambda$ 、$v(0) = \frac{3}{2}\lambda$ そして $v_x(0) = 0$ のとき

（＊）の解　v（x）は

$$v(x) = \frac{3\lambda}{2\cosh^2 \mu x},$$

$$\mu = \frac{1}{2}\sqrt{\frac{\lambda}{\beta}}$$

となり

$$\int_{-\infty}^{\infty} |v_x(x)|^2 dx = \beta^2$$

となる。

次に（＊＊）の解を求める。

（2）の両辺に $\varphi_t(t)$ を掛けて0からtまで積分すると

$$3\varphi_t^2(t) = 2\lambda\varphi^3(t) - 2\lambda\varphi^3(0) + 3\varphi_t^2(0)$$

ここで $-2\lambda\varphi^3(0) + 3\varphi_t^2(0) = 0$ とすれば

$$(\varphi(t))^{-(3/2)}\varphi_t(t) = \pm\sqrt{\frac{2}{3}\lambda}$$

さらに0からtまで積分すると

$$\varphi(t) = (\varphi^{-1/2}(0) \pm \sqrt{\frac{\lambda}{6}}t)^{-2}$$

以上のことより

定理　2

（2）の解は $-2\lambda\varphi^3(0) + 3\varphi_t^2(0) = 0$ の条件の下で

$$\varphi(t) = (\varphi_0^{-1/2} \pm \sqrt{\frac{\lambda}{6}}t)^{-2}$$

である。

参考文献

1. AROSIO A.& SPAGNOLO S., Global Solutions of the Cauchy Problem for a Non Linear Hyperbolic Equation, Universita di Pisa, Dipartamento di Matematica, Roma(1982).
2. BALL J.M., Initial boundary value problems for an extensible beam, J. math. Analysis Applic. 42, 61-90(1973).
3. BERNSTEIN I.N., Sobre umaclasse deequacoes funcionais. Izv. Acad. Nauk SSSR-Math, 4, 17-26(1960).
4. DICKEY R.W., The initial value problem for a nonlinear semi infinite string, Proc. R. Soc. Edinb. 82A, 19-26(1978).
5. DICKEY R.W., Infinite system of nonlinear oscillation equations with linear damping, SIAMJ. appl. Math. 19, 459-468(1970).
6. EBIHARA Y. &MEDEIROS L.A., MIRANDA M.M., Local solutions for a nonlinear degenerate hyperbolic equation, Nonlinear Analysis. 10, 27-40(1986).
7. IKEHATA R., On solutions to some quasilinear hyperbolic equations with nonlinear inhomogeneous terms, Nonlinear Analysis. 17, 181-203(1991).
8. LIONS J.L., On some questions in boundary value problems of mathematical physics, Instituto de Matematica, UFRJ, Rio de Janeiro, RJ(1978).
9. MEDEIROS L., On a new class of nonlinear wave equations, J. math. Analysis Applic. 69, 252-262(1979).
10. MEDEIROS L.& MILLA MIRANDA M., Local solutions for a nonlinear unilateral problem, (to appear).
11. MENZALA G.P., On Classical Solutions of a quasi linear hyperbolic equation, Nonlinear Analysis 3, 613-619(1979).
12. NISHIDA T., A note on nonlinear vibrations of the elastic string, Mem. Fac. Engng Kyoto Univ. 34, 329-341, (1971).
13. NISHIHARA K., On a global solutions of some quasilinear hyperbolic equation (to appear).
14. NISHIHARA K., Global existence and asymptotic behaviour of the solution of some quasilinear hyperbolic equation with linear damping, Funkcial. Ekvac. 32, 343-355, (1989).
15. NISHIHARA K. and YAMADA Y., On a global solution of some degenerate quasilinear hyperbolic equations with dissipasive terms, Funkcial. Ekvac. 33, 151-159, (1990).
16. POHOZAEV S.I., On a class of quasilinear hyperbolic equations, Mat. USSR Sbornik 25, 145-158(1975).

集中効果を持った反応拡散方程式系の解の挙動について

桑村雅隆

広島大学理学部数学科学生

バクテリアの中には大腸菌のように、自分たちの仲間を呼び寄せる働きを持ったにおいのような化学物質(走化性物質)を分泌しながら活動しているものがある。そのような生物の時間空間的な分布を記述する方程式のひとつに次のものがある [3]。

$$
(1)\qquad \begin{cases} u_t = a \triangle u - b \nabla (u \nabla v), & t > 0,\ x \in \mathbf{R}^n \\ v_t = c \triangle v - dv + eu & t > 0,\ x \in \mathbf{R}^n. \end{cases}
$$

ここで、$u = u(x,t)$ は生物の個体数密度を表わし、$v = v(x,t)$ は生物の分泌する走化性物質の濃度を表わす。また、a, b, c, d, e は正の定数であって、それぞれ具体的な意味を持つパラメーターであるが、これらの説明をする前に、方程式 (1) の表す内容を簡単に説明しよう。

今ここで考えている生物は、自然に拡散していくとともに、彼ら自身の分泌する走化性物質の多いところに集まろうとする傾向を持っている。実際、そのことは次のように考えるとわかる。(1) の第1式を $\mathbf{R}^n$ 内の任意の十分滑らかな有界領域 B において積分し、Gauss-Green の公式を適用すると

$$
(2)\qquad \frac{d}{dt}\int_B u = \int_{\partial B} a \nabla u \cdot \mathbf{n} ds + \int_{\partial B} (-bu \nabla v) \cdot \mathbf{n} ds
$$

を得る。ただし、$\mathbf{n}$ は ∂B 上の外向き単位法線ベクトルである。(2) の左辺の第1項を見ると、もしも B の内部のほうが、B の外部に比べて、生物の個体数密度 u が大きければ、

$$
\int_{\partial B} a \nabla u \cdot \mathbf{n} ds < 0
$$

となり、u は B の内部において減少する。つまり、生物は自然に拡散していることがわかる。(2) の左辺の第2項を見ると、もしも B の内部のほうが、B の外部に比べて走化性物質の濃度 v が大きければ、

$$
\int_{\partial B} (-bu \nabla v) \cdot \mathbf{n} ds > 0
$$

となり、生物の個体数密度 u は B の内部において増加する。つまり、生物は走化性物質の多いところに集中しようとする傾向を持っていることがわかる。

一方、生物の分泌する走化性物質の量は生物の個体数に比例して多くなると考えられるので、その濃度は生物の個体数密度に比例して増加する。このことが、(1) の第2式の第3項で表わされている。また、化学物質は自然に拡散したり分解したりする。このことは、それぞれ (1) の第2式の第1項と第2項で表わされている。以上が (1) の第2式の意味することである。

これで、方程式 (1) についての説明を終わる。また、今まで述べてきたことから a は生物の拡散を、b は生物が走化性物質の多いところに集中しようとする効果を、c は走化性物質の拡散を、d は走化性物質の分解を、e は生物の分泌する走化性物質の量をそれぞれ制御しているパラメーターであることがわかる。

(1) に関する初期値問題や、$\mathbf{R}^n$ の適当な領域において、適当な境界条件のもとでの (1) に関する初期値-境界値問題については、解の有界性、爆発、定常解の分岐など多くの研究がある。(例えば、[2],[4]) ここでは、(1) に関連した次の方程式の初期値-境界値問題を考える。

$$
(3) \qquad \begin{cases} u_t = a \triangle u - b \nabla (u \nabla v) + \lambda g(u), & t > 0,\ x \in \Omega \\ v_t = c \triangle v - dv + eu & t > 0,\ x \in \Omega. \end{cases}
$$

$$
(4) \qquad \frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} = 0,\ x \in \partial\Omega
$$

$$
(5) \qquad \begin{cases} u(x,0) = \phi(x) \\ v(x,0) = \psi(x) \end{cases}
$$

ただし、Ω は十分滑らかな境界を持つ $\mathbf{R}^n$ $(n = 1,2,3)$ 内の有界領域であって、g は

$$
(6) \qquad g(u) = u(1-u)(u-\mu),\ 0 < \mu < 1
$$

とする。また、λ は正の定数であって、非線形項 g の強さを制御するパラメーターである。さらに、初期値は十分滑らかであって (4) および

$$
\phi(x), \psi(x) \geq 0
$$

を満たすとする。(3) は (1) の第1式の左辺に非線形項 λg を付け加えた形になっている。これは、生物が増殖して個体数を増やすためには、一定量の個体数が必要であるということ、つまり、生物の個体数密度はある一定値（いき値）μ を越えなければ減少するが、μをこえれば、増加して高い安定な状態にとどまるという効果を表わすものである。

まず、方程式(3)-(5)の（古典）解の存在と一意性について調べよう。

定理1： 任意の正の数 T に対して [0,T] で (3)-(5) の古典解が一意的に存在する。

注意： $\sup\{|u(x,t)|; 0<x<1, 0<t<\infty\}<\infty$
　$\sup\{|v(x,t)|; 0<x<1, 0<t<\infty\}<\infty$ であるかどうかはよくわからない。

H.Amann [1] により、(3)-(5) の古典解の時間的局所解の存在と一意性は直ちにわかる。任意の有限時間内において (3)-(5) の古典解が存在するかどうかを示すには、次の補題が必要である。

補題： $u(x,t), v(x,t)$ を (3)-(5) の [0,T] における古典解とするとき

$$\| u(\cdot,t)\|_{L^4(\Omega)} \leq \| u(\cdot,0)\|_{L^4(\Omega)} \exp(\gamma t)$$

$$\| v(\cdot,t)\|_{L^4(\Omega)} \leq C \| v(\cdot,0)\|_{\alpha} \exp(-dt) + C \| u(\cdot,0)\|_{L^4(\Omega)} \exp(\gamma t)$$

ただし、C は $a,b,c,d,e,\alpha,\lambda,\mu,n$ に依存する定数であり、$\gamma = C(1+\| v(\cdot,0)\|_{\alpha}^{1/2})$ である。また、$\|\cdot\|_{\alpha}$ は Neumann 条件下で $L^4(\Omega)$ 上の Laplacian が定義する fractional power space である。この補題を用いると、再び H.Amann [1] により、定理1が示せる。

次に、生物の拡散と集中効果が小さい場合における (3)-(5) の解の挙動を調べよう。簡単のため、ここでは $a=\epsilon^2, b=\epsilon, \lambda=1$ とし、ϵ は十分小さな正の数としよう。このとき (3)-(5) の解の u-成分 は、適当な初期条件下で interface を持つことが数値計算上知られている [5]。interface とは、下図1のように Ω の内部において u の値が空間的に急激に変化している部分である。

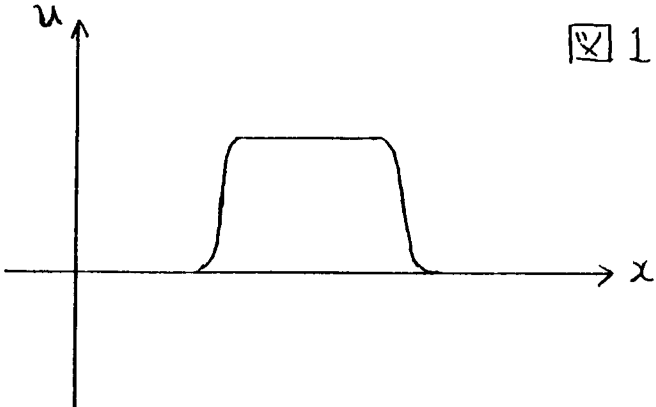
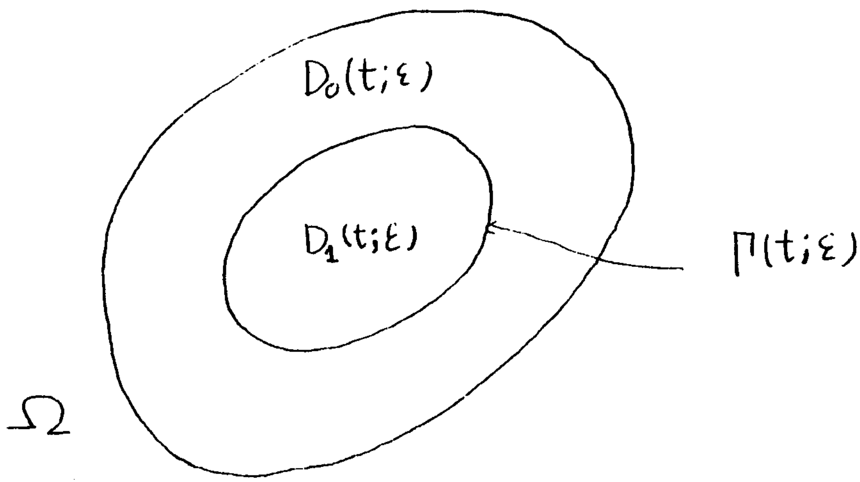


図1

このような interface の運動を考えるために、時間 t とともに発展しているΩ 内の超曲面 $\Gamma(t;\varepsilon)$ で次のものを考える。$\partial\Omega\cap\Gamma(t;\varepsilon)=\phi$ としΩ は $\Gamma(t;\varepsilon)$ によって下図2のような2つの領域 $D_0(t;\varepsilon)$ と $D_1(t;\varepsilon)$ とに分けられているとする。

図2



さらに、$\Gamma(t;\varepsilon)$ は次の方程式にしたがっているとする。

$$
(7)\qquad \begin{cases} \dfrac{\partial\Gamma}{\partial t}=\varepsilon[\dfrac{\partial v}{\partial \mathbf{n}}+c(\mu)]-\varepsilon^2\kappa \\ \Gamma(0;\varepsilon)=\Gamma_0 \\ c\triangle v-dv+eu_\Gamma=0 \quad in \quad \Omega \\ \dfrac{\partial v}{\partial \mathbf{n}}=0 \quad on \quad \partial\Omega \end{cases}
$$

ただし、

$$
(8)\qquad u_\Gamma(x)=\begin{cases} 0 & x\in D_0(t;\varepsilon) \\ 1 & x\in D_1(t;\varepsilon) \end{cases}
$$

および、$c(\mu)=-\sqrt{2}(1/2-\mu)$,

κ は$\Gamma(t;\varepsilon)$ の平均曲率、

$\dfrac{\partial\Gamma}{\partial t}$ は$\Gamma(t;\varepsilon)$ の外向き法線方向の速度、

$\frac{\partial v}{\partial \mathbf{n}}$は$\Gamma(t;\varepsilon)$ における外向き法線方向の微分であり、

Γ_0 は十分滑らかなΩ内の超曲面であって、$\partial\Omega \cap \Gamma_0 = \phi$ をみたすとする。(Γ_0 は $t = 0$ における interface の位置を表わしている)

このような超曲面の族 $\Gamma(t;\varepsilon)$ が (3)-(5) の解の u-成分に現れる interface の挙動を記述していることを主張するためには、(7) の解がどのような意味で存在し一意的であるかということと、(7),(8) で定義される $u_\Gamma(x)$ と (3)-(5) の解の u-成分 との差をとって適当な ノルムで評価しなければならない。しかし、これらはまだできていません。

参考文献

[1] H.Amman, Quasilinear evolution equations and parabolic systems, Trans. American Math. Soc. vol.293, No.1, pp.191-227.

[2] S.Childress and J.K.Percus, Nonlinear aspects of Chemotaxis, Math. Biosciences 56, pp.217-237.

[3] E.F.Keller and L.A.Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret. Biol. 26, pp.399-415.

[4] R.Schaaf, Stationary solutions of chemotaxis systems, Trans. American Math. Soc. vol.292, No.2, pp.531-556.

[5] 辻川 亨、三村 昌泰、集合パターンのダイナミクスについて、Mathematical Topics in Biology 、 研究集会、京都大学数理研、1991.

Asymptotic stability for heat equations with hysteresis in source term.

Tetsuya KOYAMA

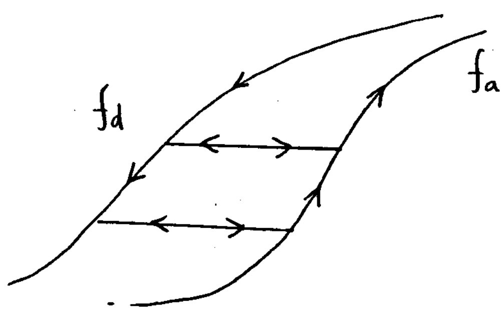
§0. Introduction.

