第１２回

発展方程式若手セミナー

報告集

１９９１年４月

序

第１２回発展方程式若手セミナーは、平成２年度科学研究費補助金（総合A．研究代表者，東北大学・加藤順二教授，研究課題「微分方程式の総合的研究」）の援助のもとに、平成２年８月１７日より８月２０日までの４日間、広島大学西条共同研修センターで行われました。

本報告集は、この研究集会で研究成果を発表された方々に講演内容の概要を書いていただいたものをまとめたものです。

このセミナーは、「発展方程式の将来の方向を探る若手研究者の勉強会」という趣旨で１９７９年夏に始まりました。今回も全国から多数の参加者を得て、盛況のうちに無事終えることができました。講演のテーマは、半群論、発展方程式論から種々の偏微分方程式、数理物理の問題など多岐にわたり、連日活発な討論と意見交換が行われました。

この報告集が発展方程式及び周辺分野に少しでも貢献できますように、またこのセミナーが若手研究者の交流の場として今後とも引き継がれていきますように願っております。

おわりに、この研究集会の開催にあたってお骨折りいただいた関係者の方々、特に大阪大学・丸尾健二先生、千葉大学・剣持信幸先生、佐賀大学・久保雅弘さんに厚くお礼を申し上げます。

１９９１年4月

第１２回発展方程式若手セミナー幹事

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Stationary Problem for the Navier-Stokes Equations in Exterior Domains.

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Introduction.

Let Ω be an exterior domain in $R^n (n \geq 2)$, i.e., a domain having a compact complement R^n/Ω, and assume that the boundary $\partial\Omega$ is of class $C^{2+\mu}$ with $0 < \mu < 1$. Consider the following boundary value problem for the Navier-Stokes equations in Ω:

$$\text{(N-S)} \qquad \begin{aligned} &-\Delta u + u\cdot\nabla u + \nabla p = f \quad \text{in} \quad \Omega, \\ &\operatorname{div} u = 0 \quad \text{in} \quad \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $u = (u_1(x), \cdots, u_n(x))$ and $p = p(x)$ denote the unknown velocity and pressure, respectively; $f = (f_1(x), \cdots, f_n(x))$ denotes the given external force.

The purpose of this report is to give global L^q-bounds and a uniqueness criterion for the weak solutions of (N-S). On account of the nonlinear term $u\cdot\nabla u$, we need a certain density property for solenoidal vector fields not only in L^q but also in the intersection $L^q \cap L^r$ for $1 < q, r < \infty$, because we have the different behavior at infinity of Δu and $u\cdot\nabla u$ for weak solutions u of (N-S).

First we shall prove regularity at infinity of weak solutions u and its associated pressure p of (N-S). To this end, the same problem on the linearlized equations of (N-S), i.e., the Stokes equations will be also investigated. For bounded domains, Cattabriga [7] showed the most general result in L^q on the Stokes equations. Our result (Lemma 2.5) clarifies a typical difference between interior and exterior problems. When $n = 3$, Fujita [9] gave an explicit representation formula of weak solutions of (N-S) for smooth f decreasing rapidly at infinity, which seems to give a similar application to ours. However, our method enables us to treat a much wider class of f. Our second result is on a uniqueness criterion for weak solutions of (N-S). This criterion in the stationary problem seems to be closely related to that of Serrin's [20] in non-stationary case.

1. Results.

Before stating our results we introduce some notations. For $1 < q < \infty$, $q' = q/(q-1)$. $\| \ \|_q$ and $(\cdot, \ \cdot)$ denote the usual norm of $L^q(\Omega)$ and the inner product between $L^q(\Omega)$ and $L^{q'}(\Omega)$, respectively. $\hat{H}_o^{1,q}(\Omega)$ is the completion of $C_o^\infty(\Omega)$ with respect to the norm $\|\nabla u\|_q$. Since Ω is an exterior domain, $\hat{H}_o^{1,q}(\Omega)$ is larger than $H_o^{1,q}(\Omega)$. $\hat{H}^{-1,q}(\Omega) := \hat{H}_o^{1,q'}(\Omega)^*$($X^*$; dual space of X). $\| \ \|_{-1,q}$ denotes the norm of $\hat{H}^{-1,q}(\Omega)$ defined by $\|f\|_{-1,q} := \sup\{ |\langle f, \ \Phi\rangle| / \|\nabla\Phi\|_{q'} ; \ \Phi \in C_o^\infty(\Omega), \ \Phi \neq 0\}$, where $\langle\cdot,\cdot\rangle$ is the duality paring of $\hat{H}^{-1,q}(\Omega)$ and $\hat{H}_o^{1,q'}(\Omega)^*$. $C_o^\infty(\Omega)^n$, $L^q(\Omega)^n, \cdots$, and $C_o^\infty(\Omega)^{n^2}$, $L^q(\Omega)^{n^2}, \cdots$ denote the corresponding spaces for the vector-valued and the matrix-valued functions, respectively. In such spaces, we shall also use the same notations $\| \ \|_q$ and $(\cdot, \ \cdot)$. $C_{o,\sigma}^\infty(\Omega)$ is the set of all C^∞-vector functions $\Phi = (\Phi_1, \cdots, \Phi_n)$ such that $\operatorname{div} \Phi = 0$.

By the Sobolev inequality, the homogeneous Sobolev space $\hat{H}_o^{1,q}(\Omega)$ can be chracterized concretly as follows.

For $1 < q < n$, we have

$$(1.1) \quad \hat{H}_o^{1,q}(\Omega) = \{u \in L^{nq/(n-q)}(\Omega); \ \nabla u \in L^q(\Omega)^n, \ u\big|_{\partial\Omega} = 0\}.$$

For $n \leq q < \infty$, we have

$$(1.2) \quad \hat{H}_o^{1,q}(\Omega) = \{u \in L^q_{loc}(\bar{\Omega}); \ \nabla u \in L^q(\Omega)^n, \ u\big|_{\partial\Omega} = 0\}.$$

Moreover, we have the following assertions on a density property for solenoidal vector fields.

PROPOSITION 1. Let $\hat{X}^q_\sigma(\Omega) \equiv \{u \in \hat{H}^{1,q}_o(\Omega)^n;\ \mathrm{div}\, u = 0\}$. Then for all $1 < q < \infty$, $1 < r < \infty$, $C^\infty_{o,\sigma}(\Omega)$ is dense in $\hat{X}^q_\sigma(\Omega) \cap \hat{X}^r_\sigma(\Omega)$ under the norm $\|\nabla u\|_q + \|\nabla u\|_r$.

PROPOSITION 2. Let q and r satisfy the following cases (i) or (ii):

(i) $1 < q < n$ and $1 < r < \infty$;

(ii) $n \le q < r < \infty$.

Then $C^\infty_{o,\sigma}(\Omega)$ is dense $\hat{X}^q_\sigma(\Omega) \cap L^r(\Omega)^n$.

For the proof, see Kozono-Sohr [16].

REMARKS 1. In case $q = r = 2$, Heywood [14] showed the same result of Proposition 1.

2. When Ω is the whole space R^n or a bounded domain, Masuda [17] and Giga [12] proved that $C^\infty_{o,\sigma}(\Omega)$ is dense in $H^{1,2}_{o,\sigma}(\Omega) \cap L^r(\Omega)^n$, where $H^{1,2}_{o,\sigma}(\Omega)$ is the closure of $C^\infty_{o,\sigma}(\Omega)$ in $H^{1,2}(\Omega)$. In Remark after the proof of Giga [12, p.210], he gave a conjecture that one can prove the same result even in unbounded domains.

Our definition of a weak solution of (N-S) is as follows.

DEFINITION. Let $f \in \hat{H}^{-1,2}(\Omega)^n$. Then a measurable function u on Ω is called a weak solution of (N-S) if

(i) $u \in \hat{X}^2_\sigma(\Omega)$;

(ii) $(\nabla u, \nabla\Phi) + (u\cdot\nabla u, \Phi) = \langle f, \Phi\rangle$ for all $\Phi \in C^\infty_{o,\sigma}(\Omega)$.

Concerning the existence of weak solutions, see, e.g., Temam [24, p.169, Theorem 1.4]. For every weak solution u of (N-S) there is a scalar function $p \in L^1_{loc}(\bar{\Omega})$, unique up to an additive constant, such that

$$(\nabla u, \nabla \Psi) + (u \cdot \nabla u, \Psi) - (p, \operatorname{div} \Psi) = \langle f, \Psi \rangle$$

holds for all $\Psi \in C_0^\infty(\Omega)^n$. This means that the pair {u, p} satisfies (N-S) in the sense of distributions. We call such p <u>the pressure associated with</u> u (see Fujita [9, Definition 2.3]).

Our result on regularity of weak solutions of (N-S) reads:

THEOREM A. (1)(<u>associated pressure</u>) <u>Let</u> $n \geq 3$ <u>and</u> $f \in \hat{H}^{-1,2}(\Omega)^n$. <u>Suppose that</u> u <u>is a weak solution of</u> (N-S). <u>Then the pressure</u> p <u>associated with</u> u <u>can be chosen in the class</u> $p \in L^2(\Omega) + L^{n/(n-2)}(\Omega)$.

(2)(<u>more regularity</u>)(i) <u>Let</u> $n = 3$ <u>and</u> $f \in \hat{H}^{-1,2}(\Omega)^3 \cap \hat{H}^{-1,q}(\Omega)^3$ <u>for</u> $3 \leq q < \infty$. <u>Suppose that</u> u <u>is a weak solution of</u> (N-S) <u>and</u> <u>that</u> p <u>is the pressure associated with</u> u. <u>Then we have</u>

$$\nabla u \in L^r(\Omega)^{3^2} \quad \text{for} \quad 2 \leq r \leq q, \quad u \in L^s(\Omega)^3 \quad \text{for} \quad 6 \leq s < \infty,$$
$$p \in L^q(\Omega).$$

<u>In particular, if</u> $q > 3$, <u>we have also</u> $u \in L^\infty(\Omega)^3$.

(ii) <u>Let</u> $n \geq 5$ <u>and</u> $f \in \hat{H}^{-1,2}(\Omega)^n \cap \hat{H}^{-1,q}(\Omega)^n$ <u>for</u> $n/(n-1) < q \leq n/(n-2)$. <u>Let</u> u <u>and</u> p <u>be as above</u>. <u>Then it holds</u>

$$\nabla u \in L^r(\Omega)^{n^2} \quad \text{for} \quad q \leq r \leq 2,$$
$$u \in L^s(\Omega)^3 \quad \text{for} \quad nq/(n-q) \leq s \leq 2n/(n-2), \quad p \in L^q(\Omega).$$

Next we shall proceed to the uniqueness criterion for the weak solutions of (N-S).

THEOREM B. Let $n \geq 3$ and $f \in \hat{H}^{-1,2}(\Omega)^n$. Let u and v be weak solutions of (N-S). Suppose also that u satisfies the energy inequality

$$\text{(E.I.)} \qquad \|\nabla u\|_2^2 \leq \langle f, u \rangle$$

and that $v \in L^n(\Omega)^n$. Then there is a positive constant λ such that if $\|v\|_n \leq \lambda$, we have $u \equiv v$ in Ω.

REMARKS. 1. If Ω is a bounded domain in R^n with $n \leq 4$, then every weak solution u belongs to $L^n(\Omega)^n$ and satisfies the energy equality $\|\nabla u\|_2^2 = \langle f, u \rangle$. Hence in such a case, we have $u \equiv v$ under the assumption that $\|f\|_{-1,2}$ is sufficiently small(see Temam [24, p.167,Theorem 1.3]).

2. In the non-stationary Navier-Stokes equations, Sohr-von Wahl [23] and Masuda [17] improved Serrin's uniqueness criterion [20] for the weak solutions on $\Omega\times(0, T)$ in the spaces $C([0, T]; L^n(\Omega)^n)$ and $L^\infty(0, T; L^n(\Omega)^n)$, respectively. So Theorem B may be regarded as the similar criterion of Serrin's [20] for the stationary problem.

2. Preliminaries.

2.1. First we consider the boundary-value problem of the equation:

$$(2.1) \qquad \operatorname{div} u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

The following lemma is essentially due to Bogovski [4]; for the special formulation and extension, see Borchers-Sohr [6].

LEMMA 2.1. *Let* $1 < q < \infty$. (i) *There is a bounded operator* $f \to u$ *from* $L^q(\Omega)$ *to* $\hat{H}_o^{1,q}(\Omega)^n$ *such that* $\operatorname{div} u = f$.

Using the well-known closed range theorem and the Sobolev inequality, we obtain immediately from this lemma the following result.

COROLLARY 2.2. (i) *Let* $1 < q < \infty$ *and let* $\hat{X}_\sigma^q(\Omega) = \{u \in \hat{H}_o^{1,q}(\Omega)^n;\ \operatorname{div} u = 0\}$. *Suppose that* $f \in \hat{H}^{-1,q'}(\Omega)^n$ *satisfies* $\langle f, u\rangle = 0$ *for all* $u \in \hat{X}_\sigma^q(\Omega)$. *Then there is a unique* $p \in L^{q'}(\Omega)$ *such that* $f = \nabla p$, i.e., $\langle f, \Phi\rangle = -(p, \operatorname{div} \Phi)$ *for all* $\Phi \in \hat{H}_o^{1,q}(\Omega)^n$ *and that* $\|p\|_{q'} \le C\|f\|_{-1,q'}$ *with* C *independent of* f.

(ii) *Let* $1 < q < n$ *and let* $u \in L^1_{loc}(\bar{\Omega})$ *with* $\nabla u \in L^q(\Omega)^n$. *Then there is a constant* K_u *such that* $u + K_u \in L^{q^*}(\Omega)$ *with* $1/q^* = 1/q - 1/n$ *and* $\|u + K_u\|_{q^*} \le C\|\nabla u\|_q$ *with* C *independent of* u. *Here* $\bar{\Omega}$ *is the closure of* Ω *and* $u \in L^1_{loc}(\bar{\Omega})$ *means that* $u \in L^1(\Omega \cap B)$ *for all balls* $B \subset R^n$ *with* $\Omega \cap B \neq \Phi$.

For the proof, see Giga-Sohr [13, Corollary 2.2].

REMARK 2.3. Using Corollary 2.2(i), we conclude from Proposition 1 that for each $f \in \hat{H}^{-1,q}(\Omega)^n$ there is a unique $p \in L^q(\Omega)$ satisfying $f = \nabla p$ if and only if $\langle f, \Phi\rangle = 0$ for all $\Phi \in C^\infty_{o,\sigma}(\Omega)$.

The following variational inequality in L^q is simple but plays an important role for our purpose; see also Simader-Sohr [22].

Let $1 < q < \infty$. Then there is a constant $C = C(n, q) > 0$ such that

$$\|\nabla u\|_q \le C\sup\{|(\nabla u, \nabla\Phi)|/\|\nabla\Phi\|_{q'} ; 0 \ne \Phi \in C^\infty_o(R^n)\} \tag{2.2}$$

holds for all $u \in L^q_{loc}(R^n)$ with $\nabla u \in L^q(R^n)$.

Indeed, note that the space $H \equiv \{\Delta\Psi; \Psi \in C^\infty_o(R^n)\}$ is dense in $L^{q'}(R^n)$. Then using the Calderon-Zygmund inequality $\|\nabla\nabla\Psi\|_{q'} \le C\|\Delta\Psi\|_{q'}$ $(\Psi \in C^\infty_o(R^n))$, we have for each $i = 1,\cdots, n$

$$\begin{aligned}
&\sup\{|(\nabla u, \nabla\Phi)|/\|\nabla\Phi\|_{q'} ; \Phi \in C^\infty_o(R^n), \Phi \ne 0\}\\
&\ge \sup\{|(\nabla u, \nabla(\partial_i\Psi))|/\|\nabla(\partial_i\Psi)\|_{q'} ; \Psi \in C^\infty_o(R^n), \Psi \ne 0\}\\
&\ge C\sup\{|(\partial_i u, \Delta\Psi)|/\|\Delta\Psi\|_{q'} ; \Psi \in C^\infty_o(R^n), \Psi \ne 0\}\\
&= C\sup\{|(\partial_i u, g)|/\|g\|_{q'} ; g \in L^{q'}(R^n), g \ne 0\}\\
&= C\|\partial_i u\|_q
\end{aligned}$$

with $C = C(n, q)$ and (2.2) follows.

Let $L^{1,q} \equiv \{u \in L^q_{loc}(R^n);\ \nabla u \in L^q(R^n)^n\}$. For $u \in L^{1,q}$ we denote by $[u]$ the set of all $v \in L^{1,q}$ such that $u - v$ is constant in R^n and define the space $L^{1,q}/R = \{[u];\ u \in L^{1,q}\}$ with norm $\|[u]\|_{L^{1,q}/R} := \|\nabla u\|_q$. Clearly $L^{1,q}/R$ is isometric to the space $G_q := \{\nabla u;\ [u] \in L^{1,q}/R\} (\subset L^q(R^n)^n)$. By the theory of Helmholtz decomposition(see Simader-Sohr [22] and Miyakawa [18]), G_q is a <u>closed subspace</u> in $L^q(R^n)^n$. Therefore, G_q is a <u>reflexive</u> Banach space. Let us consider a linear operator B_q: $\nabla u \in G_q \to B_q(\nabla u) \in G^*_{q'}$, defined by

$$\langle B_q(\nabla u), \nabla v\rangle = (\nabla u, \nabla v) \quad \text{for} \quad \nabla v \in G_{q'},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between $G^*_{q'}$ and $G_{q'}$. Then by (2.2) we see that B_q is injective and that its range is closed in $G^*_{q'}$. Since $B^*_q = B_{q'}$ (T^*; adjoint operator of T), it follows from the closed range theorem that B_q is surjective and hence bijective. For the proof of solvability of the Stokes equations in R^n, we shall make fully use of the bijectivity of the map B_q: $G_q \to G^*_{q'}$.

2.3. Let us consider the Stokes equations:

$$(2.3) \qquad -\Delta u + \nabla p = f, \quad \text{div}\ u = 0 \quad \text{in} \quad R^n$$

Recall that $\hat{X}^q_\sigma(R^n) = \{u \in \hat{H}^{1,q}_0(R^n)^n;\ \text{div}\ u = 0\}$. Then we have

LEMMA 2.4. <u>Let</u> $1 < q < \infty$, $1 < r < \infty$. <u>For every</u> $f \in \hat{H}^{-1,q}(R^n)^n \cap \hat{H}^{-1,r}(R^n)^n$, <u>there is a unique pair</u> $\{u, p\}$ <u>with</u> $u \in \hat{X}^q_\sigma(R^n) \cap \hat{X}^r_\sigma(R^n)$ and $p \in L^q(R^n) \cap L^r(R^n)$ <u>such that</u>

$$(2.4) \qquad (\nabla u, \nabla\Psi) - (p, \operatorname{div} \Psi) = \langle f, \Psi\rangle$$

for all $\Psi \in C_0^\infty(R^n)^n$. <u>Such</u> $\{u, p\}$ <u>is subject to the inequality</u>

$$(2.5) \qquad \|\nabla u\|_q + \|\nabla u\|_r + \|p\|_q + \|p\|_r \le C(\|f\|_{-1,q} + \|f\|_{-1,r}),$$

<u>where</u> $C = C(n, q, r)$.