This work is concerned with the initial-boundary value problem of heat equation which source term has nonlinear memory of hysteresis type:

$$(IBVP) \qquad \begin{cases} \dfrac{\partial}{\partial t}u(x,t) - \Delta u(x,t) + w(x,t) = 0 \quad \text{in } Q, \\ u(x,t) = g(x) \quad \text{in } \Sigma_0, \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \\ w(x,t) = \mathcal{H}(u(x,\cdot); w_0(x))(t) \quad \text{in } Q. \end{cases}$$

Here Ω is a bounded domain in $\mathbf{R}^N$ ($N \geq 1$) with smooth boundary Γ, $Q := \Omega \times (0,\infty)$, Γ_0 is a subset of Γ with positive surface measure and $\Sigma_0 := \Gamma_0 \times (0,\infty)$. g is boundary value, and u_0 and w_0 are initial values for u and w respectively.

w is a controle term with memory of hysteresis type, that is, $w(x,t)$ is determined depending on $\{u(x,s)\}_{0\leq s\leq t}$ and $w_0(x)$. This dependence is irrustrated as follows.



There are two functions f_a and f_d which are monotone nondecreasing, Lipschitz continuous and $f_a(\xi) \leq f_d(\xi)$ for all $\xi \in \mathbf{R}$. For each "input" function $\xi \in C([0,\infty))$ and "initial output" $w_0 \in \mathbf{R}$ with consistency condition $f_a(\xi(0)) \leq w_0 \leq f_d(\xi(0))$, "output" $w(t) := \mathcal{H}(\xi; w_0)(t)$ is determined by the following rules:

$w(0) = w_0,$
$f_a(\xi(t)) \leq w(t) \leq f_d(\xi(t)) \quad$ for all $t \geq 0,$
$w(t)$ can increase (or decrease) only when
$\quad w(t) = f_a(\xi(t))$ (or $w(t) = f_d(\xi(t))$ respectively).

Such operator $\mathcal{H}$ is called (Lipschitz) hysteron, and systematically studied in [K-P].

Existence and uniqueness of a solution of $(IBVP)$ is obtained in [K-K] and [K-K-V]. The aim of this note is to prove that the solutions $u(x,t)$ and $w(x,t)$ of (P) converge when $t \to \infty$ in $L^2(\Omega)$.

§1. Statement of a result.

We begin with the construction of hysteron operator $\mathcal{H}$. Let f_a and f_d be functions on $\mathbf{R}$ with the property

$$(f) \qquad \begin{array}{l} f_a \text{ and } f_d \text{ are Lipschitz continuous with Lipshitz constant} \\ \text{less than } L > 0, \text{ nondecreasing and } f_a(\xi) \le f_d(\xi) \text{ for all } \xi \in \mathbf{R}. \end{array}$$

Put

$$D(\mathcal{H}) := \{(\xi, w) \in C([0,\infty)) \times \mathbf{R}\, ;\ f_a(\xi(0)) \le w_0 \le f_d(\xi(0))\}$$

and define operator $\mathcal{H}; D(\mathcal{H}) \to C([0,\infty))$ as follows. Firstly when $(\xi, w) \in D(\mathcal{H})$ and ξ is piecewise linear, that is,

there are points $0 = t_0 < t_1 < t_2 < \cdots$ such that $\lim_{n\to\infty} t_n = \infty$ and ξ is linear on each interval $[t_{j-1}, t_j]$, $j = 1, 2, \cdots$,

define

$$\mathcal{H}(\xi, w)(t) := \begin{cases} w & \text{if } t = 0, \\ \min\{f_d(\xi(t)), \max\{f_a(\xi(t)), \mathcal{H}(\xi, w)(t_{j-1})\}\} \\ \qquad \text{if } t \in (t_{j-1}, t_j],\ j = 1, 2, \cdots. \end{cases}$$

for each $t \ge 0$. Then for any pair (ξ_1, w_1) and $(\xi_2, w_2) \in D(\mathcal{H})$ with piecewise linear ξ_i's, the estimate

$$(1.1) \qquad \begin{array}{c} \max_{[s,t]}(\mathcal{H}(\xi_1, w_1)(t) - \mathcal{H}(\xi_2, w_2)(t)) \\ \le \max \left\{ \begin{array}{l} \max_{[s,t]}(f_a(\xi_1) - f_a(\xi_2)), \\ \max_{[s,t]}(f_a(\xi_1) - f_a(\xi_2)), \\ \mathcal{H}(\xi_1, w_1)(s) - \mathcal{H}(\xi_2, w_2)(s) \end{array} \right\} \\ \text{for all } s \text{ and } t \text{ with } 0 \le s \le t \end{array}$$

holds (for this, see [K-K]), and this leads to Lipschitz continuity of $\mathcal{H}$

$$(1.2) \qquad \|\mathcal{H}(\xi_1, w_0) - \mathcal{H}(\xi_2, w_0)\|_{C([0,T])} \le L\|\xi_1 - \xi_2\|_{C([0,T])}$$

for all $T \ge 0$ when ξ_1 and ξ_2 are piecewise linear. Because the space of all piecewise linear continuous functions is dense in $C([0,\infty))$ with compact convergence topology, the operator $\mathcal{H}$ is extended uniquely to the operator on whole $D(\mathcal{H})$, and estimates (1.1) and (1.2) again hold.

Let u_0 and w_0 satisfy the conditions

(1.3)
$$u_0 \in W^{1,2}(\Omega), \quad u_0 = g \text{ on } \Gamma_0,$$

(1.4)
$$w_0 \in L^2(\Omega),$$

(1.5)
$$f_a(u_0(x)) \le w_0(x) \le f_d(u_0(x)), \quad \text{for a.e. } \quad x \in \Omega.$$

Put for each $T > 0$,

$$X(T) := L^2(\Omega; C([0,T])),$$
$$X_0(T) := \{u \in X(T);\ u(x,0) = u_0(x) \ \text{ for a.e. } x \in \Omega\},$$

and define operator

$$G; X_0(T) \to X(T)$$

by

(1.6)
$$G(u)(x,t) := \mathcal{H}(u(x,\cdot); w_0(x))(t).$$

Then this operator is well defined and the estimate

$$\|G(u_1) - G(u_2)\|_{X(T)} \le L\|u_1 - u_2\|_{X(T)} \ \text{ for all } T > 0 \text{ and } u_1, u_2 \in X_0(T)$$

holds.

Put

$$H := L^2(\Omega) \text{ with norm } \|\cdot\| := \|\cdot\|_{L^2(\Omega)},$$
$$V := W^{1,2}(\Omega),$$
$$K := \{z \in V;\ z = g \text{ on } \Gamma_0\}$$

and define a functional φ on H by

$$\varphi(z) := \begin{cases} \frac{1}{2}\int_\Omega \sum_{i=1}^{N} \frac{\partial z}{\partial x_i}\, dx & \text{if } z \in K, \\ \infty & \text{otherwise.} \end{cases}$$

Then the problem $(IBVP)$ is reformulate as the following Cauchy problem in H:

(CP)
$$\begin{cases} u'(t) + \partial\varphi u(t) + G(u)(t) \ni 0 & \text{for } t \ge 0, \\ u(0) = u_0. \end{cases}$$

Next theorem is a direct consequence of the results in [K-K] and [K-K-V].

THEOREM 1. Suppose that $\Omega \subset \mathbf{R}^N$ is bounded domain with smooth boundary, and that the assumptions $(f), (1.3), (1.4)$ and (1.5) hold. Let G be an operator defined by (1.6). Then (CP) has a unique solution

$$u \in L^\infty_{loc}(0,\infty;V) \cap W^{1,2}_{loc}(0,\infty;H).$$

Our aim is to show the following theorem.

THEOREM 2. Under the same assumptions as in Theorem 1,

$$u_\infty := \lim_{t\to\infty} u(t) \tag{1.8}$$

and

$$w_\infty := \lim_{t\to\infty} G(u)(t) \tag{1.9}$$

exist and satisfy

$$\partial\varphi u_\infty + w_\infty \ni 0. \tag{SP}$$

§2. Proof of Theorem 2.

LEMMA 3. Let $\xi \in AC([0,T])$ for some $T > 0$ and $(\xi, w) \in D(\mathcal{H})$, then $\mathcal{H}(\xi, w) \in AC([0,T])$ and

$$\left|\frac{d}{dt}\mathcal{H}(\xi,w)(t)\right| \le L\left|\frac{d}{dt}\xi(t)\right| \quad \text{for a.e. } 0 \le t \le T, \tag{2.1}$$

$$\frac{d}{dt}\xi(t)\frac{d}{dt}\mathcal{H}(\xi,w)(t) \ge 0 \quad \text{for a.e. } 0 \le t \le T. \tag{2.2}$$

PROOF. Fix s and t so that $0 \le s \le t \le T$, and put

$$\bar\xi(\tau) := \xi(s) \quad \text{for all } 0 \le \tau \le T,$$
$$\bar w := \mathcal{H}(\xi, w)(s).$$

Then $(\bar\xi, \bar w) \in D(\mathcal{H})$. And by (1.1) and absolute continuity of ξ, we have

$$\begin{aligned}|\mathcal{H}(\xi;w)(t) - \mathcal{H}(\xi;w)(s)| &= |\mathcal{H}(\xi;w)(t) - \mathcal{H}(\bar\xi;\bar w)(t)| \le L\|\xi - \bar\xi\|_{C([s,t])} \\ &= L\max\nolimits_{[s,t]}|\xi(\cdot) - \xi(s)| \le L\max\nolimits_{[s,t]}\int_s^{\cdot}\left|\frac{d\xi}{d\tau}\right|d\tau \le L\int_s^t\left|\frac{d\xi}{d\tau}\right|d\tau.\end{aligned}$$

This implies $\mathcal{H}(\xi;w) \in AC([0,T])$ and (2.1).

To show (2.2), we assume that ξ and $\mathcal{H}(\xi; w)$ are differentiable at t, $\mathcal{H}(\xi; w)(t) = f_a(w(t))$ and $\xi'(t) > 0$ without loss of generality. Therefore for each sufficiently small $h > 0$, we have

$$\mathcal{H}(\xi; w)(t+h) \geq f_a(\xi(t+h)) \geq f_a(\xi(t)) = \mathcal{H}(\xi; w)(t),$$

and thus

$$\left(\frac{d}{dt}\right)^{+} \mathcal{H}(\xi; w)(t) = \frac{d}{dt}\mathcal{H}(\xi; w)(t) \geq 0.$$

□

LEMMA 4. Let $u \in AC([0,T]; H)$ for some $T > 0$, then we have $G(u) \in AC([0,T]; H)$ and

$$\|G(u)'(t)\| \leq L\|u'(t)\| \quad \text{for a.e.} \quad 0 \leq t \leq T, \tag{2.3}$$

$$(G(u)'(t), u'(t)) \geq 0 \quad \text{for a.e.} \quad 0 \leq t \leq T. \tag{2.4}$$

This Lemma is a direct consequence of Lemma 3.

PROOF OF THEOREM 2. Because $G(u) \in W^{1,2}_{loc}(0, \infty; H)$, by [B. Theorem 3.7], u is right differentiable at each $t > 0$.

By Poincaré's Lemma, there exists a number $\gamma > 0$ such that

$$\gamma\|z\|^2 \leq \sum_{i=1}^{N} \|\frac{\partial z}{\partial x_i}\|^2$$

for all $z \in V$ with $z = 0$ on Γ_0.

For each s, t, T and h with $0 < s \leq t \leq T$, and $h > 0$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u(t+h) - u(t)\|^2 \\ &= -(u(t+h) - u(t), -(u'(t+h) + G(u)(t+h)) + (u'(t) + G(u)(t))) \\ &\quad -(u(t+h) - u(t), G(u)(t+h) - G(u)(t)) \\ &\leq -\gamma\|u(t+h) - u(t)\|^2 - (u(t+h) - u(t), G(u)(t+h) - G(u)(t)) \end{aligned} \tag{2.10}$$

because of

$$\begin{aligned} &u(t+h) - u(t) = 0 \quad \text{on } \Gamma_0, \\ &u'(t+h) + G(u)(t+h) = \Delta u(t+h) \\ &u'(t) + G(u)(t) = \Delta u(t). \end{aligned}$$

By integrating (2.10) we have

$$\begin{aligned} &\frac{1}{h^2}\|u(t+h) - u(t)\|^2 - \frac{1}{h^2}\|u(s+h) - u(t)\|^2 \\ &\leq -\gamma\frac{2}{h^2}\int_s^t \|u(\tau+h) - u(\tau)\|^2 d\tau \\ &\quad -\frac{2}{h^2}\int_s^t (u(\tau+h) - u(\tau), G(u)(\tau+h) - G(u)(\tau)) d\tau \end{aligned} \tag{2.11}$$

Integrating (2.3) gives

$$\|G(u)(\tau+h)-G(u)(\tau)\| \le L\int_{\tau}^{\tau+h}\|u'(\eta)\|d\eta,$$

and (2.10) gives

$$\frac{d}{dt}\|u(t+h)-u(t)\| \le \|G(u)(t+h)-G(u)(t)\|.$$

Thus we have

$$\frac{1}{h}\|u(t+h)-u(t)\|-\frac{1}{h}\|u(s+h)-u(t)\| \le L\int_{s}^{t+h}\|u'(\tau)\|d\tau,$$

and, by letting $h \downarrow 0$ we have

$$\|\left(\frac{d}{dt}\right)^{+}u(t)\|-\|\left(\frac{d}{dt}\right)^{+}u(s)\| \le L\int_{0}^{T}\|u'(\tau)\|d\tau.$$

Therefore $\|\left(\frac{d}{dt}\right)^{+}u(\cdot)\|$, $\{\frac{1}{h}\|u(\cdot+h)-u(\cdot)\|\}_{h>0}$ and $\{\frac{1}{h}\|G(u)(\cdot+h)-G(u)(\cdot)\|\}_{h>0}$ have a common bound on $[s,T]$. By Lebesgue's dominated convergence theorem for Bochner integrals, letting $h \downarrow 0$ in (2.11) gives

$$\|\left(\frac{d}{dt}\right)^{+}u(t)\|^2-\|\left(\frac{d}{dt}\right)^{+}u(s)\|^2 \le -\gamma\int_{s}^{t}\|\left(\frac{d}{dt}\right)^{+}u(\tau)\|^2d\tau-\int_{s}^{t}(u'(\tau),G(u)'(\tau))d\tau.$$

By (2.4) and Gronwall's lemma, we have

$$\|\left(\frac{d}{dt}\right)^{+}u(t)\| \le e^{-\gamma(t-s)}\|\left(\frac{d}{dt}\right)^{+}u(s)\|.$$

Therefore $\|\left(\frac{d}{dt}\right)^{+}u(t)\| \in L^1(s,\infty)$ and thus (1.8) exists. Similarly by (2.3), (1.9) also exists. Because $\lim_{t\to\infty}u'(t)=0$, letting $t\to\infty$ in (CP) gives (SP). □

References

[B] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, Math. Studies 5, North-Holland, Amsterdam, 1973.

[K-K] N. Kenmochi and T. Koyama, Nonlinear functional variational inequalities governed by time-dependent subdifferentials, to appear in Nonlinear Anal.

[K-K-V] N. Kenmochi, T. Koyama and A. Visintin, On a class of variational inequalties with memory terms, preprint.

[K-P] M. A. Krasnoselskiǐ and a. v. Pokrovskiǐ, Systems with hysteresis (Russian), Nauka, Moskow, 1983. English translation: Springer, Berlin, 1989.

Existence of periodic solutions to a multi-phase Stefan problem

Junichi SHINODA

Department of Mathematics,
Graduate School of Science and Technology,
Chiba University, Chiba, 260 JAPAN

0. Introduction

Let T be a given positive constant, say period. In this note, we consider the T-periodic solutions of the following problem

$$\text{(P)}\quad \begin{cases} u_t - \Delta\beta(u) = 0 & \text{in } Q = I \times \Omega, \\ \dfrac{\partial\beta(u)}{\partial n} + g(t,x,\beta(u)) = 0 & \text{on } \Sigma = I \times \Gamma. \end{cases}$$

Here I is an interval, Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary Γ, and g has T-periodicity in time t. If the nonlinear flux g is monotone nondecreasing with respect to the third argument, it was obtained in Aiki *et al* [1] that the periodic solutions are constructed as the limit of $u(nT+\cdot)$, where u is a solution of (P) on $[0,\infty)$ with the specific initial value, that $\Theta_T := \{\beta(\omega); \omega$ is a T-periodic solution of (P) on $\mathbf{R}\}$ is a totally ordered set with respect to the usual order of functions on $\mathbf{R}\times\Omega$, that $\{\partial v/\partial n; v \in \Theta_T\}$ is a singleton, and that $\beta(\omega)$ is uniquely determined by the quantity $\int_\Omega \omega(0,x)dx$. In this case the comparison result, which is proved by means of the monotonicity of β and g, plays an important role in the construction of periodic solutions. But if g is nonmonotone, this result does not hold. So we shall show later the existence of periodic solutions of (P) through a fixed point theorem. And also we will give an example such that Θ_T is not totally ordered. For the results to the other types of boundary conditions, see Damlamian-Kenmochi [2] and Haraux-Kenmochi [4].

1. Assumptions and definitions

Throughout this paper, we make following assumptions.

$\beta : \mathbf{R} \to \mathbf{R}$ is a nondecreasing Lipschitz continuous function such that $\beta(0) = 0$ and $\liminf_{|r|\to\infty} \beta(r)/r > 0$. And a function $g = g(t,x,\xi) : \mathbf{R}\times\Gamma\times\mathbf{R} \to \mathbf{R}$ satisfies the following five conditions:

(g1) $g(t,x,\cdot)$ is nondecreasing with respect to ξ for a.e. $(t,x) \in \mathbf{R}\times\Gamma$;

(g2) $g(\cdot,\cdot,\xi) \in L^2_{loc}(\mathbf{R}; L^2(\Gamma))$ for any $\xi \in \mathbf{R}$;

(g3) for each $M > 0$ there is a constant $C_g(M) > 0$ such that

$$|g(t,x,\xi) - g(t,x,\xi')| \le C_g(M)|\xi - \xi'| \tag{1}$$

for any $\xi, \xi' \in [-M, M]$ and a.e. $(t,x) \in \mathbf{R} \times \Gamma$;

(g4) there exist two constants M_1 and M_2 with $M_1 \le M_2$ such that

$$g(t,x,\beta(M_1)) \le 0, \quad g(t,x,\beta(M_2)) \ge 0 \quad \text{for a.e.}(t,x) \in \mathbf{R} \times \Gamma; \tag{2}$$

(g5) $g(t+T,x,\xi) = g(t,x,\xi)$ for any $\xi \in \mathbf{R}$ and a.e. $(t,x) \in \mathbf{R} \times \Gamma$.

Now we state definitions of solutions to problem (P). For the sake of simplicity, we set $H = L^2(\Omega)$ and $V = H^1(\Omega)$.

Definition 1. Let I be a compact interval of the form $[t_0, t_1]$. Then $u : I \to H$ is said to be a weak solution of (P) on I when the following two conditions are fulfilled.

(w1) $u \in L^\infty(Q) \cap C_w(I; H)$, $\beta(u) \in L^2(I; V)$;

(w2) for any $\varphi \in W_0 = \{\varphi \in H^1(Q); \varphi(0,\cdot) = \varphi(T,\cdot) = 0 \text{ a.e. in } \Omega\}$,

$$-\int_Q u\varphi_t dxdt + \int_Q \nabla\beta(u)\nabla\varphi dxdt + \int_\Sigma g(\cdot,\cdot,\beta(u))\varphi d\Gamma dt = 0. \tag{3}$$

If the interval I is of the form $[t_0, \infty)$ or $\mathbf{R}$, then u is called a weak solution of (P) on I when, for any compact interval I' contained in I, u is a weak solution of (P) on I'.

Definition 2. Let I be the interval of the form $[t_0, t_1]$ or $[t_0, \infty)$. Then we call $u : I \to H$ a solution to the Cauchy problem CP(u_0) on I if u is a weak solution of (P) on I which verifies the initial condition $u(t_0) = u_0$.

Next we mention the definition of T-periodic weak solutions.

Definition 3. Let $u : \mathbf{R} \to H$. Then u is called a T-periodic weak solution of (P) on $\mathbf{R}$ provided that u is a weak solution of (P) on $\mathbf{R}$ and satisfies the periodic condition $u(t+T) = u(t)$ for all $t \in \mathbf{R}$.

2. A result and its proof

First we state our result.

Theorem. *There exists at least one T-periodic weak solution of (P) on* $\mathbf{R}$.

Before proving the theorem, we quote a result for the Cauchy problem CP(u_0).

Proposition (cf. [4,5,6]). *Let t_0 be a real number and u_0 a function in $L^\infty(\Omega)$. furthermore, let be $\widetilde{M}_1$ and $\widetilde{M}_2$ constants such that $\widetilde{M}_1 \le M_1$, $\widetilde{M}_2 \ge M_2$ and $\widetilde{M}_1 \le u_0 \le \widetilde{M}_2$ a.e. in Ω. Then, there exists a unique weak solution u for CP(u_0) such that*

$$\widetilde{M}_1 \le u \le \widetilde{M}_2 \text{ a.e. in } \Omega. \tag{4}$$

Next, for later use, we define a closed convex set K and a mapping $P : K \to K$. That is, for M_1 and M_2 given in (g4), K is defined as

$$K = \{z \in H; M_1 \le z \le M_2 \text{ a.e. in } \Omega\}. \tag{5}$$

And, for each $z \in K$, we assign to $P(z)$ the value at $t = T$ of the unique weak solution for CP(z).

Remark. By virtue of the proposition, P is well-defined. And K is metrizable with respect to the induced weak topology of $L^2(\Omega)$ (cf. Dunford-Schwartz [3;p. 434]).

The next lemma is crucial.

Lemma. *P is weakly continuous on K.*

PROOF OF LEMMA: Let $\{z_n\}$ be a sequence in K such that z_n converges to some z_0 weakly in $L^2(\Omega)$, and u_n be a weak solution to CP(z_n) for $n \ge 1$. It is noted here that the weak solution u_n satisfies the identity

$$\langle u_n'(t), z\rangle + \int_\Omega \nabla\beta(u_n(t))\nabla z dx + \int_\Gamma g(t,\cdot,\beta(u_n(t)))z d\Gamma = 0 \tag{6}$$

for a.e. $t \in \mathbf{R}$ and for any $z \in V$, where $\langle\cdot,\cdot\rangle$ is the duality pairing between V' and V. Then we can make uniform estimates with respect to n, that is,

$$|u_n|_{L^\infty(Q)} \le \max\{|M_1|, |M_2|\}, \tag{7}$$

$$|u_n|_{W^{1,2}(0,T;V')} \le C, \tag{8}$$

$$|\beta(u_n)|_{L^2(0,T;V)} + |\beta(u_n)|_{H^1_{loc}(Q)} \le C, \tag{9}$$

and

$$|g(\cdot,\cdot,\beta(u_n))|_{L^2(\Sigma)} \le C, \tag{10}$$

where C is a positive constant independent of n. From these estimates, we find a subsequence $\{n_k\}$ of $\{n\}$ such that $(u_{n_k}, \beta(u_{n_k}), g(\cdot,\cdot,\beta(u_{n_k})))$ converges to some element (u, v, g) in the following sense:

$$u_{n_k} \to u \text{ weakly in } W^{1,2}(0,T;V') \text{ and weakly}^* \text{ in } L^\infty(Q), \tag{11}$$

$$\beta(u_{n_k}) \to v \text{ weakly in } L^2(0,T;V) \text{ and in } H^1_{loc}(Q), \tag{12}$$

and

$$g(\cdot,\cdot,\beta(u_{n_k})) \to g \text{ weakly in } L^2(\Sigma). \tag{13}$$

By (12) and the uniform boundedness of $\beta(u_{n_k})$ on Q, it is derived that $\beta(u_{n_k})$ converges to v in $L^2(Q)$. Therefore we have $v = \beta(u)$ since β is a maximal monotone operator on $L^2(Q)$.

On the other hand, it is well-known that for any $\varepsilon > 0$ there exists a positive constant $C(\varepsilon)$ such that

$$|w|_{L^2(\Gamma)} \le \varepsilon |\nabla w|_H + C(\varepsilon)|w|_H \text{ for any } w \in V. \tag{14}$$

Substituting $\beta(u_{n_k}) - \beta(u)$ as w to (14) and integrating over $[0,T]$, it implies that $\beta(u_{n_k})$ converges to $\beta(u)$ in $L^2(\Sigma)$ hence $g(\cdot,\cdot,\beta(u_{n_k}))$ to $g(\cdot,\cdot,\beta(u))$. So, we get $g = g(\cdot,\cdot,\beta(u))$.

It is easily seen that u is a weak solution for CP(u_0) on $[0,T]$. By the uniqueness of the weak solution, we can assert that

$$u_n \to u \text{ weakly in } W^{1,2}(0,T;V') \text{ and weakly* in } L^\infty(Q), \tag{15}$$

$$\beta(u_n) \to \beta(u) \text{ weakly in } L^2(0,T;V) \text{ and in } L^2(Q), \tag{16}$$

and

$$g(\cdot,\cdot,\beta(u_n)) \to g(\cdot,\cdot,\beta(u)) \text{ in } L^2(\Sigma). \tag{17}$$

In particular, $P(u_n) = u_n(T)$ converges to $P(u_0) = u(T)$ weakly in H. Thus the lemma has been proved. ∎

PROOF OF THEOREM: By the lemma, we can apply Tychonoff's fixed point theorem. It ensures that there exists $u_0 \in K$ such that $P(u_0) = u_0$. Let us denote by u the unique weak solution for CP(u_0). Then, it is easily seen that the T-periodic extension $\tilde{u}$ of u is a desired one. ∎

Finally we give an example as was proposed in the introduction.

Example. Let $\Omega = (0,2)$,

$$\beta(r) = \begin{cases} r-1 & r \ge 1, \\ 0 & 0 < r < 1, \\ r & r \le 0, \end{cases} \tag{18}$$

and $g(t,x,\xi)=\xi^3-2\xi$. Then,

$$\omega_1(t,x)\equiv 0 \text{ and } \omega_2(t,x)=\begin{cases} x & x\geq 1, \\ x-1 & x<1 \end{cases} \tag{19}$$

are T-periodic solutions of (P) on $\mathbf{R}$, and we have $\beta(\omega_1(t,x))\equiv 0$ and $\beta(\omega_2(t,x))=x-1$. Moreover we easily see that Θ_T is no longer a totally ordered set and that $g(\cdot,\cdot,\beta(\omega_1))\neq g(\cdot,\cdot,\beta(\omega_2)$ on $\mathbf{R}\times\Gamma$.

References

[1] T. Aiki, N. Kenmochi, and J. Shinoda, *Periodic stability for a class of degenerate parabolic equations with nonlinear flux.* to appear in Nonlinear Anal. T.M.A.

[2] A. Damlamian and N. Kenmochi, *Periodicity and almost periodicity of solutions to a multi-phase Stefan problem in several space variables*, Nonlinear Anal. T.M.A. **12** (1988), 921-943.

[3] N. Dunford and J.T. Schwartz, "Linear operators, part I," Interscience, New York, 1964, (second edition).

[4] A. Haraux and N. Kenmochi, *Asymptotic behaviour of solutions to some degenerate parabolic equations*, Funk. Ekvac. **34** (1991), 19-38.

[5] M. Niezgodka and I. Pawlow, *A generalized Stefan problem in several space variables*, Appl. Math. Optim. **9** (1983), 193-224.

[6] M. Niezgodka, I. Pawlow and A. Visintin, *Remarks on the paper by A. Visintin "Sur le problème de Stefan avec flux non linéaire"*, Boll. U.M.I., Anal. Funz. Appl. Serie V **18** (1981), 87-88.

[7] A. Visintin, *Sur le problème de Stefan avec flux non linéaire*, Boll. U.M.I., Anal. Funz. Appl. Serie V **18** (1981), 63-86.

Controllability for retarded system with nonlinear term in Hilbert space

JIN-MUN JEONG

We consider the problem of control for the following retarded functional differential equation of parabolic type

$$(1)\qquad \frac{\partial u}{\partial t}(x,t) + \mathcal{A}(x,D_x)u(x,t) + \mathcal{A}_1(x,D_x)u(x,t-h) + \int_{-h}^{0} a(s)\mathcal{A}_2(x,D_x)u(x,t+s)ds = (\Phi_0 w(t))(x), \quad x\in\Omega,\ t\in(0,T],$$

$$(2)\qquad u(x,t) = 0, \qquad x\in\partial\Omega,\ t\in(0,T],$$

$$(3)\qquad u(x,0) = g^0(x),\ u(x,s) = g^1(x,s), \qquad x\in\Omega,\ s\in[-h,0).$$

Here, Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, $\mathcal{A}(x,D_x)$, $\mathcal{A}_\iota(x,D_x)$, $\iota = 1, 2$, are second order linear differential operators with real coefficients, and $\mathcal{A}(x,D)$ is elliptic in $\overline{\Omega}$. The function $a(\cdot)$ is real valued and Hölder continuous in $[-h,0]$, where h is some fixed positive number. The controller Φ_0 is a bounded linear operator from some Banach space U to $L^1(\Omega)$; $w(\cdot)$ is some function with values in U, and $g^0(\cdot)$, $g^1(\cdot,\cdot)$ are given functions defined in Ω and $\Omega\times[-h,0)$ respectively.

In view of Sobolev's imbedding theorem we may consider $L^1(\Omega)\subset W^{-1,p}(\Omega)$, if $1<p<n/(n-1)$. Hence we investigate the problem (1)~(3) in the space $W^{-1,p}(\Omega)$ choosing p in this way and considering Φ_0 as an operator into $W^{-1,p}(\Omega)$. Necessarily we realize the operators $\mathcal{A}(x,D_x)$, $\mathcal{A}_\iota(x,D_x)$, $\iota=1, 2$, in the space $W^{-1,p}(\Omega)$ by

$$A_0 u = -\mathcal{A}(x,D_x)u, \quad A_\iota u = -\mathcal{A}_\iota(x,D_x)u, \quad \iota = 1,\ 2, \quad \text{for } u\in W_0^{1,p}(\Omega)$$

in the distribution sense. It will be shown that A_0 generates an analytic semigroup in $W^{-1,p}(\Omega)$. Thus, the problem (1)~(3) is formulated as

$$(4)\qquad \frac{d}{dt}u(t) = A_0u(t) + A_1u(t-h) + \int_{-h}^{0} a(s)A_2u(t+s)ds + \Phi_0 w(t), \qquad t\in(0,T],$$

$$(5)\qquad u(0) = g^0, \quad u(s) = g^1(s) \qquad s\in[-h,0),$$

and the adjoint problem as

$$(6)\qquad \frac{d}{dt}v(t) = A_0^*v(t) + A_1^*v(t-h) + \int_{-h}^{0} a(s)A_2^*v(t+s)ds, \qquad t\in(0,T],$$

$$(7)\qquad v(0) = \phi^0, \quad v(s) = \phi^1(s) \qquad s\in[-h,0),$$

where $A_\iota^* : W_0^{1,p'}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$, $\iota = 0,\ 1,\ 2, p' = p/(p-1)$, are the adjoint operators of $A_\iota, \iota = 0,\ 1,\ 2$, respectively.

the space $W^{-1,p}(\Omega)$ is ζ-convex. Furthermore, with the aid of a result by R. T. Seeley [13] it is easily seen that the inequality

$$\|(-A_0)^{is}\|_{B(W^{-1,p}(\Omega))} \le Ce^{\gamma|s|}, \quad -\infty < s < \infty,$$

holds for some constants $C > 0$ and $\gamma \in (0, \pi/2)$. Consequently, in view of the maximal regularity result by G. Dore and A. Venni [6] the initial value problem

$$\frac{d}{dt}u(t) = A_0 u(t) + f(t), \quad t \in (0, T],$$
$$u(0) = u_0$$

has a unique solution u in the class $L^q(0,T; W_0^{1,p}(\Omega)) \cap W^{1,q}(0,T; W^{-1,p}(\Omega))$ for any $u_0 \in H_{p,q} = (W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{1/q,q}$ and $f \in L^q(0,T; W^{-1,p}(\Omega))$, $1 < q < \infty$. Therefore, we can apply the method of G. Di Blasio, K. Kunisch and E. Sinestrari [5] to the problem (4), (5) with a more general element f in place of $\Phi_0 w$ to show the existence and uniqueness of the solution

$$u \in L^q(-h,T; W_0^{1,p}(\Omega)) \cap W^{1,q}(0,T; W^{-1,p}(\Omega)) \subset C([0,T]; H_{p,q})$$

for any $g = (g^0, g^1) \in Z_{p,q} = H_{p,q} \times L^q(-h, 0; W_0^{1,p}(\Omega)$ and $f \in L^q(0,T; W^{-1,p}(\Omega))$.

Since we are assuming that $a(\cdot)$ is Hölder continuous, the fundamental solution $W(t)$ of (4), (5) exists [17].