<u>Proof</u>. By the definition of the space $\hat{H}_0^{1,q'}(R^n)$, we see that the operator $-\nabla$: $\hat{H}_0^{1,q'}(R^n) \to L^{q'}(R^n)^n$ is injective and has a closed range. Hence by the closed range theorem, the adjoint operator $\operatorname{div} = (-\nabla)^*$: $L^q(R^n)^n \to \hat{H}^{-1,q}(R^n)$ is surjective. Since the null space Ker(div) of div is a closed subspace in $L^q(R^n)^n$, for each $h \in \hat{H}^{-1,q}(R^n)$, there is at least one $u \in L^q(R^n)^n$ such that

$$(2.6) \qquad -(u, \nabla\Phi) = \langle h, \Phi\rangle \text{ for all } \Phi \in C_0^\infty(R^n) \text{ and that}$$
$$\|u\|_q \le C\|h\|_{-1,q}$$

with C independent of h. Let us recall the space G_q and the bijective operator B_q: $G_q \to G_{q'}^*$, in the previous subsection. Since $u \in L^q(R^n)^n$, the map $\nabla\Phi \in G_{q'} \to -(u, \nabla\Phi) \in R$ is an element in $G_{q'}^*$, so we can choose $\pi \in \hat{H}_0^{1,q}(R^n)$ so that

$$(2.7)\qquad (\nabla\pi,\ \nabla\Phi) = \langle B_q(\nabla\pi),\ \nabla\Phi\rangle = -(u,\ \nabla\Phi) = \langle h,\ \Phi\rangle$$

for all $\Phi \in C_0^\infty(R^n)$. By (2.2) such π is uniquely determined by h and so we can define a bounded linear operator S_q: $h \in \hat{H}^{-1,q}(R^n) \to \pi \in \hat{H}_0^{1,q}(R^n)$ by the relation (2.7). If in addition, $h \in \hat{H}^{-1,r}(R^n)$, we have also $\pi \in \hat{H}_0^{1,r}(R^n)$. Indeed, with q replaced by r, we see by the above argument that there is a unique $\eta \in \hat{H}_0^{1,r}(R^n) = L^{1,r}/R$ such that $(\nabla\eta,\ \nabla\Phi) = \langle h,\ \Phi\rangle$ for all $\Phi \in C_0^\infty(R^n)$ and $\|\nabla\eta\|_r \le C\|h\|_{-1,r}$ with $C = C(n,\ r)$ independent of h. Thus we get $\pi \in L^{1,q}$, $\eta \in L^{1,r}$ and $\pi - \eta \in L^1_{loc}(R^n)$ satisfies $\Delta(\pi - \eta) = 0$ in the sense of distributions. Then it follows from Weyl's lemma that $\pi - \eta$ is of class C^∞ and harmonic in R^n; so is $\nabla\pi - \nabla\eta$. Applying the mean value property and the Hölder inequality, we get

$$(2.8)\qquad |\nabla\pi(x) - \nabla\eta(x)| \le C(\|\nabla\pi\|_q |x|^{-n/q} + \|\nabla\eta\|_r |x|^{-n/r})$$

for all $x(\ne 0) \in R^n$ with C independent of x. Then the classical Liouville theorem yields that $\nabla\pi - \nabla\eta \equiv 0$ and hence $\pi - \eta \equiv$ const. in R^n. This shows that $\pi \in L^{1,r}$ and hence $\pi \in \hat{H}_0^{1,q}(R^n)\cap\hat{H}_0^{1,r}(R^n)$ for π in (2.7). From this we conclude now that S: $h \to \pi$ is a bounded operator from $\hat{H}^{-1,q}(R^n)\cap \hat{H}^{-1,r}(R^n)$ to $\hat{H}_0^{1,q}(R^n)\cap\hat{H}_0^{1,r}(R^n)$ with

$$\|\nabla\pi\|_q + \|\nabla\pi\|_r \le C(\|h\|_{-1,q} + \|h\|_{-1,r}),$$

where $C = C(n,\ q,\ r)$ is independent of h.

Using S, we give an explicit formula for the pair $\{u,\ p\}$ of solution in (2.3). For each $f \in \hat{H}^{-1,q}(R^n)^n\cap\ \hat{H}^{-1,r}(R^n)^n$, we define $\{u,\ p\}$ by

$$u = Sf + S(\nabla \mathrm{div}\, Sf), \quad p = -\mathrm{div}\, Sf.$$

Here $Sf = S(f_1, \cdots, f_n) \equiv (Sf_1, \cdots, Sf_n)$ and correspondingly for $S(\nabla \mathrm{div}\, Sf)$. Now it is easy to see that such $\{u, p\}$ satisfies (2.4). To show that $\mathrm{div}\, u = 0$, we observe that

$$(\mathrm{div}\, S(\nabla\psi) + \psi, \Delta\Phi) = 0 \quad \text{for all} \quad \psi \in L^q(R^n), \ \Phi \in C_o^\infty(R^n).$$

Since the space $H = \{\Delta\Phi;\ \Phi \in C_o^\infty(R^n)\}$ is dense in $L^{q'}(R^n)$, the above identity yields that $\mathrm{div}\, S(\nabla\psi) = -\psi$ for all $\psi \in L^q(R^n)$. Then we get $\mathrm{div}\, u = 0$ and see that the above pair $\{u, p\}$ has the desired properties.

Now it remains to show the uniqueness. Let $\{u', p'\}$ with $u' \in \hat{X}_\sigma^q(R^n) \cap \hat{X}_\sigma^r(R^n)$ and $p' \in L^q(R^n) \cap L^r(R^n)$ satisfy (2.4). Then $\bar{u} \equiv u - u'$, $\bar{p} \equiv p - p'$ satisfies (2.4) with $f = 0$. Applying the operator div to both sides of the first equation, we get $\Delta\bar{p} = 0$ in the sense of distributions in R^n. Hence $\bar{p}$ is of class C^∞ and harmonic. Since $\bar{p} \in L^q(R^n) \cap L^r(R^n)$, it follows from the Liouville theorem that $\bar{p} \equiv 0$ in R^n. Therefore $(\nabla\bar{u}, \nabla\psi) = 0$ for all $\psi \in C_o^\infty(R^n)^n$. From (2.2) we obtain $\bar{u} = 0$. This completes the proof. ∎

2.4. In this subsection we show a regularity property at infinity for solutions of the Stokes equations in Ω:

$$
(2.9) \qquad \begin{aligned} & -\Delta u + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in} \quad \Omega, \\ & u = 0 \quad \text{on} \quad \partial\Omega. \end{aligned}
$$

Compared with the case when $\Omega = \mathbf{R}^n$, we have restriction on r in Lemma 2.4. Let us recall the trace theorem for vector functions. Take $R > 0$ so that $B_R \equiv \{x \in \mathbf{R}^n; |x| < R\} \supset \partial\Omega$ and set $\Omega_R \equiv \Omega \cap B_R$, $E^q(\Omega_R) \equiv \{u \in L^q(\Omega_R)^n; \operatorname{div} u \in L^q(\Omega_R)\}$ $(1 < q < \infty)$. Then it follows from Fujiwara-Morimoto [10, Lemma 1] that the boundary value $u\cdot\nu$ of the normal component to $\partial\Omega_R = \partial\Omega\cup\{|x| = R\}$ exists as element belonging to $W^{-1/q,q}(\partial\Omega_R) \equiv W^{1/q,q'}(\partial\Omega_R)^*$ and that the following generalized Stokes formula holds:

$$
(2.10) \qquad (\operatorname{div} u, \Phi)_{\Omega_R} + (u, \nabla\Phi)_{\Omega_R}
$$

$$
= -\langle u\cdot\nu, \Phi|_{\partial\Omega}\rangle_{\partial\Omega} + \langle u\cdot\nu_R, \Phi|_{\partial B_R}\rangle_{\partial B_R} \quad \text{for} \quad \Phi \in W^{1,q'}(\Omega_R).
$$

Here ν and ν_R denote unit outer normals to $\partial\Omega$ and $\partial B_R \equiv \{|x| = R\}$, respectively; $\langle\cdot,\cdot\rangle_{\partial D}$ denotes the duality paring of $W^{-1/q,q}(\partial D)$ and $W^{1/q,q'}(\partial D)$. Moreover, the map $u \in E^q(\Omega_R) \to u\cdot\nu \in W^{-1/q,q}(\partial\Omega_R)$ is surjective. Our regularity result now reads:

LEMMA 2.5. <u>Let</u> $1 < q < \infty$, <u>and</u> $n'(=n/(n-1)) < r < \infty$. <u>Let</u> $f \in \hat{H}^{-1,q}(\Omega)^n \cap \hat{H}^{-1,r}(\Omega)^n$. <u>Suppose that</u> $\{u, p\} \in \hat{X}^q_\sigma(\Omega)\times L^q(\Omega)$ <u>and satisfies</u> (2.9) <u>in the sense of distributions in</u> Ω. <u>Then we have</u> $\nabla u \in L^r(\Omega)^{n^2}$ <u>and</u> $p \in L^r(\Omega)$. <u>In case</u> $1 < q < n$, <u>we have</u>

in particular, $u \in \hat{X}^r_\sigma(\Omega)$. *In case* $n \le r$ *for* $n \ge 3$ *and in case* $2 < r$ *for* $n = 2$, *we have also* $u \in \hat{X}^r_\sigma(\Omega)$.

Proof. *Step* 1. We shall first show the local regularity

(2.11) $$u \in H^{1,r}(\Omega_R)^n, \quad p \in L^r(\Omega_R).$$

This is trivial if $n' < r \le q$. Suppose that $q < r < \infty$. Let us first assume that $1/q - 1/n \le 1/r$. Choose $N(>R)$ sufficiently large and take $\chi \in C_0^\infty(R^n)$ with $0 \le \chi \le 1$, $\chi(x) \equiv 1$ for $|x| \le N$, $\chi(x) \equiv 0$ for $|x| \ge N+1$. From (2.9) we get the following equation on $\Omega_{N+1} = \Omega \cap B_{N+1}$:

(2.12) $$-\Delta(\chi u) + \nabla(\chi p) = \hat{f}, \quad \mathrm{div}\,(\chi u) = \hat{g} \quad \text{in} \quad \Omega_{N+1},$$
$$\chi u = 0 \quad \text{on} \quad \partial\Omega_{N+1},$$

where $\hat{f} = \chi f - 2\nabla\chi\cdot\nabla u - \Delta\chi\cdot u + \nabla\chi\cdot p$, $\hat{g} = \nabla\chi\cdot u$. Since $1/q - 1/n \le 1/r$, by the Sobolev inequality we have the continuous embeddings $L^q(\Omega_{N+1}) \subset H^{-1,r}(\Omega_{N+1})(\equiv H_0^{1,r'}(\Omega_{N+1})^*)$, $H^{1,q}(\Omega_{N+1}) \subset L^r(\Omega_{N+1})$. Hence from the assumption, $\hat{f} \in H^{-1,r}(\Omega_{N+1})^n$ and $\hat{g} \in L^r(\Omega_{N+1})$. Since $\int_{\Omega_{N+1}} \hat{g}\,dx = -\int_{\partial\Omega} u\cdot\nu\,dS + \int_{\partial B_{N+1}} \chi u\cdot\nu\,dS = 0$, it follows from Cattabriga [7] and Kozono-Sohr [15, Proposition 2.10] that $\chi u \in H^{1,r}(\Omega_{N+1})^n$ and $\chi p \in L^r(\Omega_{N+1})$. Since $\chi \equiv 1$ on Ω_N, we obtain (2.11).

We next consider the case $1/q - 2/n \le 1/r < 1/q - 1/n$. From the above argument we have $u \in H^{1,q^*}(\Omega_N)^n$ and $p \in L^{q^*}(\Omega_N)$ with $1/q^* = 1/q - 1/n$. Taking q^* instead of q and then using the same argument as above, we get $u \in H^{1,r}(\Omega_{N-1})^n$ and $p \in$

$L^r(\Omega_{N-1})$ for $r > n'$ with $1/r \geq 1/q - 2/n$. Proceeding in the same way to the case $1/r < 1/q - 2/n$, by the bootstrap argument with finite steps, we get (2.11) for all $r > n'$.

Step 2. Since $f \in \hat{H}^{-1,q}(\Omega)^n \cap \hat{H}^{-1,r}(\Omega)^n$, there is a function $F \in L^q(\Omega)^{n^2} \cap L^r(\Omega)^{n^2}$ such that $f = \operatorname{div} F$, i.e., $\langle f, \Phi\rangle = -(F, \nabla\Phi)$ holds for all $\Phi \in C_0^\infty(\Omega)^n$.

Indeed, $\hat{H}_0^{1,q'}(\Omega) \cap \hat{H}_0^{1,r'}(\Omega)$ is dense in $\hat{H}_0^{1,q'}(\Omega)$ and in $\hat{H}_0^{1,r'}(\Omega)$. Hence $(\hat{H}_0^{1,q'}(\Omega) + \hat{H}_0^{1,r'}(\Omega))^* = \hat{H}^{-1,q}(\Omega) \cap \hat{H}^{-1,r}(\Omega)$ (see Aronszajn-Gagliardo [2, Theorem 8.3]). Consider the bounded operator $-\nabla$: $\hat{H}_0^{1,q'}(\Omega) + \hat{H}_0^{1,r'}(\Omega) \to L^{q'}(\Omega)^n + L^{r'}(\Omega)^n$. Using the closed range theorem for the adjoint operator $\operatorname{div} = (-\nabla)^*$: $L^q(\Omega)^n \cap L^r(\Omega)^n \to \hat{H}^{-1,q}(\Omega) \cap \hat{H}^{-1,r}(\Omega)$ in a similar manner as in (2.6), we get a function $F \in L^q(\Omega)^{n^2} \cap L^r(\Omega)^{n^2}$ with $f = \operatorname{div} F$.

Now the first equation of (2.9) can be rewritten in the following divergence form:

$$\operatorname{div}\,(T(u, p) + F) = 0 \quad \text{in} \quad \Omega, \tag{2.13}$$

where $T(u, p) = \{T_{ij}(u, p)\}_{1\leq i,j\leq n}$; $T_{ij}(u, p) = -\delta_{ij}p + (\partial_i u_j + \partial_j u_i)$. From the assumption and the argument in Step 1, we see $T(u, p) + F \in E^q(\Omega_R)^n \cap E^r(\Omega_R)^n$ and hence we can take $H \in E^q(\Omega_R)^n \cap E^r(\Omega_R)^n$ such that

$$H\cdot\nu\big|_{\partial\Omega} = (T(u, p) + F)\cdot\nu\big|_{\partial\Omega}, \quad H\cdot\nu\big|_{|x|=R} = 0. \tag{2.14}$$

Set $\tilde{H}(x) = H(x)$ for $x \in \Omega_R$, $\tilde{H}(x) = 0$ for $|x| > R$. Then we have $\tilde{H} \in L^q(\Omega)^{n^2} \cap L^r(\Omega)^{n^2}$ with $\operatorname{div} \tilde{H} \in L^q(\Omega)^n \cap L^r(\Omega)^n$. Take

$s \in (1, \infty)$ so that $1/s = 1/r + 1/n$. Then we have also $\tilde{H} \in L^s(\Omega)^{n^2}$ with $\operatorname{div} \tilde{H} \in L^s(\Omega)^n$, since $s < r$ and since $\tilde{H}$ has a compact support. Now it follows from Lemma 2.1(i) that there exists $G \in \hat{H}_0^{1,s}(\Omega)^{n^2}$ such that

$$\operatorname{div} G = \operatorname{div} \tilde{H} \quad \text{in} \quad \Omega. \tag{2.15}$$

By the Sobolev inequality we have also $G \in L^r(\Omega)^{n^2}$. Set $V \equiv F - \tilde{H} + G$. Then $V \in L^r(\Omega)^{n^2}$ and from (2.13)-(2.15) we obtain that

$$(2.16) \qquad \begin{aligned} &\operatorname{div} (T(u, p) + V) = 0 \quad \text{in the sense of distributions on } \Omega, \\ &(T(u, p) + V)\cdot\nu\big|_{\partial\Omega} = 0 \quad \text{in} \quad W^{-1/r,r}(\partial\Omega)^n. \end{aligned}$$

Let us define the function $\tilde{u}$ on R^n by $\tilde{u}(x) = u(x)$ for $x \in \Omega$, $\tilde{u}(x) = 0$ for $x \in R^n/\Omega$. In the same way, we define also $\tilde{p}$ and $\tilde{V}$ on R^n. Clearly $\tilde{u} \in \hat{X}_\sigma^q(R^n)$, $\tilde{p} \in L^q(R^n)$ and $\tilde{V} \in L^r(R^n)^{n^2}$. Moreover, it holds

$$\operatorname{div} (T(\tilde{u}, \tilde{p}) + \tilde{V}) = 0 \tag{2.17}$$

in the sense of distributions on R^n. To see this, we take a function $\eta \in C^\infty(R^n)$ with $0 \le \eta \le 1$ so that $\eta(x) \equiv 0$ near R^n/Ω, $\eta(x) \equiv 1$ for $|x| \ge R$. By the generalized Stokes formula (2.10) and (2.16), we have

$$\begin{aligned} &(T(\tilde{u}, \tilde{p}) + \tilde{V}, \nabla\Phi)_{R^n} \\ &= (T(u, p) + V, \nabla(\eta\Phi))_\Omega + (T(u, p) + V, \nabla(1 - \eta)\Phi))_{\Omega_R} \\ &= - \langle (T(u, p) + V)\cdot\nu\big|_{\partial\Omega}, \Phi\big|_{\partial\Omega}\rangle = 0 \end{aligned}$$

for all $\Phi \in C_0^\infty(R^n)^n$. This implies (2.17).

On the other hand, since $\tilde{V} \in L^r(R^n)^{n^2}$, it follows from Lemma 2.4 that there is a pair $\{u', p'\}$ with $u' \in \hat{X}_\sigma^r(R^n)$ and $p' \in L^r(R^n)$ satisfying $\mathrm{div}\,(T(u', p') + \tilde{V}) = 0$ in the sense of distributions on R^n. Applying the theory of harmonic functions for $\bar{u} \equiv \tilde{u} - u'$ and $\bar{p} = \tilde{p} - p'$ with such an aid of inequality as (2.8), we get as in Lemma 2.4 $\tilde{u} = u'$ and $\tilde{p} = p'$. From this it follows that $\nabla u \in L^r(\Omega)^{n^2}$ and $p \in L^r(\Omega)$.