In view of the above result we can define the solution semigroup for the problem (4), (5) following [5; Theorem 4.1]:

$$S(t)g = (u(t;g), u_t(\cdot, g))$$

where $g = (g^0, g^1) \in Z_{p,q}$, $u(t;g)$ is the solution of (4), (5) with $f(t) = 0$ and $u_t(\cdot; g)$ is the function $u_t(s;g) = u(t+s;g)$ defined in $[-h, 0]$. $S(t)$ is a C_0 semigroup in $Z_{p,q}$. The solution semigroup $S_T(t)$ of (6), (7) is defined by

$$S_T(t)\phi = (v(t;\phi), v_t(\cdot, \phi))$$

for $\phi = (\phi^0, \phi^1) \in Z_{p',q'}$, where $v(t;\phi)$ is the solution of (6), (7) and $v_t(\cdot;\phi)$ is the function $v_t(s;\phi) = v(t+s;\phi)$, $s \in [-h, 0]$.

The structural operator $F : Z_{p,q} \longrightarrow Z^*_{p',q'}$ is defined by

$$[Fg]^0 = g^0, \quad [Fg]^1(s) = A_1 g^1(-h-s) + \int_{-h}^0 a(\tau) A_2 g^1(\tau - s) d\tau.$$

As in S. Nakagiri [10] we have $FS(t) = S_T^*(t)F^*$ and $F^* S_T(t) = S^*(t)F^*$.

We define the set of attainability by

$$R = \{ (\int_0^t W(t-\tau)\Phi_0 w(\tau)d\tau, \int_0^t W(t+\cdot-\tau)\Phi_0 w(\tau)d\tau) : w \in L^q([0,t);U),\ t \geq 0\}.$$

DEFINITION 1. *(1) The problem (4), (5) is approximate controllable if* $\overline{R} = Z_{p,q}$, *where* $\overline{R}$ *is the closure of* R *in* $Z_{p,q}$.
(2) The problem (6), (7) is observable if for $\phi \in Z_{p',q'}$ $\Phi_0^*[S_T(t)\phi]^0 = 0$ *a.e. implies* $\phi = 0$.

THEOREM 1. *Let F be an isomorphism. Then the problem (4), (5) is approximate controllable if and only if the problem (6), (7) is observable.*

Let λ be a pole of the resolvent of A of order k_λ and let P_λ be the spectral projection. Then the generalized eigenspace corresponding to λ is given by

$$P_\lambda Z_{p,q} = \mathrm{Ker}(\lambda I - A)^{k_\lambda}.$$

For $\lambda \in \mathbf{C}$ set

$$\Delta(\lambda) = \lambda - A_0 - e^{-\lambda h}A_1 - \int_{-h}^0 e^{\lambda s}a(s)A_2 ds,$$

$$\Delta_T(\lambda) = \lambda - A_0^* - e^{-\lambda h}A_1^* - \int_{-h}^0 e^{\lambda s}a(s)A_2^* ds.$$

THEOREM 2. *The problem (6), (7) is ovservable if and only if* $\mathrm{Ker}\,\Phi_0^* \cap \mathrm{Ker}\,\Delta_T(\lambda) = 0$ *for any* $\lambda \in \sigma_p(A_T)$.

In the system (4), (5) we consider that the control space is a finite dimensional space and the controller $\Phi_0 : \mathbf{C}^N \longrightarrow L^1(\Omega)$ is expressed as

$$\Phi_0 w = \sum_{i=1}^N w_i b_i^0,$$

where $w = (w_1, \dots, w_N) \in \mathbf{C}^N$ and b_i^0, $i = 1, \dots, N$, are some fixed elements of $L^1(\Omega)$. The adjoint operator $\Phi_0^*; L^\infty(\Omega) \longrightarrow \mathbf{C}^N$ of Φ_0 is given by

$$\Phi_0^* u = ((u, b_1^0), \dots, (u, b_N^0)),$$

for any $u \in L^\infty(\Omega)$.

We suppose that the basis $\{\phi_{\lambda 1}, \dots, \phi_{\lambda m_\lambda}\}$ of $P_\lambda^T Z_{p',q'}$ is arranged so that $\{\phi_{\lambda 1}, \dots, \phi_{\lambda d_\lambda}\}$ span $\mathrm{Ker}(\lambda - A_T)$ where $d_\lambda = \dim \mathrm{Ker}(\lambda - A_T)$. Then $\{\phi_{\lambda i}^0 : i = 1, \dots, d_\lambda\}$ is a basis of $\mathrm{Ker}\,\Delta_T(\lambda)$ and $\phi_{\lambda i} = (\phi_{\lambda i}^0, e^{\lambda s}\phi_{\lambda i}^0)$ for $i = 1, \dots, d_\lambda$. We assume that

RANK CONDITION: For any $\lambda \in \sigma_p(A_T)$

$$\mathrm{rank} \begin{pmatrix} (b_1^0, \phi_{\lambda 1}^0) & \cdots & (b_1^0, \phi_{\lambda d_\lambda}^0) \\ \vdots & \ddots & \vdots \\ (b_N^0, \phi_{\lambda 1}^0) & \cdots & (b_N^0, \phi_{\lambda d_\lambda}^0) \end{pmatrix} = d_\lambda.$$

Since $\phi_{\lambda j}^0 \in L^\infty(\Omega)$ each $(b_i^0, \phi_{\lambda j}^0)$ is meaningful.

THEOREM 3. *If the Rank Condition is satisfied, then the problem (7), (8) is observable.*

REFERENCES

1. S. Agmon, *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure. Appl. Math. **15** (1963), 119–147.
2. J. Bourgain, *Some remarks on Banach spaces in which martingale difference sequences are unconditional*, Ark. Mat. **21** (1983), 163–168.
3. D. L. Burkholder, *A geometric condition that implies the existence of certain singular integrals of Banach space valued functions*, Proc. Conf. Harmonic Analysis, University of Chicago (1981), 270–286.
4. P. L. Butzer and H. Berens, "Semi-Groups of Operators and Approximation," Springer-Verlag, Belin-Heidelberg-New York, 1967.
5. G. Di Blasio, K. Kunisch and E. Sinestrari, *L^2-regularity for parabolic partial integrodifferential equations with delay in the highest-order derivatives*, J. Math. Anal. appl. **102** (1984), 38–57.
6. G. Dore and A. Venni, *On the closedness of the sum of two closed operators*, Math. Z. **196** (1987), 189–201.
7. J. M. Jeong, *Spectral properties of the operator associated with a retarded functional differential equation in Hilbert space*, Proc. Japan Accad. **65A** (1989), 98–101.
8. J. L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Études Sci. Publ. Math. **19** (1964), 5–68.
9. S. Nakagiri, *Spectral mode controllability and observability for linear systems with time delay in Hilbert space*, (preprint).
10. S. Nakagiri, *Structural operators and spectral theory for differential equations with unbounded time delay in Hilbert spase*, Seminar note at Osaka University, 1988(in Japanese).
11. S. Nakagiri, *Structural properties of functional differential equations in Banach spaces*, Osaka J. Math. **25** (1988), 353–398.
12. S. Nakagiri and M. Yamamoto, *Controllability and observability of linear retarded systems in Banach space*, to appear in Int. J. control.
13. R. Seeley, *Norms and domains of the complex power A_B^Z*, Amer. J. Math. **93** (1971), 299–309.
14. R. Seeley, *Interpolation in L^p with boundary conditions*, Studia Math. **44** (1972), 47–60.
15. T. Suzuki and M. Yamamoto, *Observability, controllability and feedback stabilizability for evolution equations I*, Japan J. Appl. Math. **2** (1985), 211–228.
16. H. Tanabe, "Equations of Evolution," Pitman-London, 1979.
17. H. Tanabe, *On fundamental solution of differential equation with time delay in Banach space*, Proc. Japan Accad. **64A** (1988), 131–134.
18. H. Tanabe, *Structural operators for linear delay-differential equations in Hilbert space*, Proc. Japan Accad. **64A** (1988), 265–266.
19. H. Triebel, "Interpolation Theory, Function Spaces, Differential Operators," North-Holland, 1978.
20. K. Yosida, "Functional Analysis," 3rd ed., Springer, Berlin-Göttingen-Heidelberg, 1980.

ある種の退化する2階楕円型作用素の準楕円性について

鈴木　道治（筑波大・院）

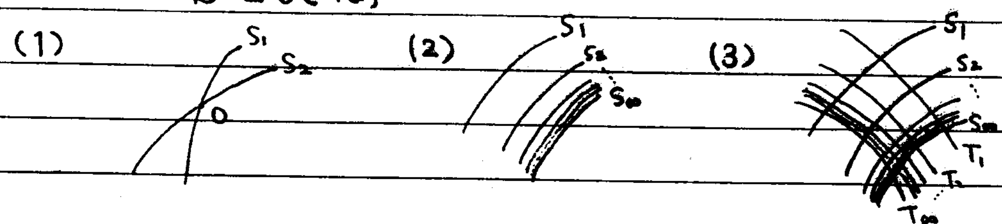
ここで我々は，

$$P = D_1^2 + \alpha(x)D_2^2 + \beta(x,D) \quad \text{in } \mathbb{R}^2$$

（ただし　$\alpha(x) \geqq 0$，　$\alpha(x) \in C^\infty(\mathbb{R}^2)$
$\beta(x,D)$ は，1st. order classical, properly supported Ψ.d.op.）

の (micro-) hypoellipticity を，$S = \{x \in \mathbb{R}^2 \mid \alpha(x) = 0\}$ の形状を様々にかえて調べた結果を述べる。

たとえば S としては，



のような場合を考える。

S が上の (1)〜(3) のような場合に P の microhypoellipticity をいうために基本的な定理を用意する。

$x^0 = (x_1^0, x_2^0) \in \mathbb{R}^2$，$\Sigma_0$ を $\mathbb{R}^2$ の subset で，$x^0 \in \Sigma_0$ とし，
P は，microhypoelliptic in $U_0 \setminus \Sigma_0$（U_0 は x^0 のある nbd.）
と仮定する。

このとき 次が成り立つ。

Theorem

(i) $U_0 \cap \Sigma_0 = \{x^0\}$ の場合

$\exists U$: x^0 の nbd. $\exists C>0$ s.t.

$$(\mathrm{Re}\,\beta_1(x;0,\pm1))^2 \leq C\,\alpha(x) \quad \text{for } x \in U$$

$\Longrightarrow$

P は microhypoelliptic at x^0 である。

(ii) $U_0 \cap \Sigma_0 \subset \{x \in \mathbb{R}^2 \mid f(x)=0\}$ ただし $f \in C^1$, $f(x^0)=0$, $\frac{\partial f}{\partial x_1}(x^0) \neq 0$ の場合

$\exists U$: x^0 の nbd. $\exists \ell \in \mathbb{N}$, $\exists C>0$ s.t.

$$(\mathrm{Re}\,\beta_1(x;0,\pm1))^2 + (\mathrm{Im}\,\beta_1(x;0,\pm1))^{2\ell} \leq C\,\alpha(x) \quad \text{for } x \in U$$

$\Longrightarrow$

P は microhypoelliptic at x^0

この定理を証明するために, Proposition を用意する。それは, 梶谷-若林 [1] の Theorem 1.2 の variant である。

$L(x,D) \in S^m_{1,0}$: properly supported Ψ.d.op.

に対し,

$$L_\Lambda(x,D) = e^{-\Lambda}(x'',D)\,L(x,D)\,e^{\Lambda}(x'',D)$$

とおく。ここで

$e^{\pm\Lambda}(x'',D)$: Ψ.d.op's with the symbols $e^{\pm\Lambda(x,\xi'')}$ (複号同順)

とする。関数 $\Lambda(x'',\xi)$ を次のように定義する。

まず $z^0=(x^0,\xi^0) \in T^*\mathbb{R}^n \setminus 0$, $\xi^0=(0,\cdots,0,1)$ とする。

そして次の仮定をする。

(H) : $^\exists \mathcal{C}$: z^0 の conic nbd. in $T^*\mathbb{R}^n \setminus 0$, $^\exists \Sigma$: conic smooth mfd. in $T^*\mathbb{R}^n \setminus 0$, $^\exists n' \in \mathbb{N}$, $^\exists V$: vector subspace in $\mathbb{R}^{n-1}$ s.t.

$n' \leqq n$, $z^0 \in \Sigma$, $p(x,D)$ は microhypoelliptic in $\mathcal{C} \setminus \Sigma$,

$T_{z^0}\Sigma \cap W = \{0\}$

ここで

$W = \{(\delta x, \delta\xi', 0) \in T_{z^0}(T^*\mathbb{R}^n) \mid \delta x_j = 0\ (n' < j \leqq n),\ \delta\xi' \in V\}$ ┘

写像 η を, $\eta : \mathbb{R}^{n-1} \ni \delta\xi' \longmapsto \eta(\delta\xi') \in V^\perp$ と定義する。

さらに $\varphi(x'', \xi)$, $\lambda(\xi)$ を

$\varphi(x'', \xi) \in S^0$, positively homogeneous of degree 0 for $|\xi| \geqq 1$,

$\varphi(x'', \xi) = |x'' - x^{0\prime\prime}|^2 + |\eta(\xi')|^2/\xi_n^2$ near $\mathcal{C} \cap \{|\xi| \geqq 1\}$

(ただし $n = n'$ ならば $x'' = x^{0\prime\prime} = 0$ とする。)

$\lambda(\xi) \in S^1$, $\lambda(\xi) = \langle \xi_n \rangle$ if $\xi_n \geqq |\xi|/2 \geqq 1$

$\frac{1}{4}\langle\xi\rangle \leqq \lambda(\xi) \leqq 2\langle\xi\rangle$

とおく。そして $\Lambda(x'', \xi)$ を,

$\Lambda(x'', \xi) = \{-s + a\varphi(x'', \xi)\} \log \lambda(\xi) + N \log(1 + \delta\lambda(\xi))$

とおく。

Proposition

(H) を仮定し, さらに次を仮定する。

: $^\exists \chi_k \in S^0$ $(k=1,2)$, $^\exists l_j \in \mathbb{R}$ $(j=1,2,3)$, $^\exists a_0 \geqq 0$, $^\exists N_0 \geqq 0$, $^\exists s_0 \geqq 0$

$\chi_1 \subset\subset \chi_2$, $\chi_k = 1$ near z^0 $(k=1,2)$

s.t. " $^\forall a \geqq a_0$, $^\forall N \geqq N_0$, $^\forall s \geqq s_0$, $^\exists \psi(x,\xi) \in S^0$, $^\exists \delta_0 > 0$, $^\exists C > 0$

s.t. $\psi(x,\xi)$: pos. homo. of deg. 0 for $|\xi| \geqq 1$

$\operatorname{supp}\psi \cap \Sigma = \emptyset$

(1) $\|\chi_1(x,D)v\|_{l_1} \leqq C\{\|P_\Lambda v\|_{l_2} + \|v\|_{l_1 - 1} + \|(1-\chi_2(x,D))v\| + \|\psi(x,D)v\|_{l_3}\}$

if $v \in C_0^\infty$ and $0 < \delta \leqq \delta_0$ "

このとき

$$u \in \mathcal{D}', \quad z^0 \notin WF(Pu) \implies z^0 \notin WF(u) \quad \lrcorner$$

この Proposition を認めた上で、定理の証明を簡単にスケッチしてみる。

$$\varphi(t) = \begin{cases} 0 & \text{if } \Sigma_0 \cap \sqcup_0 = \{x^0\} \\ (t - x_2^0)^2 & \text{if } \Sigma_0 \cap \sqcup_0 \neq \{x^0\} \end{cases}$$

とおき、$\Lambda(x,\xi)$ を、

$$\Lambda(x,\xi) = \{-s + a\varphi(x_2)\} \log \lambda(\xi) + N \log(1 + \delta\lambda(\xi))$$

$$(\ 0 \leq \delta \leq 1, \quad a \geq 0, \ N \geq 0 \ \text{ and } \ s \in \mathbb{R})$$

ととる。そうすると $P_\Lambda (= e^{-\Lambda} \circ P \circ e^{\Lambda})$ の symbol $P_\Lambda(x,\xi)$ は、

$$P_\Lambda(x,\xi) = (1 + q(x,\xi)) \left[P(x,\xi) + \sqrt{-1}\,(\Lambda_{\xi_2} P_{x_2} - \Lambda_{x_2} P_{\xi_2}) + \cdots \right]$$

ここで $q(x,\xi) \in S^{-1+\rho} \ (\rho > 0)$

となるが、さらに $P_\Lambda(x,D)$ を次のように modify する。

$$\widetilde{P}_\Lambda(x,D) = \left(\frac{1}{1+q}\right)(x,D)\, P_\Lambda(x,D)$$

すると $\widetilde{P}_\Lambda(x,D)$ の symbol $\widetilde{P}_\Lambda(x,\xi)$ は、

(2) $$\widetilde{P}_\Lambda(x,\xi) = \xi_1^2 + d(x)\xi_2^2 \pm \mathrm{Re}\,\beta_1(x;0,\pm 1)\xi_2$$
$$+ e_0(x,\xi)\xi_1 + e_1(x,\xi)\, d(x) \log\lambda(\xi) + e_2(x,\xi)\, d(x) (\log\lambda(\xi))^2$$
$$\pm e_3(x,\xi)\, \mathrm{Im}\,\beta_1(x;0,\pm 1)(\log\lambda(\xi)) + e_4(x,\xi)\, d_{x_2}(x) \log\lambda(\xi)$$
$$+ i\, e_5(x,\xi)\, d(x)\xi_2 \log\lambda(\xi) + i\, r(x,\xi) + R_1(x,\xi) + R_2(x,\xi)$$
$$\text{for } |\xi| \geq 1.$$

ただし $e_j(x,\xi) \in S^0 \ (0 \leq j \leq 5)$, $e_k(x,\xi) \equiv 0 \ (1 \leq k \leq 5)$ if $\sqcup_0 \cap \Sigma_0 = \{x^0\}$,

$e_3(x,\xi) \equiv e_3(x)$, $R_1 \in S^2$, $\mathrm{supp}\, R_1 \cap \mathcal{C} \cap \{|\xi| \geq 2\} = \phi$,

$R_2 \in S^0$, $r \in S^1$ real-valued

$$|\partial_\xi^\alpha D_x^\beta R_1(x,\xi)| \leq C_{\alpha\beta} \langle\xi\rangle^{2-|\alpha|}$$

$$|\partial_\xi^\alpha D_x^\beta R_2(x,\xi)| \leq C'_{\alpha\beta} \langle\xi\rangle^{-|\alpha|}$$

ここで定数 $C_{\alpha,\beta}$, $C'_{\alpha,\beta}$ は δ によらない。

すると (2) より

$$\mathrm{Re}(\tilde{P}_\lambda(x,D)v, v) \geqq (1-\varepsilon)\{\|D_1v\|^2 + (\alpha(x)D_2v, D_2v)\}$$
$$+ \mathrm{Re}(\mathrm{Re}\,\beta_1(x;0,\pm1)D_2v, v)$$
$$+ \mathrm{Re}(e_3(x,D)\,\mathrm{Im}\,\beta_1(x;0,\pm1)(\log\lambda(D))v, v)$$
$$- C_\varepsilon\{\|v\|^2 + \|(1-\chi(x,D))v\|_2^2\} \quad \text{for } v\in C_0^\infty$$

をえる。これを basic estimate とよぶことにする。

この basic estimate から Prop. の (1) をどうみちびくかであるが、

(i) (in Th.) の場合, $e_3(x,\xi)\equiv 0$ であるから, basic estimate の右辺で問題なのは, $\mathrm{Re}(\mathrm{Re}\,\beta_1(x;0,\pm1)D_2v, v)$ の項である。しかし (i) の仮定により, 簡単に

(3) $$\mathrm{Re}(\mathrm{Re}\,\beta_1(x;0,\pm1)D_2v, v) + C\cdot\varepsilon(\alpha(x)D_2v, D_2v) \geqq -C_\varepsilon\|v\|^2 \quad \text{for } v\in C_0^\infty$$

が分る。また (ii) (in Th.) の場合, (3) の他に,

$\mathrm{Re}(e_3(x,D)\,\mathrm{Im}\,\beta_1(x;0,\pm1)(\log\lambda(D))v, v)$ を処理しなければならないが、これには次の lemma を用意する。

Lemma

(ii) の条件が成り立つならば, 次が成り立つ。

$^\forall\varepsilon>0\ ^\exists C_\varepsilon>0$ s.t.

(4) $$\|\mathrm{Im}\,\beta_1(x;0,\pm1)(\log\lambda(D))v\|^2 \leqq \varepsilon(\alpha(x)D_2v, D_2v) + C_\varepsilon\{\|v\|^2 + \|(1-\chi(x,D))v\|_2^2\} \quad \text{for } v\in C_0^\infty$$

┘

そうすると basic estimate, (3), (4) そして Poincaré's ineq. により (1) (in Prop.) をえる。 //

最後に，冒頭の例(1)～(3)のようなSの場合，このTheoremをくりかえし用いれば，S上でのmicrohypoellipticityがえられることを注意しておく。

Reference

Kajitani-Wakabayashi [1]: Propagation of singularities for several classes of pseudodifferential operators, to appear

Asymptotic Behavior and Stability of Solutions to the Exterior Convection Problem

Toshiaki HISHIDA

Department of Mathematics, Waseda University

Suppose that a viscous incompressible fluid occupies the exterior domain to a sphere of radius $R > 0$ centered at the origin in three-dimensions. We consider the convection problem for such a fluid heated at the surface $|x| = R$. The temperature at the surface and that at infinity ($|x| \to \infty$) are assumed to be maintained uniformly; they are, respectively, presented by constants T_w and T_∞ with $T_w > T_\infty \geq 0$. The gravitational field $g(x)$, which plays an important part in convection phenomena, is given by $g(x) = g_0 \nabla(1/|x|)$ with a gravitational constant $g_0 > 0$. If the temperature difference $T_w - T_\infty$ is small enough, the fluid remains motionless and heat is transported purely by conduction; such a steady state is called the conductive state. On exceeding a critical temperature difference, the buoyant force against the direction of the gravitational field overcomes the stabilizing effect of viscous force and, as a result, derives the convective state.

In this paper we are concerned with the asymptotic stability of the steady conductive state mentioned above. The stationary convection problem, governing the velocity field $u = {}^t(u^1(x), u^2(x), u^3(x))$, the temperature $T = T(x)$ and the pressure $p = p(x)$, is described by the following system of equations of motion, continuity and heat conduction (see, e.g., Chandrasekhar [2; Chapter II]):

$$u\cdot\nabla u = (1-\chi(T-T_\infty))g + \nu\Delta u - \frac{\nabla p}{\rho}, \qquad |x| > R,$$

$$(1) \qquad \nabla \cdot u = 0, \qquad |x| > R,$$

$$u \cdot \nabla T = \kappa \Delta T, \qquad |x| > R,$$

where ρ (density at infinity), χ (volume expansion coefficient), ν (kinematic viscosity) and κ (thermal conductivity) are positive constants. The system above is derived from the Boussinesq approximation: density variations are neglected except in the gravitational term (buoyancy term), in which they are assumed to be proportional to temperature variations. For details, see [2]. We consider (1) subject to boundary conditions

$$(2) \qquad \begin{aligned} &u = 0, \quad T = T_w, \qquad &|x| = R, \\ &u \to 0, \quad T \to T_\infty \qquad &\text{as } |x| \to \infty. \end{aligned}$$

We now make the following change of variables and functions:

$$x = Rx^*, \qquad u = \frac{\nu}{R} u^*, \qquad T - T_\infty = \frac{\nu \sqrt{T_w - T_\infty}}{\sqrt{\chi R g_0}} T^*$$

$$\text{and} \quad p - \frac{\rho g_0}{|x|} = \frac{\rho \nu^2}{R^2} p^*.$$

By omitting the asterisks for notational simplicity, (1) and (2) are reduced to the nondimensionalized form:

$$u \cdot \nabla u = -\sqrt{\tau}(\nabla \frac{1}{|x|})T + \Delta u - \nabla p, \qquad |x| > 1,$$

$$\nabla \cdot u = 0, \qquad |x| > 1,$$

$$(BP) \qquad u \cdot \nabla T = \frac{1}{\sigma} \Delta T, \qquad |x| > 1,$$

$$u = 0, \qquad T = \sqrt{\tau}, \qquad |x| = 1,$$

$$u \to 0, \qquad T \to 0 \qquad \text{as } |x| \to \infty,$$

where

$$\tau = \frac{\chi R g_0}{\nu^2}(T_w - T_\infty) = \text{Grashof number},$$

$$\sigma = \frac{\nu}{\kappa} = \text{Prandtl number},$$

$$\sigma\tau = \text{Rayleigh number}.$$

Not so much has been known for (BP). However, it is evident that for each Grashof number τ, (BP) has a solution exactly given by

$$u(x) = 0, \quad T(x) = \frac{\sqrt{\tau}}{|x|}, \quad p(x) = -\frac{\tau}{2|x|^2} + \text{constant},$$

which corresponds to the conductive state. In what follows we call such a solution the conduction solution. It seems to be physically reasonable to expect that there is a certain critical Rayleigh number $(\sigma\tau)_c$ such that the conduction solution is stable (resp. unstable) so long as $\sigma\tau < (\sigma\tau)_c$ (resp. $\sigma\tau > (\sigma\tau)_c$). When the conduction solution is perturbed by disturbance $v = {}^t(v^1(x,t), v^2(x,t), v^3(x,t))$, $\theta = \theta(x,t)$ and $\pi = \pi(x,t)$, they are governed by the following nonstationary problem:

$$\text{(IBP)} \quad \begin{cases} \dfrac{\partial v}{\partial t} + v\cdot\nabla v = -\sqrt{\tau}\left(\nabla\dfrac{1}{|x|}\right)\theta + \Delta v - \nabla\pi, & |x| > 1,\ t > 0, \\[2mm] \nabla\cdot v = 0, & |x| > 1,\ t \geq 0, \\[2mm] \dfrac{\partial\theta}{\partial t} + v\cdot\nabla\theta = \dfrac{1}{\sigma}\Delta\theta - \sqrt{\tau}\ v\cdot\nabla\dfrac{1}{|x|}, & |x| > 1,\ t > 0, \\[2mm] v = 0, \quad \theta = 0, & |x| = 1,\ t > 0, \\[2mm] v \to 0, \quad \theta \to 0 & \text{as } |x| \to \infty,\ t > 0, \end{cases}$$

$$v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x), \qquad |x| > 1,$$

where $\{v_0,\theta_0\}$ is given initial disturbance.

The principal purpose of this paper is to prove that the solution of (IBP) exists for all time and tends to zero as $t \to \infty$ with respect to suitable norms under smallness conditions of both the Rayleigh number and the initial disturbance. In this context, the conduction solution is said to be asymptotically stable. We are mainly interested in the decay property in H^2 of strong solution for initial disturbance of class $D(A^{1/4})\times L^2$, where A is the Stokes operator in L^2.

Up to now Galdi and Padula [3; Part I] have studied the stability of the conduction solution with respect to Dirichlet integral. They have shown among others, that (i) $(\sigma\tau)_c$ is characterized by the supremum of $\sigma\tau$ so that the linearized operator around the conduction solution is nonnegative in L^2; (ii) $(\sigma\tau)_c \geq 1/16$; (iii) when $\sigma\tau < (\sigma\tau)_c$ and $\{v_0,\theta_0\}$ is small in H^1, the Dirichlet norm of $\{v,\theta\}$ decays like $O(t^{-1/2})$ as $t \to \infty$. Although they have been also concerned with instability, we intend to concentrate our analysis on the stability problem (several decay properties of disturbance). From our viewpoint, it seems that (ii) and (iii) above are less than perfect.

In the present paper it is shown that $(\sigma\tau)_c \geq 1/4$ and that the Dirichlet norm of $\{v,\theta\}$ decays like $o(t^{-1/2})$ for small $\{v_0,\theta_0\}$ in $D(A^{1/4})\times L^2$. Moreover, we derive the L^p decay for all $2 \leq p \leq \infty$ with explicit rates as well as the H^2 decay. By using the fact that the square root of the linearized operator has an equivalent L^2-norm to the Dirichlet norm, the desired decay properties can be deduced through the following:

(a) $\{v,\theta\}$ decays in L^2,

(b) $\{\nabla v,\nabla\theta\}$ decays like $o(t^{-1/2})$ in L^2,

(c) $\{\partial v/\partial t,\partial\theta/\partial t\}$ decays like $o(t^{-1})$ in L^2.

To show (a), we treat (IBP) via the integral representation inverted by the linearized operator, making use of an estimate on the nonlinear term essentially due to Borchers and Miyakawa [1], in which L^2 decay for Navier-Stokes flows has been studied. It is also proved that there cannot be any uniform rate of L^2 decay of solutions, by improving the scaling argument of Schonbek [6]. To show (b) and (c), we appeal to the weighted energy method, which is partially similar to [3] (see also Masuda [5]).

Before stating our results, we introduce notation and some definitions. All functions in this paper are real-valued and, for simplicity, we use the same symbol for denoting the spaces of scalar and vector functions. Set $\Omega = \{x \in R^3;\ |x| > 1\}$. By $C^\infty_{0,\sigma}(\Omega)$ we denote the set of solenoidal (i.e., $\nabla\cdot v = 0$) vector fields with components in $C^\infty_0(\Omega)$. For $1 \le p \le \infty$, $\|\cdot\|_p$ stands for the norm of $L^p(\Omega)$; especially for $p = 2$ we simply write $\|\cdot\| = \|\cdot\|_2$. We define $L^2_\sigma(\Omega)$ by the completion of $C^\infty_{0,\sigma}(\Omega)$ in the norm $\|\cdot\|$. Let P be the bounded projection operator from $L^2(\Omega)$ onto $L^2_\sigma(\Omega)$ along the decomposition $L^2(\Omega) = L^2_\sigma(\Omega)\oplus L^2_\sigma(\Omega)^\perp$, where $L^2_\sigma(\Omega)^\perp = \{\nabla\pi \in L^2(\Omega);\ \pi \in L^2_{loc}(\bar\Omega)\}$. Then the Stokes operator in $L^2_\sigma(\Omega)$ is defined by $Av = -P\Delta v$ for $v \in D(A) = L^2_\sigma(\Omega)\cap H^1_0(\Omega)\cap H^2(\Omega)$. We also introduce the following operators: $B\theta = -\Delta\theta$ for $\theta \in D(B) = H^1_0(\Omega)\cap H^2(\Omega)$, $S\theta = P(\nabla\frac{1}{|x|})\theta$ and $Tv = v\cdot\nabla\frac{1}{|x|}$. It can be shown that $S\theta \in L^2_\sigma(\Omega)$ and $Tv \in L^2(\Omega)$ for all θ, $v \in \hat{H}^1_0(\Omega)$, which is the completion of $C^\infty_0(\Omega)$ in the norm $\|\nabla\cdot\|$.

In terms of the operators above, we formulate (IBP) to the following Cauchy problem for evolution equations:

$$
(\mathrm{CP})\begin{cases}
\dfrac{dv}{dt} + Av + \sqrt{\tau}\, S\theta = -P(v\cdot\nabla)v, & t > 0;\ v(0) = v_0,\\[2ex]
\dfrac{d\theta}{dt} + \dfrac{1}{\sigma} B\theta + \sqrt{\tau}\, Tv = -v\cdot\nabla\theta, & t > 0;\ \theta(0) = \theta_0.
\end{cases}
$$

We make the following hypotheses throughout this paper:

(H1) $\sigma\tau < \frac{1}{4}$, or equivalently $T_w - T_\infty < \frac{\kappa\nu}{4\chi R g_0}$,

(H2) $v_0 \in D(A^{1/4})$, $\theta_0 \in L^2(\Omega)$.

We now define the notion of strong solution of (CP).

DEFINITION. A pair of functions $\{v,\theta\}$ is called a strong solution of (CP) on $[0,\infty)$ with data given by (H2) if it belongs to the class

$$v \in C([0,\infty);D(A^{1/4})) \cap C(0,\infty;D(A)) \cap C^1(0,\infty;L^2_\sigma(\Omega)),$$

$$\theta \in C([0,\infty);L^2(\Omega)) \cap C(0,\infty;D(B)) \cap C^1(0,\infty;L^2(\Omega)),$$

and satisfies (CP) in $L^2_\sigma(\Omega)\times L^2(\Omega)$.