Now it remains to show that $u \in \hat{H}_0^{1,r}(\Omega)^n$ in case $r \geq n$ $(n \geq 3)$, $r > 2 (n = 2)$ and in case $1 < q < n$. For the former case, by (1.2), we get $u \in \hat{H}_0^{1,r}(\Omega)^n$. Suppose the latter case $1 < q < n$ and $n' < r < n$ $(n \geq 3)$. By the Sobolev inequality, we have $u \in L^{q^*}(\Omega)^n$ for $1/q^* = 1/q - 1/n$. Moreover it follows from Corollary 2.2(ii) that there is a constant vector $M \in R^n$ such that $u + M \in L^{r^*}(\Omega)^n$ for $1/r^* = 1/r - 1/n$. Since $u \in L^{q^*}(\Omega)^n$, we see $M = 0$ and hence $u \in L^{r^*}(\Omega)^n$. Then again by (1.1), we get $u \in \hat{H}_0^{1,r}(\Omega)^n$. This completes the proof of Lemma 2.5. ∎

3. L^q-gradient bounds for the Navier-Stokes equations; Proof of Theorem A.

3.1. Let us first recall some fundamental facts for interpolation couples. For closed subspace X of a Banach space E we denote by $X^{\perp}$ the annihilator of X, i.e., the set of all continuous linear functionals on E vanishing on X. By Corollary 2.2(i), we have

$$(3.1)\quad \hat{X}^q_\sigma(\Omega)^{\perp} = \{f \in \hat{H}^{-1,q'}(\Omega)^n;\ f = \nabla p \ \text{ with } \ p \in L^{q'}(\Omega)\}$$

for $1 < q < \infty (q' = q/(1-q))$. Moreover by Theorem 1, $\hat{X}^q_\sigma(\Omega) \cap \hat{X}^r_\sigma(\Omega)$ is dense in $\hat{X}^q_\sigma(\Omega)$ and $\hat{X}^r_\sigma(\Omega) (1 < q,\ r < \infty)$. Hence it follows from Aronszajn-Gagliardo [2, Theorem 8.3] that

$$(3.2)\qquad (\hat{X}^q_\sigma(\Omega) \cap \hat{X}^r_\sigma(\Omega))^{\perp} = \hat{X}^q_\sigma(\Omega)^{\perp} + \hat{X}^r_\sigma(\Omega)^{\perp}.$$

For L^q-gradient bounds of weak solutions of (N-S), we need the following variational inequality.

LEMMA 3.1. Let $u \in \hat{X}^q_\sigma(\Omega)$ for $1 < q < \infty$. Suppose that

$$\sup\ \{ |(\nabla u,\ \nabla\Phi)| / \|\nabla\Phi\|_{r'}\ ;\ 0 \neq \Phi \in C^{\infty}_{0,\sigma}(\Omega)\} < \infty$$

for some $r > n' (= n/(n-1))$. Then it follows $\nabla u \in L^r(\Omega)^{n^2}$. If in addition $1 < q < n$, we have also $u \in \hat{X}^r_\sigma(\Omega)$.

Proof. Since $C^{\infty}_{0,\sigma}(\Omega)$ is dense in $\hat{X}^{r'}_\sigma(\Omega)$ and since $\hat{X}^{r'}_\sigma(\Omega)$ is a closed subspace of $\hat{H}^{1,r'}_0(\Omega)^n$, it follows from the assumption and the Hahn-Banach theorem that there is a functional $f \in \hat{H}^{-1,r}(\Omega)^n$ such that $(\nabla u,\ \nabla\Phi) = \langle f,\ \Phi\rangle$ holds for all $\Phi \in C^{\infty}_{0,\sigma}(\Omega)$. Now by Proposition 1, $C^{\infty}_{0,\sigma}(\Omega)$ is dense in $\hat{X}^{q'}_\sigma(\Omega)\cap\hat{X}^{r'}_\sigma(\Omega)$ and therefore, from the above identity, we get $<-\Delta u - f, v> = 0$ for all $v \in \hat{X}^{q'}_\sigma(\Omega)\cap\hat{X}^{r'}_\sigma(\Omega)$. Then by (3.1) and (3.2) there are functions $p_1 \in L^q(\Omega)$ and $p_2 \in L^r(\Omega)$ such that $-\Delta u + \nabla p_1 = f + \nabla p_2$ in the sense of distributions on Ω. Since $f + \nabla p_2 \in \hat{H}^{-1,r}(\Omega)^n$ with $r > n'$, by Lemma 2.5 we get the desired result. ∎

We next consider the complex interpolation space $[X, Y]_\theta$, $0 \leq \theta \leq 1$. Note that the norms $\|\nabla u\|_q$ and $\|\nabla u\|_r$ are consistent on $C^\infty_0(\Omega)$ and that the pair $\{\hat{H}^{1,q}_0(\Omega), \hat{H}^{1,r}_0(\Omega)\}$ is an interpolation couple. Moreover from (1.1-2), we get the following concrete characterization(see, e.g., Triebel [25, 1.9]):

If $1 < q < n$, $1 < r < n$ or if $n \leq q < \infty$, $n \leq r < \infty$,

$$[\hat{H}^{1,q}_0(\Omega), \hat{H}^{1,r}_0(\Omega)]_\theta = \hat{H}^{1,s}_0(\Omega),$$

where $1/s = (1 - \theta)/q + \theta/r$, $0 \leq \theta \leq 1$. Applying duality argument[25, 1.11.2], we get

$$[\hat{H}^{-1,q}(\Omega), \hat{H}^{-1,r}(\Omega)]_\theta = \hat{H}^{-1,s}(\Omega) \tag{3.3}$$

for $n' < q < \infty$, $n' < r < \infty$, where $1/s = (1 - \theta)/q + \theta/r$, $0 \leq \theta \leq 1$.

3.2. <u>Completion of the Proof of Theorem</u> A.

(1)<u>Associated Pressure</u>. Since $u \in \hat{X}^2_\sigma(\Omega)$, we get $-\Delta u - f \in \hat{H}^{-1,2}(\Omega)^n$. By the Sobolev inequality we have the continuous embeddings $\hat{H}^{1,2}_0(\Omega)\subset L^{2n/(n-2)}(\Omega)$, $\hat{H}^{1,n/2}_0(\Omega)\subset L^n(\Omega)$, so it follows from the Hölder inequality that

$$|(u\cdot\nabla u,\ \psi)| \le \|u\|_{2n/(n-2)}\|\nabla u\|_2\|\psi\|_n \le C\|\nabla u\|_2^2\|\nabla\psi\|_{n/2}$$

for all $\psi \in \hat{H}_0^{1,n/2}(\Omega)^n$ with C independent of u and ψ. This implies that $u\cdot\nabla u \in \hat{H}^{-1,n/(n-2)}(\Omega)^n$ and hence we get

$$(3.4) \qquad -\Delta u + u\cdot\nabla u - f \in \hat{H}^{-1,2}(\Omega)^n + \hat{H}^{-1,n/(n-2)}(\Omega)^n.$$

On the other hand, by Proposition 1, $C_{0,\sigma}^\infty(\Omega)$ is dense in $\hat{X}_\sigma^2(\Omega)\cap\hat{X}_\sigma^{n/2}(\Omega)$. Now by (3.4) and the definiton of the weak solution of (N-S), we get $-\Delta u + u\cdot\nabla u - f \in (\hat{X}_\sigma^2(\Omega)\cap\hat{X}_\sigma^{n/2}(\Omega))^\perp$. Then it follows from (3.1) and (3.2) that there exist scalar functions $p_1 \in L^2(\Omega)$ and $p_2 \in L^{n/(n-2)}(\Omega)$ such that $-\Delta u + u\cdot\nabla u - f = -\nabla p_1 - \nabla p_2$, which means that

$$(\nabla u,\ \nabla\psi) + (u\cdot\nabla u,\ \psi) - (p_1 + p_2,\ \mathrm{div}\ \psi) = \langle f,\ \psi\rangle$$

for all $\psi \in C_0^\infty(\Omega)^n$. Now we see that $p_1 + p_2 \in L^2(\Omega) + L^{n/(n-2)}(\Omega)$ is the pressure associated with u.

(2) <u>More Regularity</u>. (i) Since $n = 3$, we have by (3.4) that $u\cdot\nabla u \in \hat{H}^{-1,3}(\Omega)^3$ and hence from the assumption on f with the aid of (3.3) it follows that $u\cdot\nabla u - f \in \hat{H}^{-1,3}(\Omega)^3$. Now applying Lemma 3.1, we get $\nabla u \in L^3(\Omega)^{3^2}$. By interpolation, $\nabla u \in L^r(\Omega)^{3^2}$ for $2 \le r \le 3$. Since $u \in L^6(\Omega)^3$, it follows from Corollary 2.2(ii) that $u \in L^s(\Omega)^3$ for all s with $6 \le s < \infty$. Since $2q \ge 6$, we obtain by integration by parts and the Hölder inequality $|(u\cdot\nabla u,\ \psi)| = |(u\cdot\nabla\psi,\ u)| \le \|u\|_{2q}^2\|\nabla\psi\|_{q'}$ for all $\psi \in C_0^\infty(\Omega)^3$, which implies that $u\cdot\nabla u \in \hat{H}^{-1,q}(\Omega)^3$. By

assumption $u\cdot\nabla u - f \in \hat{H}^{-1,q}(\Omega)^3$ and Lemma 3.1 yields, together with interpolation, $\nabla u \in L^r(\Omega)^{3^2}$ for $2 \le r \le q$. Now $-\Delta u + u\cdot\nabla u - f$ belongs to $\hat{H}^{-1,q}(\Omega)^3$ and vanishes on $C^\infty_{o,\sigma}(\Omega)$. By Remark 2.3 the pressure p associated with u can be chosen in the class that $p \in L^q(\Omega)$.

Suppose in particular that $q > 3$. By interpolation we have $u \in \hat{X}^{\tilde{q}}_\sigma(\Omega)\cap L^6(\Omega)^3$ for $3 < \tilde{q} < 6$. Then we have $u \in L^\infty(\Omega)^3$ because it holds

$$\|\Phi\|_\infty \le C\|\nabla\Phi\|^\alpha_{\tilde{q}}\|\Phi\|^{1-\alpha}_6 \quad \text{for all} \quad \Phi \in \hat{X}^{\tilde{q}}_\sigma(\Omega)\cap L^6(\Omega)^3, \tag{3.5}$$

where $\alpha = \tilde{q}/3(\tilde{q} - 2)$. Indeed, from Gagliardo-Nirenberg inequality(see, e.g., Friedman [8, p.24 Theorem 9.4]), we see that (3.5) holds for all $\Phi \in C^\infty_{o,\sigma}(\Omega)$. Now since $C^\infty_{o,\sigma}(\Omega)$ is dense in $\hat{X}^{\tilde{q}}_\sigma(\Omega)\cap L^6(\Omega)^3$(by Proposition 2), we get (3.5) by passage to the limit.

(ii) By (3.3) and the assumption on f, we see as in case (i) that $u\cdot\nabla u - f \in \hat{H}^{-1,n/(n-2)}(\Omega)^n$. It follows from Lemma 3.1 and interpolation that $\nabla u \in L^r(\Omega)^{n^2}$. for $n/(n - 2) \le r \le 2$. Since $u \in L^{2n/(n-2)}(\Omega)^n$, we have by Corollary 2.2(ii) that $u \in L^\gamma(\Omega)^n$ for $n/(n - 3) \le \gamma \le 2n/(n - 2)$, which yields $u\cdot\nabla u \in \hat{H}^{-1,\delta}(\Omega)^n$ for $n/2(n - 3) \le \delta \le n/(n - 2)$. Since $n/2(n - 3) \le n' < q \le n/(n - 2)$, we have in particular $u\cdot\nabla u \in \hat{H}^{-1,q}(\Omega)^n$. Then in the same way as above we have by Lemma 3.1 and Remark 2.3 that $\nabla u \in L^q(\Omega)^{n^2}$ and $p \in L^q(\Omega)$. Now, the assertion follows from interpolation and Corollary 2.2(ii). This completes the proof. ∎

4. Uniqueness for the weak solutions of the stationary Navier-Stokes equations; proof of Theorem B.

As we have seen in the proof of Theorem A(1), it holds $|(u\cdot\nabla u,\ \Phi)| \le C\|\nabla u\|_2^2\|\Phi\|_n$ for all $\Phi \in C^\infty_{o,\sigma}(\Omega)$. By Proposition 2, $C^\infty_{o,\sigma}(\Omega)$ is dense in $\hat{X}^2_\sigma(\Omega)\cap L^n(\Omega)^n$ and hence by passage to the limit we can insert $v \in \hat{X}^2_\sigma(\Omega)\cap L^n(\Omega)^n$ as a test function Φ in the definition of weak solution of (N-S). Since $(v\cdot\nabla v,\ v) = 0$, we obtain

$$(4.1)\qquad (\nabla u,\ \nabla v) + (u\cdot\nabla u,\ v) = \langle f,\ v\rangle,$$

$$(4.2)\qquad \|\nabla v\|_2^2 = \langle f,\ v\rangle.$$

Moreover, we have by the Hölder and the Sobolev inequalities that

$$|(v\cdot\nabla v,\ \Phi)| \le \|v\|_n\|\nabla v\|_2\|\Phi\|_{2n/(n-2)} \le C\|v\|_n\|\nabla v\|_2\|\nabla\Phi\|_2$$

for all $\Phi \in C^\infty_{o,\sigma}(\Omega)$. By Theorem 1, $C^\infty_{o,\sigma}(\Omega)$ is dense in $\hat{X}^2_\sigma(\Omega)$ and hence from the above inequality we can insert $u \in \hat{X}^2_\sigma(\Omega)$ as a test function defining the weak solution $v \in \hat{X}^2_\sigma(\Omega)\cap L^n(\Omega)^n$;

$$(4.3)\qquad (\nabla v,\ \nabla u) + (v\cdot\nabla v,\ u) = \langle f,\ u\rangle.$$

Adding (4.1-3) and (E.I.), we get by integration by parts

$$\begin{aligned}\|\nabla u - \nabla v\|_2^2 &\le (u\cdot\nabla u,\ v) + (v\cdot\nabla v,\ u) = (u\cdot\nabla u,\ v) - (v\cdot\nabla u,\ v)\\ &= ((u - v)\cdot\nabla(u - v),\ v).\end{aligned}$$

Here we used $((u - v)\cdot\nabla v,\ v) = 0$. Letting $w \equiv u - v$ and then using the Hölder inequality and the Sobolev inequality $\|\psi\|_{2n/(n-2)} \le C_* \|\nabla\psi\|_2$ $(\psi \in \hat{H}_o^{1,2}(\Omega))$, we have from above

$$\|\nabla w\|_2^2 \le \|w\|_{2n/(n-2)} \|\nabla w\|_2 \|v\|_n \le C_* \|\nabla w\|_2^2 \|v\|_n .$$

Take $0 < \lambda < C_*^{-1}$. Then under the assumption that $\|v\|_n \le \lambda$, we conclude $\|\nabla w\|_2^2 \le 0$, which implies $u \equiv v$ on Ω. This completes the proof. ∎

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Decaying Rate of Fundamental Solutios for a Linear Volterra Diffusion Equations

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Let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. We consider a linear Volterra integrodifferential equation in $L^p(\Omega)$ with $1 < p < \infty$:

$$\frac{dv}{dt}(t) + Av(t) = \int_0^t g(t-s)v(s)\,ds, \qquad t > 0. \tag{1}$$

In this article we will construct a fundamental solution $R(t)$ of (1), derive a uniform L^p-estimate of $R(t)$ with respect to p (§1), and apply the results to a class of semilinear Volterra diffusion equations treated in [3] (§2).

§1. Main results

Notation and hypotheses. We denote by $|||\cdot|||_p$ the operator norm of the bounded linear operators on $L^p(\Omega)$. For $0 < \gamma < \frac{\pi}{2}$ and $R > 0$, we define a sector excluding a neighborhood of the origin by

$$S_{\gamma,R} = \{z \in \mathbf{C}\ ;\ |z| > R,\ |\arg z| < \frac{\pi}{2} + \gamma\}.$$

For a measurable function $g(t)$ on $[0,\infty)$, $\widehat{g}(z)$ is an analytic continuation of its Laplace transform $\int_0^\infty e^{-zt}g(t)\,dt$, and D_g is a maximal region in which $\widehat{g}(z)$ is a single-valued and holomorphic function. The resolvent set and the spectrum of a linear operator A is denoted by $\rho(A)$ and $\sigma(A)$ respectively.

Impose the following conditions on A and g :

(H_A) A linear operator A in $L^p(\Omega)$ is defined by

$$D(A) = \left\{ u \in W^{2,p}(\Omega) \; ; \; r(x)u + \{1 - r(x)\}\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\},$$
$$Au = -\Delta u + q(x)u \quad \text{for } u \in D(A),$$

where $q(x) \in C^\infty(\bar{\Omega})$ is a real-valued function and the function $r(x) \in C^\infty(\partial\Omega)$ satisfies either

$$(a) \;\; r(x) \equiv 1 \text{ on } \partial\Omega \qquad \text{or} \qquad (b) \;\; 0 \le r(x) < 1 \text{ on } \partial\Omega.$$

(H_g) (i) $g \in L^1(0,\infty)$.

(ii) There exist $M, R > 0$ and $0 < \gamma < \frac{\pi}{2}$ such that

$$(2) \quad \begin{cases} S_{\gamma,R} \subset D_g, \\ |\hat{g}(z)| \le M \quad \text{for } z \in S_{\gamma,R}. \end{cases}$$

It is well known that the operator A has the following properties :

Lemma 1.1. *Under the assumption (H_A), $-A$ generates an analytic semigroup e^{-tA} on $L^p(\Omega)$. Moreover, for some constants $M, R > 0$ and $0 < \gamma < \frac{\pi}{2}$ which are independent of p,*

$$(3) \qquad |||(z + A)^{-1}|||_p \le \frac{M}{|z|}, \qquad z \in S_{\gamma,R},\; 1 < p < \infty$$

holds (see, e.g., [5; Chap. 17]).

Lemma 1.2. *Under the assnption (H_A), the operator A is self-adjoint and bounded from below in $L^2(\Omega)$, the spectrum $\sigma(A)$ is discrete, and a faimily $\phi_j \in C^\infty(\bar{\Omega})$ ($j = 1,2,3,\cdots$) of the corresponding eigenfunctions forms a complete orthonormal system in $L^2(\Omega)$.*

Fundamental solutions. Let $\{R(t)\}_{t\geq 0}$ be a strongly continuous one parameter family of bounded linear operators on $L^p(\Omega)$ and satisfy

$$R(t) = e^{-tA} + \int_0^t e^{-(t-s)A} \int_0^s g(s-r)R(r)\,dr\,ds, \qquad t \geq 0. \tag{4}$$

Then we call $\{R(t)\}_{t\geq 0}$ a fundamental solution of (1).

Theorem 1.3. *If (H_A) and (H_g) are satisfied, the unique fundamental solution $R(t)$ of (1) is explicitly given by*

$$R(t) = \frac{1}{2\pi i}\int_\Gamma e^{tz}(z + A - \widehat{g}(z))^{-1}\,dz, \qquad t > 0. \tag{5}$$

Here Γ is a path running in some region $S_{\gamma,R}$ from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ with $\frac{\pi}{2} < \theta < \frac{\pi}{2} + \gamma$.