For $\varepsilon > 0$ we introduce the following set of the initial disturbance $\{v_0,\theta_0\}$:

$$K_\varepsilon = \{v_0 \in D(A^{1/4}),\ \theta_0 \in L^2(\Omega);\ \|A^{1/4}v_0\| + \|v_0\| + \|\theta_0\| < \varepsilon\}.$$

Our main result on the asymptotic stability of conduction solutions reads:

Theorem 1. Suppose that (H1) and (H2) hold. Then there exists a positive constant $\varepsilon = \varepsilon(\sigma,\tau)$ such that whenever $\{v_0,\theta_0\} \in K_\varepsilon$, (CP) has a unique strong solution $\{v,\theta\}$ on $[0,\infty)$ with the following decay property:

$$\|v(t)\|_{H^2(\Omega)} + \|\theta(t)\|_{H^2(\Omega)} \to 0 \quad \text{as } t \to \infty.$$

Theorem 1 indicates that $\{v,\theta\}$ decays in L^p spaces for all $2 \le p \le \infty$. By the following theorem we give some decay rates in such spaces and further decay properties.

Theorem 2. The solution $\{v,\theta\}$ in Theorem 1 possesses the following decay properties as $t \to \infty$ with explicit rates:

(i) $$\|v(t)\|_p + \|\theta(t)\|_p = \begin{cases} o(t^{-(3/2-3/p)/2}) & \text{if } 2 \le p < 6, \\ o(t^{-1/2}) & \text{if } 6 \le p \le \infty. \end{cases}$$

(ii) $$\|\frac{dv}{dt}(t)\| + \|\frac{d\theta}{dt}(t)\| = o(t^{-1}).$$

(iii) The pressure gradient $\nabla\pi$ associated with $\{v,\theta\}$ decays like $\|\nabla\pi(t)\| = o(t^{-1/2})$.

(iv) If, in addition, $\{v_0,\theta_0\} \in L^q(\Omega)$ for some $1 \le q < 2$, then

$$\|v(t)\| + \|\theta(t)\| = \begin{cases} O(t^{-(3/q-3/2)/2}) & \text{if } \frac{6}{5} < q < 2, \\ o(t^{\eta-1/2}) & \text{if } 1 \le q \le \frac{6}{5}, \end{cases}$$

where $\eta > 0$ is an arbitrary small number.

For $\{v_0,\theta_0\} \in K_\varepsilon$ we denote by $\Phi[\{v_0,\theta_0\}]$ the solution $\{v,\theta\}$ in Theorem 1. Without the additional assumption like (iv) of Theorem 2, we have the lack of uniformity of L^2 decay of $\{v,\theta\} \in \Phi[K_\varepsilon]$ in the sense that:

Theorem 3. For all $\alpha \in (0,\varepsilon]$, there exists *no* function $H(\cdot)$: $R^+ \to R^+$ with the following properties (1) and (2):

(1) $H(t) \to 0$ as $t \to \infty$.

(2) For all $\{v,\theta\} \in \Phi[K_\alpha]$ and $t > 0$,

$$\|v(t)\| + \|\theta(t)\| \le H(t).$$

For the proof of theorems above, see [4].

References

[1] Borchers W. & Miyakawa T., L^2 decay for Navier-Stokes flows in unbounded domains, with application to exterior stationary flows, preprint.

[2] Chandrasekhar S., *Hydrodynamic and Hydromagnetic Stability*, Dover, New York (1981).

[3] Galdi G.P. & Padula M., A new approach to energy theory in the stability of fluid motion, Arch. Rational Mech. Anal. **110**, 187-286 (1990).

[4] Hishida T., Asymptotic behavior and stability of solutions to the exterior convection problem, preprint.

[5] Masuda K., On the stability of incompressible viscous fluid motions past objects, J. Math. Soc. Japan **27**, 294-327 (1975).

[6] Schonbek M.E., Large time behaviour of solutions to the Navier-Stokes equations, Commun. PDE. **11**, 733-763 (1986).

Hartree type Schrödinger 方程式の H^1-blow up する初期値について

平田 均（東大 教養）

0° 考える方程式

$$\text{(D.E.)}\quad \begin{cases} i\partial_t u = -\Delta u - (W*|u|^2)u, & u = u(t,x) \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^N) & (t,x) \in \mathbb{R}\times\mathbb{R}^N \end{cases}$$

なる非線形の Schrödinger 方程式を考える。
ここで、非線形項 $(W*|u|^2)u$ は、Hartree-type (non local-type) の非線形項と呼ばれ、自己相互作用を表わしており、例えば Herium 原子の電子に対する方程式の半古典近似に現われる。

ここでは、$W(x) = |x|^{-\gamma}$ の形に限定して、(D.E.) の"局所解の H^1-blowup" について考えることにする。

1° 積分方程式と局所解の存在

(D.E.) に対応する次の積分方程式を考える。

$$\text{(I.E.)}\quad u(t) = U(t)u_0 + i\int_0^t U(t-s)\{(W*|u|^2)u\}(s)\,ds$$

ここで、$U(t) = e^{it\Delta}$ である。

(I. E.) の局所解の存在については、次の定理が成立する。

Thm. 1. [2] $u_0 \in H^1$, $2 \leq \gamma \leq 4$. $\gamma < N$ とする。
この時、$0 < T^* \leq \infty$ と (I. E.) の解 $u \in C([0, T^*); H^1)$ が存在して次が成立する。

(i) u は $L^{\theta}_{loc}(0, T^*; W^{1,p})$ で unique.

ここで $\frac{1}{p} = \frac{1}{2} - \frac{\gamma - 2}{4N}$, $\theta = \frac{8}{\gamma - 2}$

(ii) 保存則 $\|u(t)\|_2 = \|u_0\|_2$

$$E(u) := \|\nabla u\|_2^2 - \frac{1}{2}(|u|^2, W * |u|^2) = E(u_0)$$

が $t \in [0, T^*)$ に対し成立.

(iii) $2 \leq \gamma < 4$ の時、もし $T^* < \infty$ なら

$\|\nabla u(t)\|_2 \to \infty$ as $t \uparrow T^*$

(iv) u は (D. E.) を H^{-1} で満す。

(注意) ◦ 仮定のうち、$2 \leq \gamma$ は本質的ではなく、$0 \leq \gamma \leq 4$ で局所解は構成される。しかし一意性の言える空間が複雑になるのと、H^1-blow up が起こり得ない (i.e. $T^* = \infty$ となる) ので、ここでは省略した。

◦ $\gamma < 4$ の時と、$\gamma = 4$ の時では、解の構成法が微妙に違う。そのため、(iii) は $\gamma = 4$ では分からない。

◦ (iii) の現象を H^1-blow up といい、T^* を blow up time という。

2° H^1-blow up が起こるための条件.

H^1-blow up が実際に起こるためには, $\gamma \geq 2$ は必要であるが, 十分条件としては次の結果が知られていた。([1])

Thm. 2. $u_0 \in \Sigma := \{v \in H^1 : xv \in L^2\}$, $E(u_0) < 0$ なら, Thm 1 の局所解は, 有限時間で H^1-blow up する。

この結果は, $f(t) := \|xu(t)\|_2^2$ とすると,

$$f'(t) = 4\,\mathrm{Im}\,(u, x\nabla u)$$
$$f''(t) = 8E(u_0) - (2\gamma - 4)(|u|^2, W * |u|^2) \qquad t \in [0, T^*)$$

となるので, 上の仮定の下で, $f''(t) < 0$ となり, 従って有限時間内に, $f(t) \leq 0$ となって矛盾を生ずることより従う。

上の仮定のうち, $xu_0 \in L^2$ は, 最近, 巾 type (local type) の非線形項 $|u|^{p-1}u$ をもつ非線形 Schrödinger 方程式 $i\partial_t u = -\Delta u - |u|^{p-1}u$, $\frac{4}{N} \leq p-1 \leq \frac{4}{N-2}$ の場合には, はずせる事が Tsutsumi, Ogawa [3] によって示された。(ただし, u_0 が球対称の場合)

ここでは, 彼らの方法が, Hartree type の場合でも有効な事を示す。

Thm 3. $v_0 \in H^1$, v_0 は球対称, $2<\gamma\leq 4$, $\gamma<N-1$, $E(v_0)<0$ ならば, Thm 1 の局所解は, 有限時間で H^1-blow up する。

(注意) ○ 仮定 $\gamma<N-1$ は, $\nabla(|x|^{-\gamma})\in L^1_{loc}$ となるための条件である。

○ $\gamma=2$ の場合, 巾 type での $p-1=\frac{4}{N}$ に対応しているが, Tsutsumi-Ogawa [3] では出来た方法が nonlocal interaction がじゃまをして働かない。

従って今の所, $\gamma=2$ では出来ていない。

3° Thm 3 の証明の方針と補題

まず, 2つの有用な補題をおく。

Lem. 4 (Gagliardo-Nirenberg) $N\geq 3$ の時 $v\in H^1(\mathbb{R}^N)$ に対し,

$$\|v\|_p \leq C\|\nabla v\|_2^a \|v\|_2^{1-a}$$

ここで,

$$\frac{1}{p}=a\left(\frac{1}{2}-\frac{1}{N}\right)+\frac{1-a}{2}$$

Lem. 5 (Strauss) $N\geq 2$, $v\in H^1(\mathbb{R}^N)$ が球対称の時, $p\in[2,\infty]$, $R>0$ に対して

$$\|v\|_{L^p(R<|x|)} \leq C R^{-\left(\frac{1}{2}-\frac{1}{p}\right)(N-1)} \|v\|_{L^2(R<|x|)}^{\frac{1}{2}+\frac{1}{p}} \|\nabla v\|_{L^2(R<|x|)}^{\frac{1}{2}-\frac{1}{p}}$$

（注意）
- Lem 4 は、もっと一般的な形であるが、ここでは必要とする場合のみに限った。
- Lem 5 は、"球対称函数は本質的には1次元函数" だから成立する。証明は、$r^{N-1}v^2(r)$ を微分して積分すればよい。

さて、Thm 3 では $\|xu(t)\|_2$ を取ることが出来ないので、かわりに x を適当な有界函数で近似してやる。すなわち、

$\phi \in C^\infty([0,\infty))$ を、$0 \le \phi' \le 1$ なるよう定め、

$$\phi(r) = \begin{cases} r & 0 \le r \le 1 \\ \text{smooth} & 1 \le r \le 2 \\ 3/2 & 2 \le r \end{cases}$$

$\phi_m(r) = m\,\phi(r/m)$, $\psi_m(x) = \dfrac{x}{|x|}\phi_m(|x|)$ とおく。

(D.E.) と $\psi_m \nabla u$ との L^2 内積を取って、さらにその実部を取ると、

$$-\frac{d}{dt}\,\mathrm{Im}\int \psi_m u \nabla \bar{u}\,dx = 2\int_{|x|\le m} |\nabla u|^2 dx + 2\int_{m\le|x|\le 2m} \phi_m' |\nabla u|^2 dx$$
$$-\frac{1}{2}\int_{|x|\ge m} \Delta(\nabla\psi_m)\cdot|u|^2 dx + \int \psi_m |u|^2 (\nabla W * |u|^2)\,dx$$

が得られる。ここで、

$$\int \psi_m |u|^2 (\nabla W * |u|^2)\,dx = \gamma\left(E(u_0) - \|\nabla u\|_2^2\right)$$
$$+\frac{\gamma}{2}\iint_{|x|\vee|y|\ge m} dx\,dy\,\left\{|x-y|^2 - (\psi_m(x)-\psi_m(y))(x-y)\right\}|x-y|^{-\gamma-2}|u|^2(x)\,|u|^2(y)$$

であるが、$0 \le \phi' \le 1$ より $|x-y|^2 - (\psi_m(x)-\psi_m(y))(x-y) \le 2|x-y|^2$ が成立するから

$$\int \psi_m |u|^2 (\nabla W * |u|^2)\,dx \le \gamma\left(E(u_0) - \|\nabla u\|_2^2\right) + 2\gamma\int_{|x|\ge m} |u|^2(x)\,(r^{-\gamma} * |u|^2)(x)\,dx$$

u_0 が球対称なら、解の一意性より u も球対称なので、Lem 4, 5 が用いて、

$$\int \psi_m |u|^2 \nabla W * |u|^2 dx \le \gamma(E(u_0) - \|\nabla u\|_2^2) + Cm^{-\delta}\|u\|_2^2\|\nabla u\|_2^2$$

ただし. $\delta := \gamma \cdot \frac{n-1}{n+1} > 0$

さらに. $|\Delta(\nabla\psi_m)| \le Cm^{-2}$ なので, 結局

$$\mathrm{Im}\int \psi_m u \nabla\bar{u}\, dx - \mathrm{Im}\int \psi_m u_0 \nabla \bar{u}_0\, dx$$

$$\ge \int_0^t d\tau \left[-\gamma E(u_0) + (\gamma-2)\|\nabla u\|_2^2 - Cm^{-\delta}\|u_0\|_2^2\|\nabla u\|_2^2 - Cm^{-2}\|u_0\|_2^2\right]$$

そこで m を十分大に取って. $\varepsilon := (\gamma-2) - Cm^{-\delta}\|u_0\|_2^2 > 0$

$K := -\gamma E(u_0) - Cm^{-2}\|u_0\|_2^2 > 0$ とすれば,

$$f(t) \ge f(0) + tK + A\int_0^t f(\tau)^2 d\tau$$

ただし. $f(t) := \mathrm{Im}\int \psi_m u \nabla\bar{u} dx$, $A := \frac{4\varepsilon}{9m^2}\|u_0\|_2^{-2}$

この微分不等式を解けば, f が有限時間で爆発することが分かる。

$f(t) \le \frac{3m}{2}\|u_0\|_2\|\nabla u\|_2$ なので, 結局 H^1-blowup が分かる。//

Ref. [1] T. Cazenave "An Introduction to Nonlinear Schrödinger Equations" : Textos de Metodos Matematicos do Universidade Federal do Rio de Janeiro 22.

[2] H. Hirata "The Cauchy Problem for Hartree type Schrödinger Equation in H^Δ" (Preprint)

[3] Y. Tsutsumi and T. Ogawa "Blow-up of H^1-Solution for the Nonlinear Schrödinger Equation" J. D. E. (92) 1991 P.317～330.

Reonance of The Ordinary Second Differential Operators on The Half-Line.

KAZUO WATANABE

Department of Mathematics
Gakushuin University

1. Introduction

We shall consider the following second order differential operators on a half-line $[0,\infty)$, one with the Dirichlet condition and another has a "jump condition".

$$\text{(I)} \qquad \begin{cases} Hu = -\dfrac{d^2}{dx^2}u \text{ on } L^2(0,\infty), \\ u(0) = u(1\pm 0) = 0. \end{cases}$$

$$\text{(II)} \qquad \begin{cases} H_\sigma u = -\dfrac{d^2}{dx^2}u \text{ on } L^2(0,\infty), \\ u(0) = 0,\ u(1-0) = u(1+0) \equiv u(1) \\ u'(1-0) - u'(1+0) = \sigma u(1). \end{cases}$$

It is well-known that H has embedded eigenvalues $\{n^2\pi^2\}_{n\geq 1}$ and continuous spectrum $[0,\infty)$. Then we expect that these embedded eigenvalues are "resonances", i.e, the eigenvalues of H are not stable with respect to H_σ.

In this paper we shall show $H_\sigma \to H$ in the norm resolvent sense. And we shall examine "resonance" and in fact calculate "exponential decay" of $(\exp(-itH_\sigma)\varphi,\varphi)$, where φ is the eigenfunction corresponding to the eigenvalue π^2 of H.

THEOREM 1. *Let* $\lambda_0 = 1/(\pi^2+1)$ *and* $\varphi(x) = \sqrt{2}\sin\pi x$, *or* $0 \leq x \leq 1$, $= 0$, *or* $1 < x$. *Then we have*

$$|(\exp(-itR_\sigma(-1))\varphi,\varphi)| = e^{-\Gamma\sigma^{-2}t} + o(1),\ \sigma >> 1, \tag{1.1}$$

where $R_\sigma(-1) = (H_\sigma+1)^{-1}$ *and*

$$\Gamma = \frac{2\pi^2}{(1+\pi^2)^2}.$$

THEOREM 2. *Let φ be the same as in Theorem 1. Then we have*

(1.2)

$$(\exp(-itH_\sigma)\varphi,\varphi) = \int_0^\infty \frac{e^{it\lambda^2}}{(\pi^2-(\lambda+i0)^2)^2} \frac{4\pi^2\lambda^2\sin^2\lambda d\lambda}{(\sigma\sin\lambda-\lambda e^{i\lambda})(\sigma\sin\lambda-\lambda e^{-i\lambda})}.$$

THEOREM 3. *Let φ be the same as in Theorem 1. Then we have*

$$|(\exp(-itH_\sigma)\varphi,\varphi)| = C(\sigma)e^{-\Gamma(\sigma)t}+o(1),\ \sigma >> 1, \tag{1.3}$$

where $\Gamma(\sigma)>0$.