Idea of the proof. Take the Laplace transformation for (4) and (5). This formal calculation is justified by

$$(6)\begin{cases} (z + A - \widehat{g}(z))^{-1} = (z+A)^{-1}\{I - \widehat{g}(z)(z+A)^{-1}\}^{-1}, \\ |||(z + A - \widehat{g}(z))^{-1}|||_p \leq \frac{M'}{|z|} \\ \text{for } z \in S_{\gamma,R},\ 1 < p < \infty, \end{cases}$$

with some constants M', R and γ independent of p. It is easy to see (6) from (2) and (3). (cf. [1], [4; Theorem 5.4], [7].) ∎

Setting $P(z) = (z + A - \widehat{g}(z))^{-1}$, we introduce the following sets :

$$\begin{aligned}\rho_0(A,g) &= \{z \in D_g \ ; \ P(z) \text{ is a bounded linear operator on } L^p(\Omega)\} \\ &= \{z \in D_g \ ; \ -z + \widehat{g}(z) \in \rho(A)\}, \\ \rho(A,g) &= \rho_0(A,g) \cup \{z \in \mathbf{C} \ ; \ z \text{ is an isolated removable singularity of } P(\cdot)\}, \\ \sigma(A,g) &= \mathbf{C} \setminus \rho(A,g).\end{aligned}$$

We call $\sigma(A,g)$ the *retarded spectrum* associated with (1). An analytic extension of $P(z)$ on $\rho(A,g)$ is said to be the *retarded resolvent* (cf. [2]). The next lemma is an essential part of our idea. It is derived from Lemma 1.2 and an L^2-L^p estimate for e^{-tA} :

Lemma 1.4. *Impose* (H_A) *and* (H_g)*-(i). Let* $K \subset \rho_0(A,g)$ *be a compact set. Then*

$$\sup\{|||P(z)|||_p \ ; \ z \in K, \ 2 \le p < \infty\} < \infty. \tag{7}$$

For the proof, see [4; Lemma 5.5].

Setting

$$\lambda_{A,g} = \sup_{\lambda \in \sigma(A,g)} \operatorname{Re} \lambda,$$

we obtain the *decaying rate* of $R(t)$ with respect to L^∞-topology :

Theorem 1.5. *Assume* (H_A) *and* (H_g)*. For a given positive number* ε*, the fundamental solution* $R(t)$ *is evaluated like*

$$|||R(t)|||_p \le C e^{(\lambda_{A,g}+\varepsilon)t}, \qquad 2 \le p < \infty, \ t \ge 0, \tag{8}$$

where C is a positive constant independent of p and t. In particular, the inequality

$$(9) \qquad |||R(t)|||_\infty \leq Ce^{(\lambda_{A,g}+\varepsilon)t}, \qquad t \geq 0$$

holds true.

Idea of the proof. Since $P(z)$ is analytic on $\rho(A,g)$, we have only to replace Γ properly in (5). The argument is justified by the uniform estimates (6) and (7). (See [4; Theorem 5.6].) ∎

§2. Application to semilinear Volterra diffusion equtions

As an example of the results of §1, we consider the asymptotic stability of the positive equilibrium u_∞ for a semilinear delay model in population dynamics :

$$(10) \begin{cases} \frac{\partial u}{\partial t} = \Delta u + (a - bu - f * u)u, & t > 0, \ x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & t > 0, \ x \in \partial\Omega, \\ u(0,x) = u_0(x) \geq 0 \, (\not\equiv 0), & x \in \Omega, \end{cases}$$

where a and b are positive constants, $f * u$ denotes the convolution

$$f * u(t,x) = \int_0^t f(t-s)u(s,x)\,ds,$$

and u_0 is an appropriate smooth function. The positive function $f(t)$ which we keep in mind is the type of

$$(11) \qquad f(t) = \frac{\alpha}{T}e^{-\frac{t}{T}} \qquad (T > 0, \alpha > 0)$$

$$\text{or } (12) \qquad f(t) = \frac{\alpha}{T^2}te^{-\frac{t}{T}} \qquad (T > 0, 0 < \alpha < 8b)$$

$$\text{or } (13) \qquad f(t) = \frac{\alpha\rho(\omega^2+\rho^2)}{\omega^2+\rho^2+\omega\rho}e^{-\rho t}(1+\sin\omega t) \qquad (\alpha > 0, 0 < \omega < \rho)$$

(for the biological meaning of (11) and (12), see, e.g., [3], [4]). In these cases $u(t,\cdot)$ converges to $u_\infty = \frac{a}{b+\alpha}$ as $t \to \infty$ (see, e.g., [4], [6]). Replacing $u - u_\infty$ by v in (10), the usual linearization procedure leads us to (1) with $q(x) \equiv bu_\infty$, $r(x) \equiv 0$ and $g(t) = -u_\infty f(t)$. We can obtain the following theorem from the results of §1 and [6] :

Theorem 2.1. *Let the delay kernel f be the type of (11) or (12) or (13). Then the corresponding constant $\lambda_{A,g}$ is negative and the solution $u(t,x)$ of (10) converges like*

$$\|u(t,\cdot) - u_\infty\|_\infty = O\left(e^{(\lambda_{A,g}+\epsilon)t}\right) \qquad \text{as } t \longrightarrow \infty. \tag{14}$$

Here ϵ is any number satisfying $\lambda_{A,g} < \lambda_{A,g} + \epsilon < 0$.

Idea of the proof. Put $v(t) = u(t+\tau,\cdot) - u_\infty$ for sufficiently large number τ. Then we can reduce (10) to

$$\text{(15)} \begin{cases} \frac{dv}{dt}(t) + Av(t) = \int_0^t g(t-s)v(s)\,ds + h_1[v](t) + h_2(t), & t > 0, \\ v(0) = u(\tau,\cdot) - u_\infty, \end{cases}$$

where $h_1[v]$ and h_2 are given by

$$h_1[v](t) = -v(t)\left\{ bv(t) + \int_0^t f(t-s)v(s)\,ds \right\},$$
$$h_2(t) = u(t+\tau,\cdot)\left\{ u_\infty \int_t^\infty f(s)\,ds - \int_0^\tau f(t+\tau-s)u(s,\cdot)\,ds \right\}.$$

Since

$$v(t) = R(t)v(0) + \int_0^t R(t-s)\{h_1[v](s) + h_2(s)\}\,ds, \quad t \ge 0,$$

we can get to (14) by virtue of (9) and Theorem 3.2 in [6]. The negativity of $\lambda_{A,g}$ is deduced from the fact that $\lambda \in \sigma(A,g)$ if and only if $-\lambda + \widehat{g}(\lambda)$ is an eigenvalue of A or λ is a pole of $\widehat{g}(\cdot)$. (See [4; Theorem 2.5].) ∎

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ある特異摂動問題に対する粘性解の収束の評価について

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§1. Introduction

次のような特異摂動問題を考える.

$$(1.1)_\varepsilon \qquad \begin{cases} \max\{-\varepsilon^2\Delta u_\varepsilon + u_\varepsilon - f, |Du_\varepsilon| - g\} = 0 & \text{in} \quad \Omega \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega \end{cases}$$

但し、$\varepsilon > 0$ 、Ω は R^N の有界領域とし、f、g は $\bar{\Omega}$ で定義された非負関数とする. この方程式はある確率制御問題の value function が満たす、Bellman 方程式として知られている ([8]). 本稿の目的は、$\varepsilon \to 0$ としたときに、u_ε が次の方程式の解 u_0 に収束する rate を求めることである.

$$(1.1)_0 \qquad \begin{cases} \max\{u_0 - f, |Du_0| - g\} = 0 & \text{in} \quad \Omega \\ u_0 = 0 & \text{on} \quad \partial\Omega \end{cases}$$

このような問題については過去にもいくつかの結果があるが、それらはかなり面倒な計算を必要としている（例えば [2] 、[11] を参照）. ここでは粘性解 (viscosity solution) の比較定理の議論を応用してより簡単に各点収束の評価を求めることにする. この idea は [4] で最初に与えられ、[7] では u_ε そのものに obstacle を課した場合を例とするような議論が行なわれている.

§2. The notion of viscosity solutions

粘性解の概念を簡単に説明するために、次のような 1 階偏微分方程式の境界値問題を考える. 2 階の場合でも大筋は同じである（特に後半部分）.

$$(2.1) \qquad \begin{cases} -g \cdot Du + u - f = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

但し、$\Omega \in R^N$ は滑らかな境界 $\partial\Omega$ を持つ有界領域で、$g : \Omega \to R^N$、$f : \Omega \to R$ とする. この方程式の解を常微分方程式系

$$(2.2) \qquad \begin{cases} \dfrac{dx(t)}{dt} = g(x(t)), \\ x(0) = x_0 \in \Omega \end{cases}$$

の解を用いて表現することを考える.

まず、u を (2.1) の古典解とする.

$$\frac{d}{dt}(e^{-t}u(x(t))) = e^{-t}(g\cdot Du - u)$$

であるから、両辺を $[0,T]$ で積分すると、

$$\int_0^T \frac{d}{dt}(e^{-t}u(x(t)))dt = \int_0^T e^{-t}(g\cdot Du - u)dt$$

よって

$$e^{-T}u(x(T)) - u(x_0) = -\int_0^T e^{-t}f(x(t))dt$$

を得る. ここで、T を

$$T = \inf\{t > 0 | x(t) \in \partial\Omega\} \qquad (\ x(t)\ が\ \partial\Omega\ に到達する時刻)$$

とすると、境界条件より

$$u(x_0) = \int_0^T e^{-t}f(x(t))dt \tag{2.3}$$

となる.

一般に (2.1) では Ω 全体での古典解は存在しないので、この議論は成立しないが、(2.3) で定義された u は次の意味で (2.1) の weak solution となっている.

$u \in C(\Omega)$ 、$\varphi \in C^1(\Omega)$ とし、$u - \varphi$ は $x_0 \in \Omega$ で最大値をとるとする. φ に定数を加えることによって、

$$u(x_0) = \varphi(x_0), \quad u(x) \le \varphi(x) \quad \text{in } \Omega$$

としてよい.

$t > 0$ とする. $x(t+s)$ は $s = 0$ で $x(t)$ を初期値とする (2.2) の解だから (2.3) より

$$\begin{aligned} u(x(t)) &= \int_0^{T-t} e^{-t}f(x(s+t))ds \\ &= \int_t^T e^{t-s}f(x(s))ds. \end{aligned}$$

よって

$$e^{-t}u(x(t)) = \int_t^T e^{-s}f(x(s))ds.$$

(2.3) を使って

$$e^{-t}u(x(t)) - u(x_0) = -\int_0^t e^{-s} f(x(s))ds.$$

（ここで、u が微分可能ならば、両辺を t で割って $t \to 0$ とすれば、(2.1) が得られる.）
ところで、

$$\begin{aligned} e^{-t}u(x(t)) - u(x_0) &\le e^{-t}\varphi(x(t)) - \varphi(x_0) \\ &= \int_0^t \frac{d}{ds}(e^{-s}\varphi(x(s)))ds \\ &= \int_0^t e^{-t}\{g \cdot D\varphi(x(s)) - \varphi(x(s))\}ds \end{aligned}$$

となるので、この 2 式を合わせて得られる不等式の両辺を t で割って、$t \to 0$ とすれば、

$$-g \cdot D\varphi(x_0) + \varphi(x_0) - f(x_0) \le 0$$

が得られる.

同様にして、$u - \varphi$ が x_0 で最小値をとるとして、

$$-g \cdot D\varphi(x_0) + \varphi(x_0) - f(x_0) \ge 0$$

が得られる.

さて、粘性解の定義を与えるために、次のような（退化）非線形 2 階楕円型方程式を考える.

$$\text{(2.4)} \qquad F(x, u, Du, D^2u) = 0 \quad \text{in} \quad \Omega$$

ここで、$F \in C(\Omega \times R \times R^N \times S^N)$ が（退化）楕円型であるとは、任意の $(x, r, p, X) \in \Omega \times R \times R^N \times S^N$、$Y \in S^N$ に対して Y が半正定値ならば

$$F(x, r, p, X + Y) \le F(x, r, p, X)$$

が成り立つときをいう. 但し、S^N は $N \times N$ 実対称行列全体を表わす.

定義.

$u \in C(\Omega)$ とする.

(i) u が (2.4) の粘性 sub 解であるとは、任意の $\varphi \in C^2(\Omega)$ に対して、$u - \varphi$ が $x_0 \in \Omega$ で極大値をとるならば、

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \le 0$$

が成り立つときをいう.

(ii) u が (2.4) の粘性 super 解であるとは、任意の $\varphi \in C^2(\Omega)$ に対して、$u-\varphi$ が $x_0 \in \Omega$ で極小値をとるならば、

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0$$

が成り立つときをいう.

(iii) u が (2.4) の粘性解であるとは、粘性 sub & super 解のときをいう.

注意.

(i) F が 1 階のとき、$\varphi \in C^1(\Omega)$ として、上と同様の定義を与える.

(ii) u が (2.4) の古典解であるとき、粘性解になる. 実際、任意の $\varphi \in C^2(\Omega)$ に対して $u-\varphi$ が $x_0 \in \Omega$ で極大値を取ったとすると、

$$D(u(x_0) - \varphi(x_0)) = 0, \quad D^2(u(x_0) - \varphi(x_0)) : \text{nonpositive definite}$$

であるから、楕円型の定義より

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq F(x_0, u(x_0), Du(x_0), D^2u(x_0)) = 0$$

となり、粘性 sub 解になることがわかる. 粘性 super 解になることも同様に示される.

(iii) (2.4) の粘性解 u が C^2 ならば、u は (2.4) を各点で満たすことは容易にわかる.

(iv) 詳しい定義、性質については [1] 、[3] 、[4] 、[9] 、[10] および、それらの References を参照のこと.

§3. Main result

次の仮定をおく.

(A.1) Ω は R^N の有界領域で境界は滑らか

(A.2) $f \in W^{1,\infty}(\bar{\Omega})$ かつ $f \geq 0$ on $\bar{\Omega}$

(A.3) $g \in W^{1,\infty}(\bar{\Omega})$ かつ $g \geq^{\exists} \theta > 0$ on $\bar{\Omega}$

これらの仮定のもとで [5] 、[10] の結果を使うと、境界条件を満たす $(1.1)_\varepsilon$ 及び $(1.1)_0$ の粘性解 u_ε、u_0 が一意的に存在し、$\varepsilon > 0$ に無関係な定数 C により

$$0 \leq u_\varepsilon, u_0 \leq C \quad \text{on} \quad \bar{\Omega} \tag{3.1}$$

となる．また、[1] によればこれらはリプシッツ連続であり、しかも g が有界であることより $\{u_\varepsilon\}_{\varepsilon>0}$ は同程度リプシッツ連続である．このとき、次の定理が成り立つ．

定理.

(A.1)-(A.3) を仮定し、u_ε、u_0 を各々境界条件を満たす $(1.1)_\varepsilon$、$(1.1)_0$の粘性解とする．このとき、$\mu > 0$ と $\varepsilon_0 > 0$ が存在して、次の評価が成り立つ．

$$\|u_\varepsilon - u_0\| \le \mu\varepsilon \qquad \text{for all} \quad \varepsilon \in (0, \varepsilon_0)$$

但し、$\|\cdot\|$ は $C(\bar{\Omega})$ での sup norm である．

次の例により、この評価は best であることがわかる．

例. $N = 1$ 、$\Omega = (-1, 1)$ 、$f(x) = 1 - |x|$ 、$g \equiv 1$ on $\bar{\Omega}$ とする． このとき、$(1.1)_\varepsilon$、$(1.1)_0$ の粘性解は次のように書ける．

$$u_\varepsilon(x) = \varepsilon\frac{\sinh((|x| - 1)/\varepsilon)}{\cosh(1/\varepsilon)} + 1 - |x|,$$
$$u_0(x) = 1 - |x|.$$

すると、$\tanh x < 1$ かつ $\tanh x \to 1 \quad (x \to \infty)$ であることより

$$\|u_\varepsilon - u_0\| = |u_\varepsilon(0) - u_0(0)| = \varepsilon\tanh(1/\varepsilon) \le \varepsilon \qquad \text{as} \quad \varepsilon \to 0$$

が得られる．

定理の証明:

$u_\varepsilon - u_0 \le \mu\varepsilon$ on $\bar{\Omega}$ を示す．$\varepsilon_0 = \theta/3K_gK$（但し、$K$（resp. K_g）は u_ε、u_0（resp. g）のリプシッツ定数とする）とおき、$\varepsilon \in (0, \varepsilon_0)$ に対して、

$$\Phi_\varepsilon = \rho u_\varepsilon(x) - u_0(y) - \frac{|x - y|^2}{\varepsilon} - \mu\varepsilon \qquad \text{on} \quad \bar{\Omega} \times \bar{\Omega}$$

と定義する．ただし、$\rho = 1 - 3K_gK\varepsilon/2\theta$ とし、$\mu > 0$ は後から決める定数とする．$(\bar{x}, \bar{y}) \in \bar{\Omega} \times \bar{\Omega}$ を $\Phi_\varepsilon(x, y)$ の最大値を取る点とすると、$\Phi_\varepsilon(\bar{x}, \bar{x}) \le \Phi_\varepsilon(\bar{x}, \bar{y})$ と u_0 のリプシッツ連続性より

$$\frac{|\bar{x} - \bar{y}|^2}{\varepsilon} \le u_0(\bar{x}) - u_0(\bar{y}) \le K|\bar{x} - \bar{y}|$$

となり、$|\bar{x} - \bar{y}| \le K\varepsilon$ がわかる．

(I) $\bar{x}, \bar{y} \in \Omega$ のとき

関数 $\Phi_\varepsilon(x,\bar{y})/\rho$ は $\bar{x}$ で最大値を取るので、u_ε を粘性 sub 解とみると次の不等式を得る.

(3.2) $$\max\{-\frac{2N}{\rho}\varepsilon + u_\varepsilon(\bar{x}) - f(\bar{x}), \frac{2}{\rho\varepsilon}|\bar{x}-\bar{y}| - g(\bar{x})\} \le 0$$

同様に、関数 $-\Phi(\bar{x},y)$ は $\bar{y}$ で最小値を取るので、u_0 を粘性 super 解とみて、

(3.3) $$\max\{u_0(\bar{y}) - f(\bar{y}), \frac{2}{\varepsilon}|\bar{x}-\bar{y}| - g(\bar{y})\} \ge 0$$

を得る.

ここで、(3.3) において $2|\bar{x}-\bar{y}|/\varepsilon - g(\bar{y}) < 0$ となることを言う. そうでないと仮定すれば、(3.2) で $2|\bar{x}-\bar{y}|/\rho\varepsilon - g(\bar{x}) \le 0$ であることより

$$g(\bar{y}) \le \frac{2}{\varepsilon}|\bar{x}-\bar{y}| \le \rho g(\bar{x})$$

となる. すると、(A.3) と $|\bar{x}-\bar{y}| \le K\varepsilon$ を使って

$$(1-\rho)\theta \le (1-\rho)g(\bar{y}) \le \rho(g(\bar{x}) - g(\bar{y})) < K_g|\bar{x}-\bar{y}| \le K_g K\varepsilon$$

が得られる. ところが、ρ の定義より $3/2 < 1$ となり、これは矛盾である. よって $2|\bar{x}-\bar{y}|/\varepsilon - g(\bar{y}) < 0$ が言える. 従って、(3.3) において

$$u_0(\bar{y}) - f(\bar{y}) \ge 0$$

が得られ、(3.2) より

$$-\frac{2N}{\rho}\varepsilon + u_\varepsilon(\bar{x}) - f(\bar{x}) \le 0$$

だからこの 2 つの不等式と (A.2) を使って計算すると、$\mu > 0$ を十分大きく取ることにより

$$u_\varepsilon(\bar{x}) - u_0(\bar{y}) \le \mu\varepsilon$$

となる. すると、$\Phi_\varepsilon(x,x) \le \Phi_\varepsilon(\bar{x},\bar{y}) \le 0$ on $\bar{\Omega}$ が得られ、$\{u_\varepsilon\}_{\varepsilon>0}$ の一様有界性より改めて $\mu > 0$ を大きく取り直すと $u_\varepsilon - u_0 \le \mu\varepsilon$ on $\bar{\Omega}$ が得られる.