2. Livsic matrix and Resonance

In order to prove Theorem 1, we shall use the result of [O]. Let H be a self-adjoint operator in a Hilbert space $\mathbf{H}$, P be the orthogonal projection associated to the eigenvalue λ_0 of H, $K=$ Range P ,$dim K<\infty$ and $\overline{P}:=I-P$. Let W be closed symmetric such that $D(H)\subset D(W)$. We define $H(\kappa)$ as $H(\kappa)=H+\kappa W$. Then "Livsic matrix" $B(z,\kappa)$ is the operator in K having the following form:

$$B(z,\kappa)=\lambda_0+\kappa PWP-\kappa^2PW\overline{P}(\overline{H}(\kappa)-z)^{-1}\overline{P}WP, \tag{2.1}$$

where $\overline{H}(\kappa)=\overline{P}H(\kappa)\overline{P}$.

DEFINITION 1. *(A. Orth [O].) The operator family $H(\kappa)$ has a simple resonance at λ_0, if λ_0 is nondegenerate and if there are a real neighborhood I of λ_0, a real neighborhood U of 0 and a densely embedded subspace $\mathbf{H}_+$ of $\mathbf{H}$ with its dual $\mathbf{H}_-$, such that*

(i) for $\kappa\in U$, $(\overline{H}(\kappa)-z)^{-1}$ has a continuous extention from $\mathbf{C}\setminus\mathbf{R}$ onto $z\in I$ as an operator in $B(\mathbf{H}_+,\mathbf{H}_-)$. This continuation is Lipschitz-continous with constant $L(\kappa)=o(\kappa^{-2})$;

(ii) $K\subset\mathbf{H}_+$, and $W(K)\subset\mathbf{H}_+$;

(iii) for $\kappa\in U$ and all possible eigenvalues $\mu(\kappa)\in I$ of $H(\kappa)$, the associated eigenvectors are in $\mathbf{H}_+$.

REMARK. *Lipschitz continuity of the condition (i) is weakened as below. $(PW\overline{P}(\overline{H}(\kappa)-z)^{-1}\overline{P}WP\varphi,\varphi)$ is Lipschitz continuous with constant $o(\kappa^2)$.*

THEOREM 4. *(Theorem 1.5 in [O], A. Orth). Let $H(\kappa)$ have a simple resonance at λ_0 and φ be in K with $\|\varphi\|=1$. Then for small κ there exists a unique solution $\lambda(\kappa)$ such that*

$$\lambda(\kappa)=Re(B(\lambda(\kappa),\kappa)\varphi,\varphi).$$

Furthermore we put $B(\kappa) = B(\lambda(\kappa), \kappa)$ and $\Gamma(\kappa) = -Im(B(\kappa)\varphi, \varphi)$. Then we can choose $\delta(\kappa) \geq 0$, such that for $\Gamma(\kappa) = 0$ $\delta(\kappa) = 0$, while for $\Gamma(\kappa) \neq 0$ and $\kappa \to 0$, $\max\{\delta(\kappa), \kappa^2 L(\kappa)\delta(\kappa)/\Gamma(\kappa), \Gamma(\kappa)/\delta(\kappa)\} \to 0$. Let $J(\kappa) = [\lambda(\kappa) - \delta(\kappa), \lambda(\kappa) + \delta(\kappa)]$ and E_κ be the spectral projection of H_κ. Then we obtain $E_\kappa(J(\kappa)) \to P$ strongly.

THEOREM 5. *(Theorem 1.8 and (1.8) in [O], A. Orth). Let λ_0 be a simple resonance of $H(\kappa)$ and $\varphi \in K$ with $\|\varphi\| = 1$. And we assume that $\Gamma = \lim_{Imz>0, z\to\lambda_0} Im(PW\overline{P}(H-z)^{-1}\overline{P}WP\varphi, \varphi) \neq 0$. Then for $\varphi \in K$ with $\|\varphi\| = 1$, we have*

$$|(e^{-itH(\kappa)}\varphi, \varphi)| = e^{-\Gamma\kappa^2 t} + o(1). \tag{2.2}$$

3. Propositions and Lemmas

PROPOSITION 1. *Let $\zeta \in \mathbf{C} \setminus [0, \infty)$ and $V_\sigma(\zeta) := R_\sigma(\zeta) - R(\zeta)$, where $R(\zeta) = (H - \zeta)^{-1}$ and $R_\sigma(\zeta) = (H_\sigma - \zeta)^{-1}$. Then $V_\sigma(\zeta)$ has the kernel given as*

$$v_\sigma(\zeta; x, y) = \frac{1}{p(\sigma, \zeta)} g(\zeta; x) g(\zeta; y) \tag{3.1}$$

where

$$g(\zeta; x) = \begin{cases} \sin\sqrt{\zeta}x, & 0 \leq x < 1, \\ \sin\sqrt{\zeta}e^{i\sqrt{\zeta}(x-1)}, & 1 < x, \end{cases} \tag{3.2}$$

and $p(\sigma, \zeta) = (\sqrt{\zeta}e^{-i\sqrt{\zeta}} - \sigma\sin\sqrt{\zeta})\sin\sqrt{\zeta}$, $Im\sqrt{\zeta} > 0$. $V_\sigma(\zeta)$ is the operator of rank one and the the representation is

$$V_\sigma(\zeta)u(x) = \frac{1}{p(\sigma, \zeta)}(u, \overline{g}(\zeta; \cdot))g(\zeta; x), \ u \in L^2(0, \infty) \tag{3.3}$$

where $\overline{g}$ is the complex conjugate of g.

The weighted $L^{2,s}(0, \infty)$ is defined by $\{u \in L^2_{loc}(0, \infty) : \langle x\rangle^s u \in L^2(0, \infty)\}$, where $\langle x\rangle = (1 + x^2)^{1/2}$. By Proposition 1 we easily obtain the following Lemmas 2 and 3.

LEMMA 2. *Let $\zeta \in \mathbf{C}\setminus[0, \infty)$ and $s, s' \in \mathbf{R}$. Then we have $\|V_\sigma(\zeta)\|_{s,s'} \to 0$ as $\sigma \to \infty$, where $\|\cdot\|_{s,s'}$ is the $B(L^{2,s}, L^{2,s'})$ norm.*

LEMMA 3. *For $s, s' \in \mathbf{R}$ and $\zeta \in \mathbf{C} \setminus [0, \infty)$, there exsists an operator $W(\zeta)$ such that $\sigma V_\sigma(\zeta) \to W(\zeta)$ $(\sigma \to \infty)$ in $B(L^{2,s}, L^{2,s'})$. More precisely the operator $W(\zeta)$ has the kernel*

$$w(\zeta; x, y) = \frac{-1}{\sin^2\sqrt{\zeta}} g(\zeta; x) g(\zeta; y) \tag{3.9}$$

where $g(\zeta;x)$ is the same as in Proposition 1.

LEMMA 4. *Let $\lambda_0 = 1/(\pi^2+1)$ be the eigenvalue of $R(-1)$, P be the orthogonal projection cooresponding to the eigenspace of the eigenvalue λ_0 and $\overline{P} := I - P$. For $s > 1/2$ we put $\mathbf{H}_+ = L^{2,s}(0,\infty)$ and $\mathbf{H}_- = L^{2,-s}(0,\infty)$. Then the operator $R_\sigma(-1)$ satisfies the condition of Definition 1 regarded as $\kappa = 1/p(\sigma)$.*

PROPOSITION 5. *Let P_1 be the projection from $L^2(0,\infty)$ to $L^2(0,1)$ and $P_2 = 1 - P_1$. Then we have*

(3.17)
$$\begin{aligned}(A(z)\overline{P}g,g) = &-(\|P_1\overline{P}g\|^2 + \|P_2 g\|^2)/z \\ &- \frac{1}{z^2}\sum_{n=2}^{\infty}\frac{1}{n^2\pi^2+1-1/z}|(P_1 g,\varphi_n)|^2 \\ &- \frac{|\sin i|^2}{2z^2(1-i\sqrt{-1+1/z})^2}\end{aligned}$$

where $\varphi_n(x) = \sqrt{2}\sin n\pi x$.

LEMMA 6. *For small $\varepsilon > 0$ and small $l > 0$, let σ be sufficiently large. Then the equation $\sigma\sin z - ze^{iz} = 0$ has unique solution $z(\sigma)$ in $\{z \in \mathbf{C} : \pi \le Rez \le \pi+\varepsilon,\ 0 < Imz < l\}$.*

4. Proofs of Theorems

PROOF OF THEOREM 1: We shall use the notations in Lemma 4 and its proof. Let $B(z,\sigma)$ be Livsic matrix of $R_\sigma(-1)$:

(4.1) $$B(z,\sigma) = \lambda_0 + \frac{1}{p(\sigma)}PWP - \frac{1}{p(\sigma)^2}PW\overline{P}(\overline{R}_\sigma(-1)-z)^{-1}\overline{P}WP,$$

where $\lambda_0 = 1/(\pi^2+1)$. By Theorem 5 it is sufficiently to prove that

(4.2)
$$\Gamma = \lim_{Imz>0,z\to\lambda_0}\ \lim_{\sigma\to\infty}\sigma^2 Im(PV_\sigma\overline{P}(\overline{R}_\sigma(-1)-z)^{-1}\overline{P}V_\sigma P\varphi,\varphi) > 0.$$

Since the contribution of the first and second terms is zero for (4.2), (4.2) is equal to

$$\begin{aligned}&-\left|\frac{(\varphi,g)}{\sin i}\right|^2 \lim_{Imz>0,z\to\lambda_0} Im\frac{1}{2z^2(1-i\sqrt{-1+1/z})^2} \\ &= -\frac{\pi^2}{(1+\pi^2)^2}Im\frac{1}{(1+i\pi)^2}\end{aligned}$$

$$= \frac{2\pi^3}{(1+\pi^2)^4}.$$

Here we used $\sqrt{-1+1/z} \to -\pi$, because $Im 1/z$ is negative.

PROOF OF THEOREM 2: We shall use spectral representation of H_σ. Then we have

$$\begin{aligned}
(4.3) \qquad & (\exp(-itH_\sigma)\varphi, \varphi) \\
&= \frac{1}{2\pi i}\int_0^\infty e^{it\mu}((R_\sigma(\mu - i0) - R_\sigma(\mu + i0))\varphi, \varphi)d\mu \\
&= \frac{1}{2\pi i}\int_0^\infty e^{it\mu}((V_\sigma(\mu - i0) - V_\sigma(\mu + i0))\varphi, \varphi)d\mu \\
&\quad + \frac{1}{2\pi i}\int_0^\infty e^{it\mu}((R(\mu - i0) - R(\mu + i0))\varphi, \varphi)d\mu \\
&=: I + e^{it\pi^2}.
\end{aligned}$$

Putting $k = \mu \pm i0$, we caluculate $(V_\sigma(k)\varphi, \varphi)$.

$$\begin{aligned}
(4.4) \qquad & (V_\sigma(k)\varphi, \varphi) \\
&= \frac{-\pi \sin\sqrt{k}}{p(\sigma, k)(k - \pi^2)}\left(\frac{\sin(\sqrt{k} - \pi)}{\sqrt{k} - \pi} - \frac{\sin(\sqrt{k} + \pi)}{\sqrt{k} + \pi}\right) \quad (*) \\
&= \frac{2\pi^2 \sin^2\sqrt{k}}{p(\sigma, k)(k - \pi^2)^2}
\end{aligned}$$

We substitute

$$(4.5) \qquad \frac{1}{2\pi i}\left(\frac{1}{\mu - i0 - \pi^2} - \frac{1}{\mu + i0 - \pi^2}\right) = \delta(\mu - \pi^2),$$

$\sqrt{\mu - i0} = -\sqrt{\mu + i0}$ and $(*)$ into the part of $V_\sigma(\mu - i0)$ of (4.3).

(4.6)

$$\begin{aligned}
& \frac{1}{2\pi i}\int_0^\infty e^{it\mu}(V_\sigma(\mu - i0)\varphi, \varphi)d\mu \\
&= \frac{1}{2\pi i}\int_0^\infty \frac{-e^{it\mu}\pi \sin\sqrt{\mu - i0}}{p(\sigma, \mu - i0)(\mu - i0 - \pi^2)} \\
&\quad \times \left(\frac{\sin(\sqrt{\mu - i0} - \pi)}{\sqrt{\mu - i0} - \pi} - \frac{\sin(\sqrt{\mu - i0} + \pi)}{\sqrt{\mu - i0} + \pi}\right) d\mu \\
&= -e^{it\pi^2}
\end{aligned}$$

$$+\frac{1}{2\pi i}\int_0^\infty \frac{e^{it\mu}\pi}{-\sigma\sin\sqrt{\mu}+\sqrt{\mu}e^{i\sqrt{\mu}}}\frac{1}{\mu+i0-\pi^2}\frac{2\pi\sin\sqrt{\mu+i0}}{\mu+i0-\pi^2}d\mu.$$

Hence (4.3) is equal to

(4.7)

$$\frac{1}{2\pi i}\int_0^\infty \frac{-2\pi^2 e^{it\mu}\sin^2\sqrt{\mu+i0}}{p(\sigma,\mu-i0)(\mu+i0-\pi^2)^2}d\mu$$
$$-\frac{1}{2\pi i}\int_0^\infty \frac{2\pi^2 e^{it\mu}\sin^2\sqrt{\mu+i0}}{p(\sigma,\mu+i0)(\mu+i0-\pi^2)^2}d\mu$$
$$=\frac{4\pi^2 i}{2\pi i}\int_0^\infty \frac{e^{it\mu}\sin\sqrt{\mu}\sqrt{\mu}\sin\sqrt{\mu}}{(...)(...)(...)^2}d\mu$$
$$=4\pi\int_0^\infty \frac{e^{it\lambda^2}\lambda^2\sin^2\lambda}{((\lambda+i0)^2-\pi^2)^2(\sigma\sin\lambda-\lambda e^{i\lambda})(\sigma\sin\lambda-\lambda e^{-i\lambda})}d\lambda$$

PROOF OF THEOREM 3: By Lemma 6 we shall change the integral path in Theorem 2. For arbitrary $\varepsilon > 0$ and $l > 0$ (fixed), let σ be sufficiently large. We divide the integral path into 5 parts;

$$\begin{aligned}
C_1 &= \{s : 0 \le s \le \pi-\varepsilon\},\\
C_2 &= \{\pi-\varepsilon+isl : 0 \le s \le 1\},\\
C_3 &= \{\pi-\varepsilon+2s\epsilon+il : 0 \le s \le 1\},\\
C_4 &= \{\pi+\varepsilon+i(1-s)l : 0 \le s \le 1\},\\
C_5 &= \{\pi+\varepsilon \le s < \infty\}.
\end{aligned}$$

For simplicity we put $f_\sigma(x,t)$ as below.

(4.8) $$f_\sigma(x,t)=\frac{e^{itx^2}x^2\sin^2 x}{\sigma^2\sin^2 x-2\sigma x\sin x\cos x+x^2}$$

C_3 part: We shall consider the numerator of $f_\sigma(x,t)$. For $0 \le s \le 1$, we have

$$\begin{aligned}
&|e^{it(\pi-\varepsilon+2s\varepsilon+il)^2}(\pi-\varepsilon+2s\varepsilon+il)^2\sin^2(\pi-\varepsilon+2s\varepsilon+il)|\\
\le&|z\sin(\pi-\varepsilon+2s\varepsilon+il)|^2 e^{-2tl(\pi-\varepsilon)}.
\end{aligned}$$

Hence we have

(4.10) $$|4\pi\int_{C_3}\frac{1}{(z^2-\pi^2)^2}f_\sigma(z,t)dz|$$

$$\leq \int_{C_3} \frac{4\pi|z\sin z|^2 e^{2tl(\pi-\varepsilon)}}{|z^2-\pi^2|^2|\sigma^2\sin^2 z - 2\sigma z\sin z\cos z + z^2|}|dz|$$

Using the residue theorem, we obtain that

(4.11)

$$\begin{aligned}
&4\pi\int_0^\infty \frac{1}{((x+i0)^2-\pi^2)^2} f_\sigma(x,t)dx \\
=&8\pi^2 i \lim_{z\to z(\sigma)} (z-z(\sigma))\frac{1}{(z^2-\pi^2)^2} f_\sigma(z,t) \\
&+4\pi\sum_{n=1}^{5}\int_{C_n}\frac{1}{(z^2-\pi^2)^2} f_\sigma(z,t)dz \\
=&\frac{8\pi^2 i e^{itz(\sigma)^2} z(\sigma)^2\sin^2 z(\sigma)}{(z(\sigma)^2-\pi^2)^2} \\
&\times\frac{1}{(\sigma\sin z(\sigma) - z(\sigma)e^{-iz(\sigma)})(\sigma\cos z(\sigma) - e^{iz(\sigma)} - iz(\sigma)e^{iz(\sigma)})} \\
&+\sum_{n=1}^{5}\int_{C_n} ...dz.
\end{aligned}$$

Therefore we have

$$|(\exp(-itH_\sigma)\varphi,\varphi)| = C(\sigma,\varepsilon)\exp(-2l(\pi-\varepsilon)t) + o(1),$$

where $C(\sigma,\varepsilon) = \int_{C_3} \frac{4\pi|z\sin z|^2|dz|}{|z^2-\pi^2|^2|\sigma^2\sin^2 z - 2\sigma z\sin z\cos z + z^2|}$.