(II) $\bar{x}$ or $\bar{y} \in \partial\Omega$ のとき

$u_\varepsilon = u_0 = 0$ on $\partial\Omega$ だから、これと (3.1)、$\{u_\varepsilon\}_{\varepsilon>0}$ の同程度リプシッツ連続性を使うと容易である.

(I)、(II) より $\mu > 0$ を更に大きく取ると、$u_\varepsilon - u_0 \le \mu\varepsilon$ on $\bar{\Omega}$ が得られる.
$u_0 - u_\varepsilon \le \mu\varepsilon$ on $\bar{\Omega}$ についても同様である.

注意. 詳細は [6] を参照のこと.

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一次元非粘性流体のピストン問題の compensated compactness による解の存在について

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§0．序論

非粘性理想気体の断熱変化の一次元での運動方程式は、西田(1968)、西田―Smoller(1973)、Liu(1977)などにより、その初期値問題の弱解の存在が示され、それにともない初期値境界値問題も西田(1968)、西田―Smoller(1977)、Liu(1977,1978)などにより解かれた。これらはいずれも Lagrange 座標系において、いわゆる Glimm の差分法によるもので初期値は

$(\gamma-1)\times$（初期値の全変動）

が十分小さい、という条件のもとで解かれた。

DiPerna(1983[3])は Tartar や Murat らによる compensated compactness の方法を用いて、一般の有界な初期値に対する初期値問題を

$$\gamma=\frac{2}{2\tau+1}+1=\frac{7}{5},\frac{9}{7},\frac{11}{9},\ldots\quad(\tau\geq2:\text{整数})$$

のもと、Euler 座標系において解いた。また、Ding、Chen、Luo らは、さらに

$$1<\gamma\leq\frac{5}{3}$$

の場合の考察を行っている([1],[2])。

ここでは、DiPerna と同様の方法を使うことにより、DiPerna と同程度の条件のもとでこの方程式の初期値境界値問題（ピストン問題）の弱解の存在を示す。

§1．定義（c.f. [5]）

有界かつ可測な関数の組 $(\rho(x,t),m(x,t))$　$(\rho(x,t)\geq0$ a.e.$)$
が、Euler 座標における非粘性理想気体の断熱変化の一次元での運動方程式

$$\begin{cases}\rho_t+m_x=0,\\ m_t+\left[\dfrac{m^2}{\rho}+P(\rho)\right]_x=0,\end{cases}$$

$(\rho$:気体密度、$m=\rho u$:運動量、u:速度、$P(\rho)=\dfrac{1}{\gamma}\rho^\gamma$:圧力、

$\gamma=\dfrac{2}{2\tau+1}+1$:断熱指数、$\tau\geq1$:整数$)$

の、(a) 1-ピストン問題:

$D_1=\{(x,t):\ t>0,\ x>x_1(t)\}$　での初期値境界値問題

$$\begin{cases}(\rho(x,0),u(x,0))=(\rho_0(x),u_0(x)) & (x>0),\\ \rho(x_1(t),t)\{u(x_1(t),t)-u_1(t)\}=0 & (t>0),\end{cases}$$

の弱解であるとは、任意の $\varphi\in C_0^1=C_0^1(R_x\times R_t)$ 、及び $x=x_1(t)$ 上 0 となる任意

の $\psi\in C_0^1$ に対し

$$\begin{cases} \iint_{D_1}\left(\rho\varphi_t+m\varphi_x\right)dxdt + \int_0^\infty \rho_0(x)\varphi(x,0)dx = 0, \\ \iint_{D_1}\left\{m\psi_t+\left(\frac{m^2}{\rho}+P(\rho)\right)\psi_x\right\}dxdt + \int_0^\infty m_0(x)\psi(x,0)dx = 0, \end{cases}$$

($\rho_0(x), u_0(x), u_1(t)$ は与えられた有界可測な関数で、$\rho_0(x)\geq 0$,
$m_0(x)=\rho_0(x)u_0(x)$, $x_1(t)=\int_0^t u_1(s)ds$)

を満たすこと。

同様にして、(b)2-ピストン問題:

$D_2=\{(x,t):\ t>0,\ x_1(t)<x<x_2(t)\}$ での初期値境界値問題

$$\begin{cases} (\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)) & (0<x<L), \\ \rho(x_j(t),t)\{u(x_j(t),t)-u_j(t)\} = 0 & (t>0,\ j=1,2), \end{cases}$$

($\rho_0(x), u_0(x), u_1(t), u_2(t)$ は与えられた有界可測な関数で、
$\rho_0(x)\geq 0$, $m_0(x)=\rho_0(x)u_0(x)$, $x_1(t)=\int_0^t u_1(s)ds$,
$x_2(t)=L+\int_0^t u_2(s)ds$, $L>0$:定数)

の弱解も定義される。

また、2-ピストン問題(b)においては、ピストンの運動に対して次の場合を考える。

(C1) $u_1(t)\leq$(ある定数)$\leq u_2(t)$,

(C2) $x_2(t)-x_1(t)\geq\delta(t)$ $(\delta(t)>0,\ t>0)$.

(明らかに、(C2)の方がより一般の場合である。)

smooth な関数の組 $(\eta(U), q(U))$ $(U={}^t(\rho,m))$ が <u>entropy pair</u> であるとは、方程式

$$U_t+F(U)_x=0 \qquad \left(F(U)={}^t\left(m, \frac{m^2}{\rho}+P\right)\right)$$

の smooth solution に対しては

$$\eta(U)_t+q(U)_x=0$$

を満たすこと、すなわち、

$$\text{grad } q(U) = \text{grad } \eta(U)\cdot\text{grad } F(U) \qquad \left(\text{grad}=\left(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial m}\right)\right)$$

を満たすこと ($\eta(U)$は <u>entropy</u>、$q(U)$は <u>entropy flux</u> と呼ばれる)。
また、entropy $\eta(U)$ が <u>weak</u> であるとは、η を (ρ,u) の関数とみて

$$\eta(0,u)=\left[\eta(\rho,u)\right]_{\rho=0}=0 \qquad \left(u=\frac{m}{\rho}\right)$$

を満たすことをいう。以下の議論で必要な entropy は、Darboux の公式:

$$\eta = \int_z^w \{(w-s)(z-s)\}^\tau\varphi(s)ds$$

$$= \left(\frac{2}{\theta}\right)^{\frac{1}{\theta}}\rho\int_0^1\{y(1-y)\}^\tau\varphi\left(\frac{m}{\rho}+\frac{2y-1}{\theta}\rho^\theta\right)dy$$

($w=\frac{m}{\rho}+\frac{1}{\theta}\rho^\theta$, $z=\frac{m}{\rho}-\frac{1}{\theta}\rho^\theta$, $\theta=\frac{\gamma-1}{2}$, $\varphi(s)$ は smooth な関数)

で与えられる smooth weak entropy である。

§2. 近似解の構成

この方程式に対し、Lax、西田らによる次の定理が成り立つ([4], [5])。

定理1

方程式　$U_t+F(U)_x=0$　の、

(Ⅰ) $\{(x,t):\ t>0,\ x\in R\}$ における初期値問題(Riemann 問題と呼ばれる)

$$U(x,0)=\begin{cases} U_l & (x<0), \\ U_r & (x>0) \end{cases} \qquad (U_l, U_r \text{ は定ベクトル}),$$

(Ⅱ) $\{(x,t):\ t>0,\ x>at\}$ (a は定数) における初期値境界値問題

$$\begin{cases} U(x,0)=U_r & (x>0), \\ \rho(at,t)\{u(at,t)-a\}=0 & (t>0) \end{cases} \qquad (U_r \text{ は定ベクトル }),$$

(Ⅲ) $\{(x,t):\ t>0,\ x<at\}$ (a は定数) における初期値境界値問題

$$\begin{cases} U(x,0)=U_l & (x<0), \\ \rho(at,t)\{u(at,t)-a\}=0 & (t>0) \end{cases} \qquad (U_l \text{ は定ベクトル }),$$

は、定数状態、膨張波 (smooth solution)、衝撃波 (不連続線からなる weak solution) で構成される、区分的に連続な weak solution を持つ。

この定理を使い、壁 $x=x_j(t)$ を折れ線で近似し、初期値を積分による平均をとって階段関数で近似することにより近似解 $U^\Delta(x,t)$ を構成できる。こうして構成される近似解について次の命題が成り立つ。

命題2 (近似解の一様有界性)

ピストン問題 (a),(b) の近似解 $U^\Delta(x,t)$ は、(b) においては (C1)、または (C2) のもと、任意の (x,t) で定義できて、任意の $T>0$ に対し、

$$\begin{cases} \rho^\Delta(x,t)=0 \text{ ならば } m^\Delta(x,t)=0, \\ 0\le\rho^\Delta(x,t)\le C_1(T), & (\ 0<t<T\ ) \\ |u^\Delta(x,t)|\le C_2(T), \end{cases}$$

$$\left[\ u^\Delta(x,t)=\frac{m^\Delta(x,t)}{\rho^\Delta(x,t)}\ \right]$$

を満たす。ここで $C_j(T)$ は、(a)、および (C1) のもとでの (b) に対しては T に依存しない定数ととれ、(C2) のもとでの (b) に対しては

$$C_1(T)^\theta,\ C_2(T)\le C_3\exp\left[(A-B)\int_0^T\frac{dt}{\delta(t)}\right],$$

(A、B は $A=\mathrm{ess.sup}_t u_1(t)$, $B=\mathrm{ess.inf}_t u_2(t)$ で、$A>B$、すなわち、(C1) を満たさないとき。)

と評価される。

命題3 (コンパクト性)

命題2の $U^\Delta(x,t)$、及び Darboux の公式 であたえられる smooth (weak) entropy pair に対して

$$\eta\left(U^\Delta(x,t)\right)_t+q\left(U^\Delta(x,t)\right)_x$$

$\in$ ある $H^{-1}_{loc}(\Omega)$ の compact set (Δ によらない)

となることが、任意の bounded open set Ω に対していえる。

§3. compensated compactness

次の2つの定理は Tartar らによる([6])。

定理4 (Young measure)

K : bounded set in R^N, Ω : open set in R^M,

$v_n:\Omega\longrightarrow R^N$ $(n=1,2,...)$ が $v_n(x)\in K$ a.e. $x\in\Omega$

を満たすならば

$\{v_n\}_{n=1,2,...}$ のある部分列 $\{v_{n_j}\}_{j=1,2,...}$、及びある probability measure (on R^N_y)の族 $\{\nu_x(y)\}_{\text{a.e. } x\in\Omega}$ が存在して次を満たす。

- $\operatorname{supp}\nu_x\subset\bar{K}$,
- R^N_y 上連続な関数 $G(y)$ に対して

$\bar{G}(x)=<\nu_x(y),G(y)>=\int_{R^N}G(y)\nu(dy)$ は x-可測で、

$G(v_{n_j}(x))\longrightarrow\bar{G}(x)$ in $L^\infty(\Omega)$ weak*

($\{\nu_x\}$ を $\{v_{n_j}\}$ に関する Young measureと呼ぶ)

定理5 (div-curl lemma)

Ω : bounded open set in R^N, $v_n,v,w_n,w\in L^\infty(\Omega;R^N)$ が、

$v_n\longrightarrow v$, $w_n\longrightarrow w$ in $L^\infty(\Omega)$ weak*,

div v_n, curl $w_n\in$ ある $H^{-1}_{loc}(\Omega)$ の compact set

(n によらない)

を満たすならば、$\{v_n\},\{w_n\}$ のある部分列 $\{v_{n_j}\},\{w_{n_j}\}$ があって、

$v_{n_j}\cdot w_{n_j}\longrightarrow v\cdot w$ in $L^\infty(\Omega)$ weak*

('·' は R^N の内積、

$\operatorname{curl}(u_1,u_2,...,u_N)=\left(\frac{\partial x_j}{\partial x_i}-\frac{\partial x_i}{\partial x_j}\right)_{1\le i<j\le N}$)

となる。

§4. 近似解の収束

§2で構成した近似解は、(a)、または (C1) のもとでの (b) に対しては命題2により一様有界であるから、適当な部分列 $U^{\Delta'}(x,t)$ をとることにより、ある有界可測な関数

$$\bar{U}(x,t)={}^t(\bar{\rho}(x,t),\bar{m}(x,t))$$

があって

$$U^{\Delta'}(x,t)\longrightarrow\bar{U}(x,t)\quad\text{in } L^\infty \text{ weak*}$$

で、かつ (定理4により) ある probability measure の族

$\{\nu_{(x,t)}(U)\}_{(x,t)}$

があって、任意の連続な関数 $G(U)$ に対して

$$G\left[U^{\Delta'}(x,t)\right]\longrightarrow<\nu_{(x,t)}(U),G(U)>\quad\text{in } L^\infty \text{ weak*}$$

とすることができる。いま、(x,t) という独立変数に対し、

$$\mathrm{div}\ (q,\eta) = \eta_t + q_x,$$
$$\mathrm{curl}(-\eta,q) = \eta_t + q_x$$

であるから、Darboux の公式で与えられる smooth な entropy pair

$$(\eta_1,q_1),(\eta_2,q_2)$$

に対し、commutative relation:

$$<\nu,\eta_2 q_1 - \eta_1 q_2> = <\nu,\eta_2><\nu,q_1> - <\nu,\eta_1><\nu,q_2> \quad \text{a.e. } (x,t)$$

が成り立つ。この関係式、及び Darboux の公式で得られる多くの entropy pair を使うことによって

$$\nu_{(x,t)}(U) = \delta_{\bar{U}(x,t)}(U)$$

(= 1点 $\bar{U}(x,t)$ を support にもつ δ-measure)

であることを導びきだせる。それには次の定理を使う（詳しくは [1],[2]）。

<u>定理6</u> (Lebesgue derivative)

μ : nonnegative Radon measure (on an open set $\Omega \subset R^N$),

m : N 次元 Lebesgue measure に対して

$$\underline{D}\mu(x) = \liminf_{r\downarrow 0} \frac{\mu(B_r(x))}{m(B_r(x))} = 0 \quad \text{a.e. } \mu$$

ならば $\mu \equiv 0$.

($B_r(x)$: 中心 x、半径 r の N 次元開球、 $\underline{D}\mu(x)$ を <u>Lebesgue lower derivative</u> と呼ぶ)

よって、$G(U)$ を ρ, ρ^2, m, m^2 などとれば定理4により

$$\left.\begin{array}{ll} \rho^{\Delta'} \longrightarrow \bar{\rho}, & (\rho^{\Delta'})^2 \longrightarrow (\bar{\rho})^2 \\ m^{\Delta'} \longrightarrow \bar{m}, & (m^{\Delta'})^2 \longrightarrow (\bar{m})^2 \end{array}\right\} \quad \text{in } L^\infty \text{ weak*}$$

となり、$\bar{\rho}$、$\bar{m}$ は *bounded* であるから、結局、適当な部分列 $\{U^{\Delta''}\}_{\Delta''}$ をとって

$$U^{\Delta''} \longrightarrow \bar{U} \qquad \text{a.e.}$$

とできる。

この $\bar{U}$ が次の、いわゆる<u>エントロピー条件</u>を満たすことも容易に示される:

$$\eta(\bar{U}(x,t))_t + q(\bar{U}(x,t))_x \leq 0 \qquad (\text{ distribution sense })$$

を、

$$\left\{\begin{array}{l} \eta : \text{smooth weak convex entropy}, \\ q(0,u) = [q(\rho,u)]_{\rho=0} = 0 \end{array}\right.$$

を満たす smooth entropy pair (η,q) に対し、D_j の内部で満たす（特に、力学的エネルギー

$$\eta_* = \frac{1}{2}\rho u^2 + \frac{\rho^\gamma}{\gamma(\gamma-1)} \quad \left(q_* = \frac{1}{2}\rho u^3 + \frac{\rho^\gamma u}{\gamma-1}\right)$$

に対していえる）。

(C2) のもとでの (b) の場合も、上と同様な議論を（対角線論法により）行うことができる。よって次の存在定理がいえる。

定理7（弱解の存在）

ピストン問題 (a),(b) に対し（(b) については (C1) または (C2) のもと），有界可測な弱解

$$\bar{U}(x,t) = {}^t(\ \bar{\rho}(x,t),\bar{m}(x,t)\ )$$

が存在し、

$$\begin{cases} \bar{m}(x,t)=0 \quad \text{a.e. on } \{(x,t):0<t<T,\ \bar{\rho}(x,t)=0\ \}, \\ 0 \le \bar{\rho}(x,t) \le C_1(T) \quad \text{a.e. on } \{(x,t):0<t<T\}\ , \\ |\bar{u}(x,t)| \le C_2(T) \quad \text{a.e. on } \{(x,t):0<t<T,\ \bar{\rho}(x,t)>0\ \}, \end{cases}$$

$$\left[\ \bar{u}(x,t) = \frac{\bar{m}(x,t)}{\bar{\rho}(x,t)},\ C_1(T),C_2(T) \text{ は命題２と同じもの} \right]$$

、及びエントロピー条件を満たす。この弱解は§２で構成した近似解の、ある部分列の a.e. 収束極限として得られる。

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The correspondence between a linear semigroup and a nonlinear semigroup

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We are concerned with a possibly nonlinear semigroup $\{T(t)\}$, a one-parameter family of continuous operators on a Banach space X into X. Instead of the concept of the dual semigroup and the dual Banach space, we define a linear semigroup $\{T(t)^{\otimes}\}$ on a certain continuous function space. If $\{T(t)\}$ is not a linear semigroup, we can no longer define the dual semigroup. Thus our new semigroup provides us with a new means of analysing nonlinear semigroup. By a semigroup we mean a one parameter semigroup ,denoting $\{T(t)\}$, of possibly nonlinear operators from X to itself, satisfying

(1) $T(t)$ is a Lipshitz continuous operator from X to X for every $t > 0$, i.e.

$$\|T(t)x - T(t)y\| \leq c(t)\|x - y\|,$$

where $c(t)$ is a continuous function in $t \geq 0$.

(2) $T(t + s) = T(t)T(s)$, $T(0) = I$

Let us denote by A the infinitesmal generator of $\{T(t)\}$.

Let $C(Y)$ be the set of all continuous functions on Y, which is a Banach space equipped with the norm

$$\|f\|_{C(Y)} = \sup_{y \in Y} |f(y)|.$$

The following A(1) is necessary for the strong continuity of $T(t)^{\otimes}$

DEFINITION. *The subset Y of X is said to be invariant for a semigroup $\{T(t)\}$, if $T(t)Y \subset Y$ for every $t \geq 0$.*

A(1) A semigroup $\{T(t)\}$ which is a C_0-semigroup has the positively invariant compact subset K of X.

1. Corresponding linear semigroup

We restrict the operation of $T(t)$ to a compact invariant subset, show the corresponding linear semigroup $T(t)^{\otimes}$ has the strong continuity. Thereby $T(t)^{\otimes}$ holds some good property from the linear semigroup theory.

THEOREM 1. *Suppose a semigroup $\{T(t)\}$ satisfies A(1). Then there is a corresponding linear semigroup $\{T(t)^{\otimes}\}$ on $C(K)$ as follows,*

$$T(t)^{\otimes} f(x) = f(T(t)x) \quad t \geq 0. \tag{1}$$

The linear semigroup $\{T(t)^{\otimes}\}$ is contractive, strongly continous for $t \geq 0$.

THEOREM 2.. *If f is a Frechet differentiable function on Y and belongs to the domain of $\mathcal{A}$, then*

$$\mathcal{A}f(x) = \langle Ax, f'(x) \rangle \tag{2}$$

EXAMPLE 1.. *Suppose $T(t)$ is a group representing parallel transformation in X, i.e $T(t)x = x + ta$ for $t \in \mathbb{R}$, where a is a constant element in X. Then for $f \in C(X)$, $T(t)^{\otimes} f(x) = f(x + ta)$.*

EXAMPLE. *Suppose $X = \mathbb{R}$, and for $g \in C(\mathbb{R})$ $Ax = g(x)$. We consider the equation,*

$$\frac{dx}{dt} = Ax \quad t \geq 0.$$

Then if the solution is represented by a semigroup $T(t)$, $T(t)$ satisfies

$$\int_{x_0}^{T(t)x} \frac{1}{g(x')} dx' = t + \int_{x_0}^{T(t)x} \frac{1}{g(x')} dx'$$

proof If $\{T(t)\}$ exists, also does $\{T(t)^{\otimes}\}$. Let us solve

$$\frac{df}{dt} = Af = f'(x)Ax.$$

We use the separation of variables, and pose $f(t,x) = \Phi(t)\Psi(x)$. Then applying (2),

$$g(x)\frac{\Psi'(x)}{\Psi(x)} = \frac{\Phi'(t)}{\Phi(t)} = c,$$

where c is a constant. Therefore we have

$$\Phi(t) = \Phi(0)exp(ct), \quad \Psi(x) = exp(\int_{x_0}^{x} \frac{c}{g(x')} dx')$$

Therefore the operation of $T(t)^{\otimes}$ is

$$T(t)^{\otimes} : \Phi(0)exp(\int_{x_0}^{x} \frac{c}{g(x')} dx') \rightarrow \Phi(0)exp(ct + \int_{x_0}^{x} \frac{c}{g(x')} dx'),$$

and we obtain,

$$\int_{x_0}^{T(t)x} \frac{1}{g(x')} dx' = t + \int_{x_0}^{x} \frac{1}{g(x')} dx'.$$

Next we construct the corresponding linear semigroup on $C(X)$. However in this case $\{T(t)^{\otimes}\}$ will not be a C_0-semigroup, except the very special case, if we equip $\| \; \|_{C(X)}$ norm to $C(X)$. $\{T(t)^{\otimes}\}$ will be a linear semigroup on the locally convex topological vector space $C(X)$. The content of this paragraph is based on the equi-continuous semigroup theory [see 2]. We shall assume that we are in the following situation.

S(1) Let a semigroup $\{T(t)\}$ have a family $\mathcal{K}$ of positively invariant compact subsets K_i which satisfies

$$X = \cup_{K \in \mathcal{K}} K.$$

THE TOPOLOGY OF C(X). *Suppose a semigroup $\{T(t)\}$ satisfies S(1). Then $C(X)$ is a locally convex space equipped with the family of seminorms $\{\| \quad \|_{C(K)}\}_{K\in\mathcal{K}}$,where*

$$\|f\|_{C(K)} = \sup_{x\in K} |f(x)|$$

Now from Theorem 1, in 2, the general theorem is derived.

THEOREM 3. *Let a semigroup $\{T(t)\}$ satisfy S(1). Then there is a corresponding linear semigroup $\{T(t)^{\otimes}\}$ on the locally convex topological vector space $C(X)$ as follows,*

$$T(t)^{\otimes} f(x) = f(T(t)x),$$

for every $x \in X$.

Remark4. If $\mathcal{K}$ is countable, then C(X) is a norm space.

2. The characterization of a dynamical system by a corresponding linear semigroup

Let $\{T(t)\}$ be a semigroup which has the bounded invariant subset of X. Then we can distinguish a class of $\{T(t)\}$ from others by the nature of corresponding semigroup $\{T(t)^{\otimes}\}$.

Let $D_t = T(t)X$ for every $t \geq 0$, and let $D = \cap_{t>0} D_t$. In this section we concern with the semigroup, whose D is dense in X. Since D is composed of every element whose orbit is continuing back to the infinitely past, our concerning semigroup has the "group-like" property. We show it is really a group on the certain extended space E containing X, in the next theorem. ([1])

THEOREM. *Let $\{T(t)\}$ be a semigroup on X and satisfy the conditions*

$$(i) D \text{ is dense in } X,$$
$$(ii) T(t)x = T(t)y \text{ implies } x = y.$$

Then there exists the space E which satisfies

(1) $X \subset E$

(2) There exist a group $T(t)$ $(t \in \mathbb{R})$ on E, i.e.

$$T(t)T(s) = T(t+s),$$
$$T(0) = I \quad \text{where } I \text{ is an identity map},$$
$$T(t)x = T(t)x$$
$$\text{for } x \in X, \text{ if the right side exist.}$$

E is constructed as follows,

$$E = \{(z_n) \| z_n \in D, \exists t > 0 \quad \text{such that} \lim_{n\to\infty} T(t)z_n \in X\}/\sim$$

,where $\sim$ represents the equivalent relation

$$(z_n) \sim (z'_n) \leftrightarrow \lim_{n\to\infty} T(t)z_n = \lim_{n\to\infty} T(t)z'_n$$
$$\text{for some } t > 0.$$

PROPOSITION. *A semigroup $\{T(t)\}$ on X satisfies the condition of the above theorem if and only if its corresponding linear semigroup $\{T(t)^{\otimes}\}$ on $C(X)$ has the backward uniqueness property.*

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STABILITY OF PERIODIC SOLUTIONS TO ONE-DIMENSIONAL TWO-PHASE STEFAN PROBLEMS

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0. Introduction

Let us consider a two-phase Stefan problem described as follows: Find a function $u = u(t,x)$ on $Q(t_0, t_1) = J \times [0,1]$, $J = [t_0, t_1]$ or (t_0, t_1), and a curve $x = l(t)$, $0 < l < 1$, on J such that

(0.1) $\rho(u)_t - u_{xx} = 0$ in $Q_l^+(t_0, t_1)$, and in $Q_l^-(t_0, t_1)$,

$$Q_l^+(t_0, t_1) = \{(t,x);\ t_0 < t < t_1,\ 0 < x < l(t)\},$$
$$Q_l^-(t_0, t_1) = \{(t,x);\ t_0 < t < t_1,\ l(t) < x < 1\},$$

(0.2) $\begin{cases} u(t, l(t)) = 0 & \text{for } t \in J, \\ l'(t) = -u_x(t, l(t)-) + u_x(t, l(t)+) & \text{for a.e. } t \in J, \end{cases}$

(0.3) $\begin{cases} u_x(t, 0+) \in \partial b_0^t(u(t,0)) & \text{for a.e. } t \in J, \\ -u_x(t, 1-) \in \partial b_1^t(u(t,1)) & \text{for a.e. } t \in J, \end{cases}$

where $\rho: R \to R$ is a continuous increasing function; b_i^t ($i = 0, 1$) is a proper l.s.c. convex function on R and $\partial b_i^t(\cdot)$ is its subdifferential in R.

In this paper, we denote "*SP on J*" the system (0.1) ~ (0.3) and say that a pair $\{u, l\}$ is a solution of *SP* on $[t_0, t_1]$, $-\infty < t_0 < t_1 < \infty$, if u and l satisfy that

(0.4) $\begin{cases} u \in W^{1,2}(t_0, t_1; L^2(0,1)) \cap L^\infty(t_0, t_1; W^{1,2}(0,1)), \\ l \in W^{1,2}(t_0, t_1), \end{cases}$

(0.5) $\begin{cases} b_i^{(\cdot)}(u(\cdot, i)) \in L^\infty(t_0, t_1),\ u(t,i) \in D(\partial b_i^t) \text{ for a.e. } t \in [t_0, t_1] \\ \text{and } i = 0, 1, \end{cases}$

and that (0.1) ~ (0.3) hold. Also, for $-\infty \le t_0' < t_1' \le \infty$, we say that (u, ℓ) is a solution of SP on (t_0', t_1'), if it is a solution of SP on $[t_0, t_1]$ for every $t_0' < t_0 < t_1 < t_1'$ in the above sence. Let T be a positive number, and (u, ℓ) is a solution of SP on R such that $u(T + t, x) = u(t, x)$ for any $(t, x) \in R \times [0, 1]$ and $\ell(t + T) = \ell(t)$ for any $t \in R$. Then (u, ℓ) is a called a T-periodic solution of SP on R. We put

$\mathcal{P} = \{(u, \ell);\ (u, \ell)$ is a T-periodic solution of SP on $R\}$.

In Kenmochi [2], for the initial value problems the local existence in time and uniqueness of solutions were proved as well as comparison results.

The periodicity of solutions has been studied by many authers. For example we quote Ishii [1] and Kenmochi [3]. In these papers, the case in which the set $\mathcal{P}$ is a singleton was treated. However, in our case the set $\mathcal{P}$ is not a singleton, in general. Therefore it is interesting to investigate the structure of the set $\mathcal{P}$. More precisely, we shall show the set $\mathcal{P}$ is totally ordered with respect to the usual order for functions.

1. Main results

We begin with the precise assumptions (a1), (a2) on ρ and b_i^t, $i = 0, 1$, under which Stefan problem (0.1) ~ (0.3) is discussed.

(a1) ρ: $R \to R$ is a bi-Lipschitz continuous and increasing function on with $\rho(0) = 0$.

(a2) For $i = 0, 1$ and each $t \in R$, b_i^t is a proper l.s.c. convex function on R, $b_i^{t+T} = b_i^t$ and which satisfies the following conditions (*) and (**):

(*) There are functions $\alpha_0 \in W^{1,2}_{loc}(R)$, $\alpha_1 \in W^{1,1}_{loc}(R)$ satisfying that for any $-\infty < s < t < \infty$ and $r \in D(b_i^s)$ there exists $r' \in D(b_i^t)$ such that

$$|r' - r| \le |\alpha_0(t) - \alpha_0(s)|(1 + |r| + |b_1^s(r)|^{1/2}),$$
$$b_i^t(r') - b_i^s(r) \le |\alpha_1(t) - \alpha_1(s)|(1 + |r|^2 + |b_i^s(r)|);$$

$$(**)\quad \begin{cases} \partial b_0^t(r) \subset (-\infty, 0] & \text{for any } r < 0 \text{ and } t \in R, \\ \partial b_1^t(r) \subset [0, \infty) & \text{for any } r > 0 \text{ and } t \in R. \end{cases}$$

At first, we give a sufficient condition in order that the set $\mathcal{F}$ is not empty.

THEOREM 1.1. *Assume (a1) and (a2) hold and suppose that*

$$(1.1)\quad \begin{cases} \text{there is a positive constant } \delta_0 \text{ such that} \\ D(b_0^t) \subset [\delta_0, \infty) \text{ and } D(b_1^t) \subset (-\infty, -\delta_0] \text{ for any } t \in R. \end{cases}$$

Then the set $\mathcal{F}$ is not empty.

The next theorem is concerned with the structure of the set $\mathcal{F}$.

THEOREM 1.2. *Suppose that (a1) ~ (a2) and (1.1) hold. Then we have the following results (1) ~ (5):*

(1) For any $\{u_i, l_i\} \in \mathcal{F}$, $i = 1, 2$,

$$u_{1,x}(t, 0+) = u_{2,x}(t, 0+),\ u_{1,x}(t, 1-) = u_{2,x}(t, 1-) \quad \text{for a.e. } t \in R.$$

(2) Let $\{u_i, l_i\} \in \mathcal{F}$, $i = 1, 2$. If

$$\int_0^1 \rho(u_1)(0, x)dx + l_1(0) = \int_0^1 \rho(u_2)(0, x)dx + l_2(0),$$

then

$$u_1(t, x) = u_2(t, x) \text{ for } (t, x) \in R \times [0, 1] \text{ and } l_1(t) = l_2(t) \text{ for } t \in R.$$

(3) $\mathcal{F}$ is a totally ordered set with respect to the usual oreder, that is, for any $\{u_i, l_i\} \in \mathcal{F}_0$, $i = 1, 2$, it holds that

$$u_1(t, x) \le u_2(t, x) \text{ for } (t, x) \in R \times [0, 1] \text{ and } l_1(t) \le l_2(t) \text{ for } t \in R,$$

or

$u_1(t,x) \geq u_2(t,x)$ *for* $(t,x) \in R \times [0,1]$ *and* $l_1(t) \geq l_2(t)$ *for* $t \in R$.

(4) There exist the maximal element $\{u^*, l^*\}$ *and the minimum element* $\{u_*, l_*\} \in \mathcal{P}$, *that is, for any* $\{u, l\} \in \mathcal{P}$,

$$u_*(t,x) \leq u(t,x) \leq u_*(t,x) \qquad \text{for any } (t,x) \in R \times (0,1),$$

and

$$l_*(t) \leq l(t) \leq l^*(t) \qquad \text{for any } t \in R.$$

(5) Put $C_* = \int_0^1 \rho(u_*)(0,x)dx + l_*(0)$ *and* $C^* = \int_0^1 \rho(u^*)(0,x)dx + l^*(0)$. *Then for any* $C \in (C_*, C^*)$ *there is one and only one* $\{u, l\}$ *in* $\mathcal{P}$ *such that* $C = \int_0^1 \rho(u)(0,x)dx + l(0)$.

REMARK 1.1. *Suppose that (a2) and (1.1) hold. Let*

$$\rho(r) = \begin{cases} c_0 r & \text{for } r \geq 0, \\ c_1 r & \text{for } r < 0, \end{cases}$$

where c_0 *and* c_1 *are positive constants. For any* $\{u_i, l_i\} \in \mathcal{P}$, $i = 1, 2$, *if*

$$\int_0^1 \rho(u_1)(0,x)dx + l_1(0) < \int_0^1 \rho(u_2)(0,x)dx + l_2(0),$$

then

$u_1(t,x) < u_2(t,x)$ *for* $(t,x) \in R \times [0,1]$ *and* $l_1(t) < l_2(t)$ *for* $t \in R$.

The third theorem is concerned with asymptotic stability of the solution $\{u, l\}$ of SP on (t_0, ∞), $-\infty < t_0 < \infty$.

THEOREM 1.3. *Under the same assumptions as in Theorem 1.1 for any solution* $\{u, l\}$ *of* SP *on* (t_0, ∞), $-\infty < t_0 < \infty$, *there exists* $\{\bar{u}, \bar{l}\} \in \mathcal{P}$ *such that*

$$u(t) - \bar{u}(t) \to 0 \text{ in } C([0,1]) \text{ and weakly in } W^{1,2}(0,1) \text{ as } t \to \infty,$$

and

$l(t) - \bar{l}(t) \to 0$ as $t \to \infty$.

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Shape Optimization for Multi-Phase Stefan Problems — Existence of Solutions and Applications —

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1.Formulation of an Optimization Problem

Let us consider a multi-phase Stefan problem described as follows:

$$SP(\Omega)\begin{cases} u_t - \Delta\beta(u) = f & \text{in } Q(\Omega) = (0,T)\times\Omega, \\ u(0,\cdot) = u_0 & \text{in } \Omega, \\ \beta(u) = g & \text{on } \Sigma(\Omega) = (0,T)\times\partial\Omega, \end{cases}$$

where $0 < T < \infty$, $\hat{\Omega}$ is a fixed bounded domain in $R^N (N \geq 2)$ with smooth boundary $\partial\hat{\Omega}$; Ω is a subdomain of $\hat{\Omega}$ with smooth boundary $\partial\Omega$; $\hat{Q} := (0,T)\times\hat{\Omega}$ and $\hat{\Sigma} := (0,T)\times\partial\hat{\Omega}$; $\beta : R \to R$ is a nondecreasing function on R such that

$$(1.1)\qquad \begin{cases} \beta(0) = 0, \ |\beta(r)| \geq C_0 |r| - C_0' & \text{for all } r \in R, \\ |\beta(r) - \beta(r')| \leq L_0 |r - r'| & \text{for all } r, r' \in R, \end{cases}$$

where $C_0 > 0$, $C_0' \geq 0$, $L_0 > 0$ are constants. Here we suppose that $f \in L^2(\hat{Q})$, $u_0 \in L^2(\hat{\Omega})$ and $g \in W^{2,2}(0,T; L^2(\hat{\Omega})) \cap L^2(0,T; H^2(\hat{\Omega}))$.

We use the following function spaces and notations:

(1) We denote by H the usual space $L^2(\hat{\Omega})$; $|\cdot|_H$ stands for the norm in H and $(\cdot,\cdot)$ the inner product in H. We denote by $C_w([0,T]; H)$ the space of all weakly continuous functions from $[0,T]$ into H and by X the usual space $H_0^1(\hat{\Omega})$.

(2) We define a bilinear form $a_\Omega(\cdot,\cdot)$ on $H^1(\Omega)$ by

$$a_\Omega(u,v) = \int_\Omega \nabla u \cdot \nabla v dx \quad \text{for } u, v \in H^1(\Omega).$$

We denote by $\langle\cdot,\cdot\rangle_\Omega$ the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, and by F_Ω the duality mapping from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$ which is given by the formula

$$\langle F_\Omega v, z\rangle = a_\Omega(v,z) \text{ for all } v, z \in H_0^1(\Omega)$$

Moreover, $(\cdot,\cdot)_\Omega$ denotes the inner product in $L^2(\Omega)$.

(3) We denote by $O := \{\ \Omega \subset \hat{\Omega}; \Omega \text{ is a smooth subdomain of } \hat{\Omega}\ \}$ and by $V(\Omega)$ the set $\{z \in H_0^1(\hat{\Omega}); z = 0 \text{ a.e. on } \hat{\Omega} - \Omega\}$ for each $\Omega \in O$. Clearly, $V(\Omega)$ is a closed linear subspace of $H_0^1(\hat{\Omega})$. This space is a Hilbert space with inner product $a(\cdot,\cdot) := a_{\hat{\Omega}}(\cdot,\cdot)$ and with norm

$$|v|_{\hat{\Omega}} := a(v,v)^{1/2} (= |\nabla v|_H) \quad \text{for } v \in V(\Omega).$$

(4) Now, we introduce a notion of convergence of closed convex sets in a Banach space X, which is due to Mosco [5]. Let $\{K_n\}$ be a sequence of closed convex sets in X and K be a closed convex set in X. Then we mean by "$K_n \to K$ in X as $n \to \infty$ (in the sense of Mosco)" that the following two conditions (M1) and (M2) are satisfied:

(M1) If $\{n_k\}$ is a subsequence of $\{n\}$, $z_k \in K_{n_k}$, and $z_k \to z$ weakly in X as $k \to \infty$, then $z \in K$.

(M2) For any $z \in K$ there is a sequence $\{z_n\} \subset X$ such that $z_n \in K_n, n = 1, 2, \ldots$, and $z_n \to z$ in X as $n \to \infty$.