References

[E] P. Exner, A solvable model of two-channel scattering, preprint.

[Ku] S.T. Kuroda, An Introduction to Scattering Theory, Aarhus University, Lect. Note ser. No. 51.

[O] A.Orth, Quantum mechanical resonance and limiting absorption: The many body problem, Comm. Math. Phys. **126**, (1990), 559-573.

参加者名簿

愛木 豊彦	長崎総合科学大
足立 匡義	東大 理
飯田 雅人	阪大 理
石井 克幸	神戸商船大
石村 直之	東大 理
伊藤 昭夫	千葉大 教育
伊藤 一男	九州大 工
井上 弘	足利工大
伊村 恵美	広島大 理
上野 正嗣	福岡大 理
大谷 光春	早大 理工
大西 勇	東大 理
岡 裕和	早大 理工
小川 卓克	名大 理
小澤 徹	京大 数理研
梶木屋 龍治	新潟大 工
加藤 圭一	阪大 理
加藤 俊直	東北大 理
角谷 敦	千葉大 自然
壁谷 喜継	神戸大 自然
川口 謙一	阪大 理
川中子 正	阪大 理
黒木場 正城	福岡大 理
桑村 雅隆	広島大 理
剱持 信幸	千葉大 教育
後藤 俊一	九州大 工
小林 良和	新潟大 工
小山 哲也	広島工業大
佐藤 得志	東北大 理
佐藤 直紀	千葉大 自然
塩澤 秀之	早大 理工
篠田 淳一	千葉大 自然
四宮 葉一	松下電工
鄭 震文	阪大 理
白水 淳	千葉大 教育
鈴木 宏昌	広島大 理
鈴木 道治	筑波大 数学
鈴木 龍一	都立航空高専
芹沢 久光	新潟大 教養
高野 健二	千葉大 教育
竹野 茂治	新潟大 自然
田中 祐二	福岡大 理
堤 誉志雄	名大 理
藤 知文	早大 理工
刀根 伸朗	広島大 理
鳥海 暁	早大 教育
長澤 壮之	東北大 教養
中野 史彦	東大 教養
中村 元	松江高専
新倉 保夫	東海産業短大
西岡 一男	阪大 理
野村 泰之	東北大 理
橋本 一夫	広島女学院大
菱田 俊明	早大 理工
平田 均	東大 教養
福田 勇	国士舘大 工
星野 弘喜	福岡大 理
丸尾 健二	阪大 工
峰崎 真由美	早大 理工
六車 史	東大 教養
山代 隆章	金沢大 理
山田 直記	宮崎大 工
山村 隆志	東北大 理
山本 吉孝	京大 理
四ッ谷 晶二	龍谷大 理工
和田 健志	阪大 理
渡辺 一雄	学習院大 理

(以上 67名)

平成3年度発展方程式若手セミナーのプログラム

8月21日（水）

8：00-8：20	石井克幸　（神戸商船大）
	Viscosity solutions of nonlinear elliptic PDEs with implicit obstacle
8：25-8：45	壁谷喜継　（神戸大自然）
	ある種のp-Laplace方程式のR^nにおける解について
8：50-9：10	川口謙一　（阪大理）
	非線型発展方程式の局所解の存在について

8月22日（木）

8：30-8：50	愛木豊彦　（長崎総合科学大）
	ステファン問題について
8：55-9：15	角谷　敦　（千葉大自然）
	最適形状設計問題について
9：20-9：40	篠田淳一　（千葉大自然）
	n次元ステファン問題の周期解について
10：10-11：40	堤　誉志雄（名大理）
	Nonlinear Wave Equations　（特別講演）
1：30-1：50	鄭　震文　（阪大理）
	遅れの関数微分方程式の正規性
1：55-2：15	剱持信幸　（千葉大教育）
	Evolution systems of nonlinear variational inequalities
2：20-2：40	山本吉孝　（京大理）
	抽象発展方程式のBesov空間におけるcoercivenessについて

2：45 - 3：05	岡　裕和　（早大理工）
	Integrated semigroupに対する平均エルゴード定理について
3：50 - 4：10	小澤　徹　（京大数理研）
	Nonlinear theory of long range scattering
4：15 - 4：35	平田　均　（東大教養）
	Hartree - type Schrödinger 方程式の H^1 - blow upする初期値について
4：40 - 5：00	鈴木道治　（筑波大数学）
	Hypoellipticity for a class of degenerate elliptic operators of second order
5：05 - 5：25	渡辺一雄　（学習院大理）
	一次元２階微分作用素の resonance
5：30 - 5：50	小山哲也　（広島工大）
	Memory のある外力項をもつ熱方程式について
8：00 - 9：30	修論途中経過発表

8月２３日（金）

8：30 - 8：50	黒木場正城（福岡大理）
	On solutions of some quasilinear hyperbolic equations
8：55 - 9：15	石村直之　（東大理）
	Limit shape of the section of shrinking doughnuts
9：20 - 9：40	伊藤一男　（九州大工）
	積分項のついたBurgers型方程式について
10：10 - 10：40	堤　誉志雄（名大理）
	Nonlinear Wave Equations　（特別講演）

8月24日（土）

8：30-8：50	桑村雅隆　（広島大理）
	集中効果を持った反応拡散方程式系の解の挙動について
8：55-9：15	菱田俊明　（早大理工）
	3次元外部熱対流方程式の熱伝導解の安定性
9：20-9：40	小川卓克　（名大理）
	2次元 Navier - Stokes流の外部問題の強解の減衰について

あとがき

発展方程式若手セミナーは"発展方程式およびその周辺分野の将来の方向を探るための若手研究者の討論と情報交換の場"という趣旨で１９７９年夏に始まりました。この趣旨は現在に至るまで受け継がれてきました。今年度も、全参加者の半数以上を占める大学院生を含む若手研究者が中心となって大変活気に満ちあふれた研究交流の場となりました。講演内容は、発展方程式論および偏微分方程式論の多くの分野にわたりました。特に、特別講演として、名古屋大学理学部の堤誉志雄氏は双曲型発展方程式の理論を大変興味深く、かつわかりやすく解説して下さいました。

この報告集が発展方程式論およびその周辺分野に少しでも貢献することを願っております。

最後に、このセミナーに関係されたすべての方にお世話になったことを感謝致します。特に、龍谷大学科学技術共同研究センターの山口昌哉、四ツ谷晶二両先生、新潟大学の梶木屋龍治氏、さらに田辺広城、丸尾健二両先生、飯田雅人氏をはじめ阪大関係者には多くの面で並々ならぬご協力をいただきました事を深く感謝申し上げます。

１９９１年　12月

第１３回発展方程式若手セミナー幹事

大阪大学理学部　　川中子 正

作用素 $\frac{d}{dt}+A$ の Besov 空間における coerciveness について

京大理　山本吉孝

$(E, |\cdot|)$ を Banach 空間、A を E における線型閉作用素とする。区間 $(0,T)$ 上の E 値関数に作用する作用素 $L = \frac{d}{dt}+A$ の放物型性を、L の E値 Besov 空間への作用で特徴づける。

§ Besov 空間　$B^{\theta}_{p,q}(0,T;E)$, $\overset{\circ}{B}{}^{\theta}_{p,q}(0,T;E)$

$0<\theta<\infty$, $1\le p\le\infty$, $1\le q\le\infty$, $0<T<\infty$

とする。$m=[\theta]+1$ とおき、E値 Bochner 可測関数 f に対してセミノルム

$$[f]_{B^{\theta}_{p,q}(0,T;E)} = \left\| \left| \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(t+\tfrac{j}{m}h) \right|_{L^p_t(0,T-h;E)} h^{-\theta} \right|_{L^q_*(0,\infty)}$$

を導入する。ここに、$L^q_*(0,T)$ は測度 $\frac{dh}{h}$ に関する q 乗可積分関数の空間である。

Definition

1. $f \in B^{\theta}_{p,q}(0,T;E)$

$\Leftrightarrow$　$f \in L^p(0,T;E)$,　$[f]_{B^{\theta}_{p,q}(0,T;E)} < \infty$

2. $f \in \overset{\circ}{B}{}^{\theta}_{p,q}(0,T;E)$

$\Leftrightarrow$　$f \in B^{\theta}_{p,q}(0,T;E)$,　$\left\| |f|_{L^p(0,h;E)} h^{-\theta} \right|_{L^q_*(0,T)} < \infty$

$B^{\theta}_{p,q}(0,T;E)$, $\overset{\circ}{B}{}^{\theta}_{p,q}(0,T;E)$ はそれぞれ

$$|f|_{B^{\theta}_{p,q}(0,T;E)} = |f|_{L^p(0,T;E)} + [f]_{B^{\theta}_{p,q}(0,T;E)}$$

$$|f|_{\overset{\circ}{B}{}^{\theta}_{p,q}(0,T;E)} = \left\| |f|_{L^p(0,h;E)} h^{-\theta} \right|_{L^q_*(0,T)} + [f]_{B^{\theta}_{p,q}(0,T;E)}$$

をノルムとして Banach 空間となる。 ┘

Remark. $\theta = m + \sigma$, $m = 0, 1, 2, \cdots$, $0 < \sigma \leq 1$ と表すとき.

1、 $f \in \overset{\circ}{B}{}^{\theta}_{p,q}(0,T;E)$

$$\Leftrightarrow \frac{d^k}{dt^k} f \in \overset{\circ}{B}{}^{\theta}_{p,q}(0,T;E), \quad 0 \leq k \leq m$$

$$\Leftrightarrow \frac{d^m}{dt^m} f \in \overset{\circ}{B}{}^{\sigma}_{p,q}(0,T;E), \quad f(0) = \cdots = \frac{d^{m-1}}{dt^{m-1}} f(0) = 0$$

2、 $0 < \sigma \leq 1$ のとき

$$\overset{\circ}{B}{}^{\sigma}_{p,q}(0,T;E) = \begin{cases} B^{\sigma}_{p,q}(0,T;E), & \text{when } 0 < \sigma < \frac{1}{p} \\ \{ f \in B^{\sigma}_{p,q}(0,T;E) ; f(0) = 0 \}, & \text{when } \frac{1}{p} < \sigma \leq 1 \end{cases}$$

$$\overset{\circ}{B}{}^{\sigma}_{p,q}(0,T;E) \subsetneq B^{\sigma}_{p,q}(0,T;E), \quad \text{when } \sigma = \frac{1}{p}$$

┘

§ 結果

A を E における閉作用素とする。A の定義域 $\mathcal{D}(A)$ には

グラフノルム を入れておく。

$0 < \theta < \infty$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $0 < T < \infty$ とし.

$\overset{\circ}{B}{}^{1+\theta}_{p,q}(0,T;E) \cap \overset{\circ}{B}{}^{\theta}_{p,q}(0,T;\mathcal{D}(A))$ から $\overset{\circ}{B}{}^{\theta}_{p,q}(0,T;E)$ への

連続線型作用素 $L = L(A, \theta, p, q, T)$ を

$$Lu = \frac{d}{dt} u + Au$$

で与える。

Remark、 $\theta - \frac{1}{p}$ が整数ではないとき、作用素方程式

$$Lu = f$$

を解くことは、$([\theta - \frac{1}{p}] + 1)$ 次の compatibity relation を

満足する $x \in E$, $g \in B^{\theta}_{p,q}(0,T;E)$ に対して方程式

$$\begin{cases} \frac{d}{dt}v + Av = g, & 0 < t < T \\ v(0) = x \end{cases}$$

を解のクラス $B^{1+\theta}_{p,q}(0,T;E) \cap B^{\theta}_{p,q}(0,T;\mathcal{D}(A))$ で解くことと同等である。

ここに、$x \in E$, $g \in B^{\theta}_{p,q}(0,T;E)$ が $([\theta-\frac{1}{p}]+1)$ 次の compatibity relation を満足するとは

$$F^{\sigma}_{p,q} = \left\{ \frac{d}{dt}u(0) \; ; \; u \in B^{1+\sigma}_{p,q}(0,T;E) \cap B^{\sigma}_{p,q}(0,T;\mathcal{D}(A)) \right\}, \quad \frac{1}{p} < \sigma < 1+\frac{1}{p}$$

と定めるとき、

$$v_0 \equiv x \in \mathcal{D}(A)$$

$$v_1 \equiv \frac{d}{dt}g(0) - Av_0 \in \mathcal{D}(A)$$

$$\cdots\cdots$$

$$v_{[\theta-\frac{1}{p}]} \equiv \frac{d^{[\theta-\frac{1}{p}]-1}}{dt^{[\theta-\frac{1}{p}]-1}} g(0) - Av_{[\theta-\frac{1}{p}]-1} \in \mathcal{D}(A)$$

$$v_{[\theta-\frac{1}{p}]+1} \equiv \frac{d^{[\theta-\frac{1}{p}]}}{dt^{[\theta-\frac{1}{p}]}} g(0) - Av_{[\theta-\frac{1}{p}]} \in F^{\theta-[\theta-\frac{1}{p}]}_{p,q}$$

となることである。（$[\theta-\frac{1}{p}] = -1$ のときは $v_0 \equiv x \in F^{\theta-[\theta-\frac{1}{p}]}_{p,q}$ と解釈する。）┘

Theorem 、 E における閉作用素 A に関する次の3条件は互いに同値である。

(1) すべての (θ, p, q, T) について作用素 L は isomorphism である。

(2) ある (θ, p, q, T) について作用素 L は isomorphism である。

(3) $\exists \omega \in \mathbb{R}$, $\exists M \geq 1$ such that

レゾルベント集合 $\rho(-A) \supset \{\lambda \in \mathbb{C} \; ; \; \mathrm{Re}\lambda > \omega\}$

$$\|(\lambda + A)^{-1}\| \leq M|\lambda - \omega|^{-1}, \quad \mathrm{Re}\lambda > \omega$$

┘

ここに、‖·‖ は作用素ノルムである。(3) は $-A$ が exponentially bounded analytic semigroup を生成することに他ならない。

(3) ⇒ (1) は、G. Da Prato - P. Grisvard (Jour. Math. Pures Appl. vol. 54 (1975)) による。

(1) ⇒ (2) は自明。

(2) ⇒ (3) の要点を以下述べる。非負整数 ℓ を

$$t^{\ell} \in \mathring{B}^{\theta}_{p,q}(0,T)$$

と選び、$x \in E$ に対して

$$u_{-\ell}(t:x) \in \mathring{B}^{1+\theta}_{p,q}(0,T;E) \cap \mathring{B}^{\theta}_{p,q}(0,T;\mathcal{D}(A))$$

を方程式

$$(Lu)(t) = \frac{1}{\ell!} t^{\ell} x$$

の解として定義する。さらに

$$S(t)x = \frac{d^{\ell+1}}{dt^{\ell+1}} u_{-\ell}(t:x)$$

と定義する。$\{S(t)\}$ が $-A$ を生成作用素とする解析的半群であることを示す。

$\mathring{B}^{1+\theta}_{p,q}(0,T;E) \cap \mathring{B}^{\theta}_{p,q}(0,T;\mathcal{D}(A))$ の列 $\{u_k(t:x) ; k \geq -\ell\}$ を方程式

$$L u_{k+1}(t:x) = \frac{d}{dt} u_k(t:x)$$

の解として帰納的に定めると、表現

$$\frac{1}{(n+\ell)!} t^{n+\ell} \frac{d^n}{dt^n} S(t)x = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{d}{dt} u_k(t:x), \quad n = 0, 1, 2, \cdots$$

が得られることがポイントとなる。　　　　以上.

平成４年２月１日

発行　龍谷大学科学技術共同研究センター

〒520-21　大津市瀬田大江町横谷１－５

TEL (0775)43-5111 (内742)　FAX (0775)43-7771