(5) We denote by χ_Ω the characteristic function of Ω on $\hat{\Omega}$ for any subset Ω of $\hat{\Omega}$.

Our shape optimization problem is considered for any non-empty subset O_c of O which is compact in the following sense:

$$(C)\begin{cases} \text{For any sequence } \{\Omega_n\} \subset O_c \text{ there is a subsequence } \{\Omega_{n_k}\} \text{ of } \{\Omega_n\} \text{ with } \Omega \in O_c \\ \text{such that } \chi_{\Omega_{n_k}} \to \chi_\Omega \text{ in } L^1(\hat{\Omega}) \text{ as } k \to \infty \text{ and } V(\Omega_{n_k}) \to V(\Omega) \text{ in } H_0^1(\hat{\Omega}) \\ \text{as } k \to \infty \text{ (in the sense of Mosco).} \end{cases}$$

EXAMPLE 1.1. (1) Let Θ be the class of all C^1-diffeomorphisms from $\overline{\hat{\Omega}}$ onto itself. Now, let Ω' be a subdomain of $\hat{\Omega}$ with smooth boundary $\partial\Omega'$ and $\overline{\Omega'} \subset \hat{\Omega}$. Given a non-empty compact subset Θ_c of Θ, put $O_c = \{\theta(\Omega'); \theta \in \Theta_c\}$. Then this O_c is compact in the sense of (C).

Let $\{\Omega_n := \theta_n(\Omega')\}$ be any sequence in O_c. Then, by the compactness of Θ_c, there is a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ such that $\theta_{n_k} \to \theta$ in $C^1(\overline{\hat{\Omega}})$ as $k \to \infty$ for some $\theta \in \Theta_c$. We see easily that $\chi_{\Omega_{n_k}} \to \chi_\Omega$, with $\Omega = \theta(\Omega')$, in $L^1(\hat{\Omega})$ as $k \to \infty$. Moreover, $V(\Omega_{n_k}) \to V(\Omega)$ in $H_0^1(\hat{\Omega})$ as $k \to \infty$ (in the sense of Mosco). Indeed, if $z_{k'} \to z$ weakly in $H_0^1(\hat{\Omega})$ as $k' \to \infty$ for a subsequence $\{n_{k'}\}$ and $z_{k'} \in V(\Omega_{n_{k'}})$, then $\widetilde{z_{k'}}(x) = z_{k'}(\theta_{n_{k'}} \circ \theta^{-1}(x)) \in V(\Omega)$ and $\widetilde{z_{k'}} \to z(\theta \circ \theta^{-1}) = z$ weakly in $H_0^1(\hat{\Omega})$, so that $z \in V(\Omega)$. Also, let $z \in V(\Omega)$ and put $z_k(x) := z(\theta \circ \theta_{n_k}^{-1}(x)) \in V(\Omega_{n_k})$. Then, clearly, we have then $z_k \to z$ in $H_0^1(\hat{\Omega})$.

(2) Let $\hat{\Omega} := \{x; |x| < 2\} \subset R^3$, $\Omega_a := \{x; a < |x| < 1\}$ for any $0 < a \le \frac{1}{2}$ and $\Omega := \{x; |x| < 1\}$. Put $O_c := \{\Omega_a; 0 < a \le \frac{1}{2}\} \cup \{\Omega\}$. Then, we see that this subset O_c of O satisfies compactness.

In fact, by [5; Lemma 1.8], the 2-capacity of any singleton is zero. Therefore we see that $V(\Omega_a) \to V(\Omega)$ in $H_0^1(\hat{\Omega})$ in the sense of Mosco as $a \to 0$. In the other hand, by the same argument as in (1), we obtain that $V(\Omega_{a'}) \to V(\Omega_a)$ in $H_0^1(\hat{\Omega})$ in the sense of Mosco as $a' \to a$. Hence O_c satisfies property (C). It is easy to see that this O_c can not be represented in the form of (1), since there is no C^1-diffeomorphism between Ω_a and Ω. ◇

Now, we give the weak formulation of $SP(\Omega)$.

DEFINITION 1.1. A function $u : [0, T] \to L^2(\Omega)$ is called a weak solution of $SP(\Omega)$, if the following conditions (w1) - (w3) are satisfied:

(w1) $u \in C_w([0,T]; L^2(\Omega))$, $u(0) = u_0$;

(w2) $\beta(u) \in L^2(0, T; H^1(\Omega))$ and $\beta(u) - g \in L^2(0, T; H_0^1(\Omega))$;

(w3) $-\int_{Q(\Omega)} u\eta_t dxdt + \int_0^T a_\Omega(\beta(u), \eta)dt = \int_{Q(\Omega)} f\eta dxdt$
for all $\eta \in L^2(0, T; H_0^1(\Omega))$ with $\eta_t \in L^2(Q(\Omega))$ and $\eta(0, \cdot) = \eta(T, \cdot) = 0$.

Now, we consider a shape optimization problem. For a given non-empty subset O_c of O, our optimization problem, denoted by $P(O_c)$, is formulated as follows:

$$P(O_c) \qquad \Omega_* \in O_c;\ J(\Omega_*) = \inf_{\Omega \in O_c} J(\Omega),$$

where

$$(1.2) \qquad J(\Omega) = \frac{1}{2}\int_{Q(\Omega)} |\ \beta(u_\Omega) - \beta_d\ |^2\, dxdt + \frac{1}{2}\int_{\hat{Q}-Q(\Omega)} |\ g\ |^2\, dxdt \quad \text{for } \Omega \in O,$$

u_Ω being the weak solution of $SP(\Omega)$, and β_d is a given function in $L^2(\hat{Q})$.

The main results are stated in the following theorems.

THEOREM 1.1. *Let $\{\Omega_n\} \subset O$ and $\Omega \in O$ such that $V(\Omega_n) \to V(\Omega)$ in X as $n \to \infty$ (in the sense of Mosco) and $\chi_{\Omega_n} \to \chi_\Omega$ in $L^1(\hat{\Omega})$ as $n \to \infty$. Also, denote by u_n and u the weak solutions of $SP(\Omega_n)$ and $SP(\Omega)$, respectively. Then, as $n \to \infty$,*

$$(1.3) \qquad (u_n(t), z)_{\Omega_n} \to (u(t), z)_\Omega \quad \text{for any } z \in H,\ t \in [0,T]$$

and

$$(1.4) \qquad \tilde{\beta}(u_n) \to \tilde{\beta}(u) \quad \text{in } L^2(\hat{Q}),$$

where

$$\tilde{\beta}(u_{\Omega'}) = \begin{cases} \beta(u_{\Omega'}) & \text{in } Q(\Omega') \\ g & \text{in } \hat{Q} - Q(\Omega') \end{cases} \quad \text{for any } \Omega' \in O.$$

THEOREM 1.2. *Problem $P(O_c)$ has at least one solution Ω_*.*

2.Uniform Estimates for $SP(\Omega)$

In this section, we prove the uniform estimates for weak solutions to $SP(\Omega)$ with respect to Ω. For this purpose, given $\Omega \in O$, we consider a function $j_\Omega : [0,T] \times H^{-1}(\Omega) \to R$ which is defined in the following way: for each $\Omega \in O$ and $t \in [0,T]$, we put

$$(2.1) \qquad j_\Omega(t,z) = \begin{cases} \int_\Omega \hat{\beta}(z(x))dx - (g(t), z)_\Omega & \text{for } z \in L^2(\Omega), \\ +\infty & \text{for } z \in H^{-1}(\Omega) - L^2(\Omega), \end{cases}$$

where $\hat{\beta}$ is the primitive of β with $\hat{\beta}(0) = 0$, i.e.

$$(2.2) \qquad \hat{\beta}(r) = \int_0^r \beta(s)ds \qquad \text{for } r \in R.$$

j_Ω is proper lower semicontinuous and convex on $H^{-1}(\Omega)$ and we see (c.f. [1; Proposition 2.6]) that the subdifferential $\partial j_\Omega(t,\cdot)$ in $H^{-1}(\Omega)$ is represented by

$$(2.3) \qquad \begin{aligned} &\partial j_\Omega(t,z) = F_\Omega(\beta(z) - g(t)) \\ &\text{for any } z \in D(\partial j_\Omega) = \{z \in L^2(\Omega); \beta(z) - g(t) \in H_0^1(\Omega)\}. \end{aligned}$$

According to [1; Theorem 2.1], problem $SP(\Omega)$ has one and only one solution u such that $u \in W^{1,2}(0,T;H^{-1}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and $\beta(u)-g \in L^2(0,T;H_0^1(\Omega))$. In fact, the weak solution u is obtained as a unique solution of the following evolution problem in $H^{-1}(\Omega)$:

$$(2.4) \qquad \begin{cases} u'(t) + F_\Omega(\beta(u(t)) - g(t)) = f(t) + \Delta g(t) & \text{for a.e. } t \in [0,T], \\ u(0) = u_0. \end{cases}$$

We show some uniform estimates for weak solutions of $SP(\Omega)$ with respect to $\Omega \in O$.

LEMMA 2.1 *There exists a positive constant $M_1 > 0$ such that*

$$(2.5) \qquad |\, u_\Omega \,|_{L^\infty(0,T;L^2(\Omega))} \le M_1, \quad |\, \beta(u_\Omega) \,|_{L^2(0,T;H^1(\Omega))} \le M_1$$

for all $\Omega \in O$, where u_Ω is the weak solution of $SP(\Omega)$.

Proof. By (2.4),

$$(2.6) \qquad \begin{aligned} &(u'_\Omega(t), \beta(u_\Omega(t)) - g(t))_\Omega \\ &= -(F_\Omega(\beta(u_\Omega(t)) - g(t)), \beta(u_\Omega(t)) - g(t))_\Omega + (f(t) + \Delta g(t), \beta(u_\Omega(t)) - g(t))_\Omega \\ &= -a_\Omega(\beta(u_\Omega(t)) - g(t), \beta(u_\Omega(t)) - g(t)) + (f(t) + \Delta g(t), \beta(u_\Omega(t)) - g(t))_\Omega. \end{aligned}$$

The function $t \mapsto j_\Omega(t; u(t))$ is absolutely continuous on $[0,T]$ and we have (cf. [4; (3.5) in the proof of Lemma 4])

$$(2.7) \qquad \frac{d}{dt} j_\Omega(t; u_\Omega(t)) = (u'_\Omega(t), \beta(u_\Omega(t)) - g(t))_\Omega - (g'(t), u_\Omega(t))_\Omega \quad \text{for a.e. } t \in [0,T].$$

By (2.6) and (2.7), we obtain that

$$\begin{aligned} &\frac{d}{dt}\{\int_\Omega \tilde\beta(u_\Omega(t))dx - (u_\Omega(t), g(t))_\Omega\} + \frac{1}{2} |\, \nabla(\beta(u_\Omega(t))) - g(t)) \,|_\Omega^2 \\ &\le C_1\{\int_\Omega \tilde\beta(u_\Omega(t))dx - (u_\Omega(t), g(t))_\Omega\} + C_2\{|\, g(t) \,|^2_{H^1(\Omega_0)} + |\, g'(t) \,|^2_H + |\, \Delta g(t) \,|^2_H + |\, f(t) \,|^2_H\}. \end{aligned}$$

By Gronwall's inequality, we derive (2.5). ◇

LEMMA 2.2 *There exists a positive constant $M_2 > 0$ such that*

$$(2.8) \qquad |\, t^{1/2} \frac{d}{dt} \beta(u_\Omega) \,|_{L^2(0,T;L^2(\Omega))} \le M_2, \quad |\, t^{1/2} \beta(u_\Omega) \,|_{L^\infty(0,T;H^1(\Omega))} \le M_2$$

for all $\Omega \in O$, where u_Ω is the weak solution of $SP(\Omega)$.

Proof. As was seen in [1], problem $SP(\Omega)$ is able to be approximated by non-degenerated problem $SP(\Omega)^\varepsilon$, $\varepsilon \in (0,1]$:

$$SP(\Omega)^\varepsilon \begin{cases} u_t - \Delta\beta^\varepsilon(u) = f & \text{in } Q(\Omega), \\ u(0,\cdot) = u_0 & \text{in } \Omega, \\ \beta^\varepsilon(u) = g & \text{on } \Sigma(\Omega), \end{cases}$$

where $\beta^\varepsilon(r) = \beta(r) + \varepsilon r, r \in R$.

In fact, this problem has one and only one weak solution $u^\varepsilon \in C([0,T]; L^2(\Omega))$ such that

$t^{1/2}\frac{d}{dt}\beta^\varepsilon(u^\varepsilon) \in L^2(Q(\Omega))$ and $\beta^\varepsilon(u^\varepsilon) \in L^2(0,T;H^1(\Omega))$, and besides $u^\varepsilon \to u_\Omega$ in $C_w([0,T];L^2(\Omega))$ and $\beta^\varepsilon(u^\varepsilon) \to \beta(u_\Omega)$ weakly in $L^2(0,T;H^1(\Omega))$, as $\varepsilon \to 0$. We note that there is a positive constant C' independent of ε and Ω such that

$$\sup_{0\le t\le T} | u^\varepsilon(t) |^2_{L^2(\Omega)} + \int_0^T | \nabla(\beta^\varepsilon(u^\varepsilon)) |^2_{L^2(\Omega)}\, dt \le C' . \tag{2.9}$$

In fact, (2.9) is obtained in a similar way to the proof of Lemma 2.1. Moreover, multiply both sides of $u_t - \Delta\beta^\varepsilon(u^\varepsilon) = f$ by $t\frac{d}{dt}(\beta^\varepsilon(u^\varepsilon) - g)$ and integrate over $Q(\Omega)$. Then, by (2.9), we have

$$| t^{1/2}\beta^\varepsilon(u^\varepsilon) |_{L^\infty(0,T;H^1(\Omega))} \le C'',\quad | t^{1/2}\frac{d}{dt}\beta^\varepsilon(u^\varepsilon) |_{L^2(0,T;L^2(\Omega))} \le C'', \quad \text{for any } \varepsilon \in (0,1] \text{ and } \Omega \in O, \tag{2.10}$$

where C'' is a constant independent of $\varepsilon \in (0,1]$ and $\Omega \in O$. Therefore, letting $\varepsilon \to 0$, we see that (2.8) holds. ◇

3.Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Put

$$v_n = \begin{cases} \beta(u_n) & \text{in } Q_n := Q(\Omega_n), \\ g & \text{in } \hat{Q} - Q_n. \end{cases}$$

Consider a function $u_g \in L^\infty(0,T;H)$ such that $g(t,x) = \beta(u_g(t,x))$ on $\hat{Q}$. Here, put

$$\tilde{u}_n = \begin{cases} u_n & \text{in } Q_n, \\ u_g & \text{in } \hat{Q} - Q_n. \end{cases}$$

Then, we see that $\tilde{u}_n \in L^\infty(0,T;H)$. By Lemmas 2.1 and 2.2, there exists a sequence $\{n_k\}$ of $\{n\}$ and $\tilde{u} \in L^\infty(0,T;H)$ such that

$$\begin{cases} \tilde{u}_{n_k} \to \tilde{u} & \text{weakly* in } L^\infty(0,T;H), \\ v_{n_k} \to v & \text{weakly in } L^2(0,T;H^1(\hat{\Omega})), \\ v_{n_k}(t) \to v(t) & \text{weakly in } H^1(\hat{\Omega}) \text{ for all } t \in (0,T]. \end{cases} \tag{3.1}$$

By Ascoli-Arzela's theorem, we see that $v_{n_k} \to v$ in $C_{loc}((0,T];H)$. By using Lemmas 2.1 and 2.2, we easily verify that $v_{n_k} \to v$ in $L^2(0,T;H)$. Since $v_{n_k} = \beta(\tilde{u}_{n_k})$ in $\hat{Q}$ and (3.1), we see that $v = \beta(\tilde{u})$ and that $\beta(\tilde{u}(t)) - g(t) \in V(\Omega)$ for any $t \in (0,T]$.

Next, let z be any function in $V(\Omega)$ and ρ be any function in $D(0,T)$. By the assumption, there exists a sequence $\{z_n\}$ such that $z_n \in V(\Omega_n)$ and $z_n \to z$ in X. Then, by $z_{n_k} = 0$ a.e. on $\hat{\Omega} - \Omega_{n_k}$, we obtain

$$-\int_0^T (\tilde{u}_{n_k}, z_{n_k})\rho' dt + \int_0^T a(v_{n_k}, z_{n_k})\rho dt = \int_0^T (f, z_{n_k})\rho dt.$$

Letting $k \to \infty$ since $z = 0$ on $\hat{\Omega} - \Omega$, we see

$$-\int_0^T (\tilde{u}, z)_\Omega \rho' dt + \int_0^T a_\Omega(v, z)\rho dt = \int_0^T (f, z)_\Omega \rho dt.$$

This shows that $u = \tilde{u}\,|_{Q(\Omega)}$ is the solution of $SP(\Omega)$. By the uniqueness, we obtain (1.4). $\diamond$

Proof of THEOREM 1.2. Let $\{\Omega_n\}$ be a sequence in O_c such that $J(\Omega_n) \to J_* :=$ $\inf\{J(\Omega); \Omega \in O_c\}$. Then, by assumption, we may assume that $V(\Omega_n) \to V(\Omega_*)$ in X (in the sense of Mosco) for some $\Omega_* \in O_c$ and $\chi_{\Omega_n} \to \chi_{\Omega_*}$ in $L^1(\hat{\Omega})$ as $n \to \infty$. Now, denote by u_n the weak solution of $SP(\Omega_n)$ and by u_* the weak solution of $SP(\Omega_*)$. Then put

$$v_n = \begin{cases} \beta(u_n) & \text{in } Q_n = Q(\Omega_n), \\ g & \text{in } \hat{Q} - Q_n, \end{cases}$$

and

$$v = \begin{cases} \beta(u_*) & \text{in } Q = Q(\Omega_*), \\ g & \text{in } \hat{Q} - Q. \end{cases}$$

From Theorem 1.1, it follows that $v_n \to v$ in $L^2(0,T;H)$ and hence $J(\Omega_n) \to J(\Omega_*)$. Therefore $J(\Omega_*) = J_*$ and Ω_* is a solution of $P(O_c)$. $\diamond$

4.Approximation for $P(O_c)$

In this section, from some numerical points of view, we discuss approximations of $SP(\Omega)$ and $P(O_c)$ by smooth problems.

Let $\{\beta^\epsilon\} = \{\beta^\epsilon; 0 < \epsilon \le 1\}$ be a family of (smooth) functions $\beta^\epsilon : R \to R$ such that

$$\begin{cases} \beta^\epsilon(0) = 0,\ |\beta^\epsilon(r) - \beta(r)| \le \epsilon(|r| + 1) & \text{for all } r \in R, \\ |\beta^\epsilon(r) - \beta^\epsilon(r')| \le \tilde{L}_0 |r - r'| & \text{for all } r, r' \in R, \\ \dfrac{d}{dr}\beta^\epsilon(r) \ge \epsilon & \text{for a.e. } r \in R, \end{cases}$$

where $\tilde{L}_0 > 0$ is constant independent of ϵ.

Next, let $\{\chi_\Omega^\nu\} = \{\chi_\Omega^\nu; 0 < \nu \le 1, \Omega \in O_c\}$ be a family of smooth functions on $\hat{\Omega}$ and suppose the following conditions $(\chi1) - (\chi3)$ hold :

$(\chi1)$ $0 \le \chi_\Omega \le \chi_\Omega^\nu \le 1$ in $\hat{\Omega}$ and supp $(\chi_\Omega^\nu) \subset \{x \in \hat{\Omega}; dist(x, \Omega) \le \nu\}$ for any $\nu \in (0,1]$ and $\Omega \in O_c$.

$(\chi2)$ For each $\nu \in (0,1]$, $\{\chi_\Omega^\nu; \Omega \in O_c\}$ is compact in $L^1(\hat{\Omega})$.

$(\chi3)$ If $\nu_n \in (0,1]$, $\nu_n \to 0$ for $n \to \infty$ and $\Omega_n \in O_c$, then there are a subsequence $\{n_k\}$ and $\Omega \in O_c$ such that $\chi_{\Omega_{n_k}}^{\nu_{n_k}} \to \chi_\Omega$ in $L^1(\hat{\Omega})$ as $k \to \infty$.

Now, we consider approximate problem $SP(\Omega)^{\epsilon\nu\mu}$, $\epsilon, \nu, \mu \in (0,1]$, for $SP(\Omega)$:

$$SP(\Omega)^{\epsilon\nu\mu} \begin{cases} u_t - \Delta\beta^\epsilon(u) = f - \dfrac{1-\chi_\Omega^\nu}{\mu}(\beta^\epsilon(u) - g) & \text{in } \hat{Q}, \\ u(0,\cdot) = u_0 & \text{in } \hat{\Omega}, \\ \beta^\epsilon(u) = g & \text{on } \hat{\Sigma}. \end{cases}$$

DEFINITION 4.1 A function $u : [0,T] \to H$ is called a solution of $SP(\Omega)^{\epsilon\nu\mu}$, if the following conditions $(aw1)$ – $(aw3)$ are satisfied:

$(aw1)$ $u \in C([0,T];H) \cap W^{1,2}_{loc}((0,T];H) \cap L^2(0,T;H^1(\hat{\Omega}))$, $u(0) = u_0$ in $\hat{\Omega}$;

$(aw2)$ $\beta^\epsilon(u(t)) - g(t) \in X$ for a.e. $t \in [0,T]$;

$(aw3)$ $\langle u'(t), z\rangle_{\hat{\Omega}} + a(\beta^\epsilon(u(t)), z) = (f(t) - \frac{1-\chi^\nu_\Omega}{\mu}(\beta^\epsilon(u(t)) - g(t)), z)$

for any $z \in X$, a.e. $t \in [0,T]$.

According to [3; Chapter 2], problem $SP(\Omega)^{\epsilon\nu\mu}$ has a unique solution u.

Our approximate optimization problem $P(O_c)^{\epsilon\nu\mu}$, associated with $SP(\Omega)^{\epsilon\nu\mu}$, is formulated as follows:

$$P(O_c)^{\epsilon\nu\mu} \quad \Omega^{\epsilon\nu\mu}_* \in O_c;\ J^{\epsilon\nu\mu}(\Omega^{\epsilon\nu\mu}_*) = \inf_{\Omega \in O_c} J^{\epsilon\nu\mu}(\Omega),$$

where $u^{\epsilon\nu\mu}_\Omega$ is the solution of $SP(\Omega)^{\epsilon\nu\mu}$ and

$$J^{\epsilon\nu\mu}(\Omega) = \frac{1}{2}\int_{\hat{Q}} \chi^\nu_\Omega \mid \beta^\epsilon(u^{\epsilon\nu\mu}_\Omega) - \beta_d \mid^2 dxdt + \frac{1}{2}\int_{\hat{Q}} (1-\chi^\nu_\Omega) \mid g \mid^2 dxdt.$$

THEOREM 4.1. *(1) For each $\epsilon, \nu, \mu \in (0,1]$, $P(O_c)^{\epsilon\nu\mu}$ has at least one solution.*

(2) Let $\{\epsilon_n\}, \{\nu_n\}, \{\mu_n\}$ be null sequences and let $\{\Omega_n\} \subset O_c$ and $\Omega \in O_c$ such that $V(\Omega_n) \to V(\Omega)$ in X as $n \to \infty$ (in the sense of Mosco), $\chi^{\nu_n}_{\Omega_n} \to \chi_\Omega$ in $L^1(\hat{\Omega})$ as $n \to \infty$. Denote by u_n the solution of $SP(\Omega_n)^{\epsilon_n\nu_n\mu_n}$. Then,

$$\begin{cases} \chi_{\Omega_n} \cdot u_n \to \chi_\Omega \cdot u & \text{weakly* in } L^\infty(0,T;H) \text{ as } n \to \infty, \\ \beta^{\epsilon_n}(u_n) \to v & \text{in } L^2(0,T;H) \text{ and weakly in } L^2(0,T;H^1(\hat{\Omega})) \text{ as } n \to \infty, \\ v = \beta(u) & \text{in } Q = (0,T) \times \Omega, \\ v = g & \text{in } \hat{Q} - Q. \end{cases}$$

and u is the weak solution of $SP(\Omega)$. In particular, if Ω_n is a solution of $P(O_c)^{\epsilon\nu\mu}$ with $\epsilon = \epsilon_n, \nu = \nu_n$ and $\mu = \mu_n$ for $n = 1, 2, \ldots$, then Ω is a solution of $P(O_c)$.

Proof. See [2] for details. ◇

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非線形保存則の解に対する一注意

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§1. 序

まず最初に初期値問題

$$(1)\quad \begin{cases} u_t + uu_x = 0 & (t,x)\in[0,\infty)\times\mathbb{R} \\ u(0,x) = h(x) & x\in\mathbb{R} \end{cases}$$

を考えよう。ただし、$h(x) = \begin{cases} 1 & x\le 0 \\ 0 & x>0 \end{cases}$ である。

(1)の解としては、弱解を考えるのがふつうである。ここで弱解とは

$$(2)\quad \begin{cases} -\displaystyle\int_0^{+\infty}\int_{-\infty}^{+\infty}\{u\varphi_t + \tfrac{1}{2}u^2\varphi_x\}\,dx\,dt = \int_{-\infty}^{+\infty} h(x)\varphi(0,x)\,dx \\ \qquad\qquad \text{for } \forall\varphi\in C_0^\infty([0,+\infty)\times\mathbb{R}) \end{cases}$$

を満たす $u(t,x)\in L^\infty_{loc}([0,\infty)\times\mathbb{R})$ のこととする。(1)に対して(2)の意味の弱解は無数に存在する。最も標準的なものとしては、

$$u(t,x) = \begin{cases} 1 & x\le \frac{1}{2}t \\ 0 & x>\frac{1}{2}t \end{cases}$$

があるが、$a>0$、$b>1$として

$$u(t,x) = \begin{cases} 1 & x\le \frac{1-a}{2}t \\ -a & \frac{1-a}{2}t < x \le \frac{b-a}{2}t \\ b & \frac{b-a}{2}t < x \le \frac{b}{2}t \\ 0 & x > \frac{b}{2}t \end{cases}$$

とすれば、これは(2)をみたしており、a, bを変えると解を無数に作ることができる。従って(2)の意味の弱解を考える場合は、何らかの付加条件を与えなければ、一意性が成り立たない。通常エントロピー条件というものを導入し、一意性を保証する。このエントロピー解u_Eの利点は、

$$
(3)\begin{cases} u_t + \varepsilon u_{xx} + u u_x = 0 \\ u(0,x) = h(x) \end{cases}
$$

の解を $u(t,x,\varepsilon)$ とすると

$$
\lim_{\varepsilon \downarrow 0} u(t,x,\varepsilon) = u_E(t,x)
$$

を満す点であり、悪い点は初期値に連続函数を与えても解uが連続にならない点である。

ここでは、(3)の解の極限にはなっていないが、連続函数は時間発展ののちも連続函数であるような解を考えたいと思う。ただし、この場合解は多価函数となる。

$h(x)$を次のような多価函数と考える。(図1)

$$
h(x) = \begin{cases} 1 & x < 0 \\ \{0 \le s \le 1\} & x = 0 \\ 0 & x > 0 \end{cases}
$$

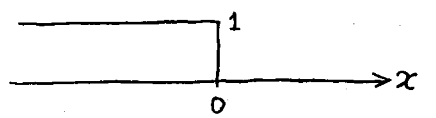

図 1

$u(t,x)$を

$$
(4)\quad u(t,x) = \begin{cases} 1 & x < 0 \\ 0,\ 1 & x = 0 \\ 0,\ x/t,\ 1 & 0 < x < t \\ 0,\ 1 & x = t \\ 0 & x > t \end{cases}
$$

$t = t_0$での切り口

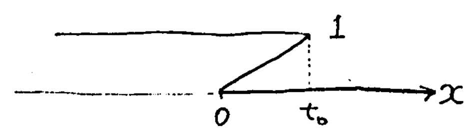

図 2

とおくと(図2)、$u(t,x)$は $x=0$, $x=t$を除いて(1)をみたしている。私がこの小文で主張したいのは、非線形双曲型方程式の解を考える場合に、弱解ばかりを考えるのではなく、(4)のような解を考えることも必要なのではないかということである。次の節で、(4)のような解の作り方と性質について述べたいと思う。

§2. 解の構成とその性質

(5)
$$\begin{cases} u_t + f'(u)u_x = 0 & \text{in } (0,\infty)\times\mathbb{R} \\ u(0,x) = u_0(x) & x\in\mathbb{R} \end{cases}$$

を考える。ここで $f\in C^\infty(\mathbb{R})$ とする。今、独立変数が (t,x) で従属変数が u であるが、これを独立変数 (t,u) 従属変数 x と変換しよう。新独立変数を $v=u(t,x(t,v))$ とおき、両辺を t に関し偏微分すると

$$0 = u_t + u_x x_t$$

を得る。(5)を満たしていることから

$$0 = -f'(u)u_x + u_x x_t$$

したがって新しい方程式

(6)
$$x_t = f'(u)$$

を得る。これは容易に積分できて

$$x(t,u) - x(0,u) = f'(u)t$$

$x(0,u) = u_0^{-1}(u)$ だから 3変数 (t,u,x) の関係式

(7)
$$x - u_0^{-1}(x) = f'(u)t$$

または

(8)
$$u_0\big(x - f'(u)t\big) = u$$

を得る。解 $u(t,x)$ は (7) または (8) を満たす陰函数として定義する。

$$X = \{x(s) \mid x(s) : (-\infty,\infty)\to\mathbb{R}^2,\ s\text{ に関し絶対連続}\}$$

$$X_0 = \{x(s) \mid x(s) : (0,1)\to\mathbb{R}^2,\ s\text{ に関し絶対連続、 } x(0)=x(1)\}$$

$\mathcal{U}(t)$: (5) に対する前ページの意味での発展作用素

即ち、$u_0 \in X$ (or X_0) に対し、

$$\mathcal{U}(t)u_0 = u_0(x - f'(\mathcal{U}(t)u_0)t)$$

を満たすものとする。そのとき、次が成り立つ。

<u>Prop. 1</u>　$\mathcal{U}(t)$ は X (or X_0) 上 group である。

<u>Prop. 2</u>　$x \in X_0$, $m(x) := \{x$ で囲まれる領域の面積$\}$

とすると

$$m(x) = m(\mathcal{U}(t)x)$$

(注意 1) 今、t はいつも独立変数と考えているが、x や u はそのつど、独立変数と思ったり、従属変数と思ったりしているので、(5) は u を従属変数、x を独立変数としたときの局所的な表示と考えなければならない。

(注意 2) このようにして作った解の singularity は、線形のときと同様の伝播の法則に従い、新しい singularity が生じたりはしない。

Prop. 1, 2 は解の作り方から、容易に証明することができる。

同様のことは、$u = {}^t(u_1(t,x), u_2(t,x), \cdots, u_n(t,x))$, $x \in \mathbb{R}^n$ の方程式

$$\begin{cases} u_t + f(u)\cdot\nabla u = 0 \\ u(0,x) = u_0(x) \end{cases}$$

の場合にも成立する。

ただし、

$$f(u) = (f^1(u), \ldots, f^n(u))$$

$$f_i \in C^\infty(\mathbb{R}^n) \quad \text{for } 1 \le i \le n$$

である。この場合

$$u(t,x) = u_0(x - f(u)t)$$

となる。

以上、tは本質的な独立変数であるが、uとxについては、(u,x)空間の1方向を従属変数とみるという考え方に従って論を進めてきた。高階の準線形双曲型方程式においても、同様の考え方が役に立つのではないかと思っているが、まだ、適当な例を見つけていない。

§3 おわびと言い訳

講演を申し込んだ時点では、「係数にsingularityをもつ波動方程式の解のなめらかさについて」という題で講演する予定でした。この話題に突如変えたのは、「係数に…」はこのセミナーにあわないのではないか、そして、多少ゴチャゴチャしているので夏向きではないのではないかという理由からです。非線形保存則を主題に選んだのは、この問題に対する弱解に異和感を感じていたからです。

しかし、この選択は失敗でした。講演内容が、雑談に近いものになってしまったからです。そのため、この文をまとめる作業も、何とか新しい内容を付加しようと思い遅れてしまいました。結局、新しい内容を付加することができず、

いたずらに原稿が遅れるという結果に終ってしまいました。
この場を借りて おわびします。(梶木屋さん どうもすみません。)

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Large Time Behavior of Solutions of Quasi-linear Heat Conduction Equations

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We consider the large time behavior of weak solutions of the following initial-boundary value problem:

$$
\text{(I)} \quad \begin{cases} u_t = \Delta\phi(u) & \text{in} \quad \Omega \times \mathbf{R}^+, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times \mathbf{R}^+, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases}
$$

Here Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. We assume that

(A1) $\phi \in C^1(\mathbf{R})$ and there exists a constant $K_0 > 0$ such that $k(r) = \phi'(r) \geq K_0$ for any $r \in \mathbf{R}$,

(A2) $k(0) = 1$ (for simplicity).

In this situation, equations related to (I) have been studied by [1], [3], [5] and others. In what follows, $\|\cdot\|_p$ denotes the norm of $L^p(\Omega)$ and we denote by $\{\lambda_\nu\}_{\nu=1}^{\infty}$ $(0 < \lambda_1 < \lambda_2 < \cdots)$ all eigenvalues of $-\Delta$ with zero-Dirichlet condition and by P_j the orthogonal projection of the eigenspace of λ_j. It is known that in large time solutions of (I) behave like those of the linear equation: $u_t = \Delta u$. Berryman and Holland [1] and Nagasawa [5] considered the behavior of classical solutions of one-dimensional equations related to (I). In [1] Berryman and Holland first obtained the asymptotic profile of solutions of (I). Let ϕ be sufficiently smooth. They proved that classical solutions of (I) satisfies

$$e^{\lambda_1 t}u(t) \underset{t\to\infty}{\longrightarrow} Ae_1 \quad \text{in} \quad H_0^1(\Omega), \tag{1}$$

where $A \in \mathbf{R}$ is a constant depending on u_0 (and $A > 0$ if $u_0 \geq 0$, $\not\equiv 0$ in Ω), and $e_1 > 0$ is the normalized eigenfunction of $-\Delta$ associated with λ_1. In [5] Nagasawa obtained an estimate for the rate of convergence and an asymptotic formula on $P_1 u(t)$ by using the following expression of A in (1):

$$A = e^{\lambda_1 t}(u(t), e_1) - \lambda_1 \int_t^\infty e^{\lambda_1 s}(\phi(u(s)) - u(s), e_1)ds \quad \text{for} \quad \text{any} \quad t \geq 0. \tag{2}$$

Let $k'(0) = \phi''(0)$ exist. According to [5], we have

$$\|u(t) - Ae^{-\lambda_1 t}e_1\|_{H_0^1} \leq C\exp[-\min\{\lambda_2 - \varepsilon, 2\lambda_1\}t] \quad \text{for} \quad t \geq 0, \tag{3}$$

$$P_1 u(t) = Ae^{-\lambda_1 t}e_1 + \frac{A^2 k'(0)}{2}e^{-2\lambda_1 t}P_1 e_1^2 + o(e^{-2\lambda_1 t}), \tag{4}$$

where $C > 0$ is a constant and $\varepsilon > 0$ is any small constant. The author [3] studied the behavior for weak solutions of multi-dimensional equations related to (I). By [3] we see that the following condition:

(5) there exist constants θ, $\rho > 0$ such that $|k(r) - 1| \leq \theta/(-\log|r|)^{1+\rho}$ for $r \in (-1, 1)$

is a sufficient and an almost necessary condition for (1), and also that the corresponding results hold for the Neumann problem. We remark that in fact equations of more general form than (I) are considered in [3] and [5].

However, we can not see fully from the above result how *any* solution behave in large time. Indeed, if $A = 0$, (1) and (3) do not give sufficient information on large time behavior. And there are, as in the linear case, infinitely many solutions with $A = 0$. In this note we intend to find the asymptotic profile for *every* weak solution and to establish precise estimates for the rate of convergence. In what follows we shall give our results for the following more general problem:

$$\text{(II)} \begin{cases} u_t = \Delta\phi(u) - f(u) & \text{in} \quad \Omega \times \mathbf{R}^+, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times \mathbf{R}^+, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases}$$

Here we assume that

(A3) There exist $a > 0$ and $\alpha \in (0, \infty)$ such that $|k(r) - 1| \leq a|r|^{\alpha}$ for any $r \in \mathbf{R}$,

(A4) $f : \mathbf{R} \to \mathbf{R}$ is a locally Lipschitz continuous function satisfying that $f(0) = 0$ and that there exist constants $b \geq 0$ and $p > 1$ such that $0 \leq rf(r) \leq b|r|^{p+1}$ for $r \in \mathbf{R}$,

(A5) $u_0 \in L^{\infty}(\Omega)$ and u_0 does not identically vanish in Ω.

We assume throughout the assumptions (A1-5), to which we shall refer collectively as assumption (A). We shall define weak solutions.

Definition. A weak solution u of (II) on $\mathbf{R}^+$ is a locally Hölder continuous function in $\Omega \times \mathbf{R}^+$ with the properties:

(i) $u(x,t) \in L^{\infty}(\Omega \times \mathbf{R}^+)$,

(ii) $$\int_{\Omega} \{u_0(x)\eta(x,0) - u(x,T)\eta(x,T)\} dx + \int_0^T dt \int_{\Omega} \{u\eta_t + \phi(u)\Delta\eta - f(u)\eta\} dx$$

for any $T > 0$ and for any $\eta \in C^2(\overline{\Omega} \times [0,T])$ such that $\eta(x,t) = 0$ on $\partial\Omega \times [0,T]$.

Proposition 1. *We assume* (A). *Then* (II) *has a unique weak solution* u.

Our main result is as follows:

Theorem 1. *We assume* (Λ). *Let* u *be the weak solution of* (II).

(i) *There exist* $m \in \mathbf{N}$ *and a non-zero eigenvector* ω_m *of* $-\Delta$ *associated with* λ_m *satisfying*

$$(6) \qquad e^{\lambda_m t} u(t) \underset{t\to\infty}{\longrightarrow} \omega_m \quad in \quad H_0^1(\Omega).$$

(ii) *More precisely, let* $\kappa = \min\{\alpha+1, p\}$ *and* $n \in \mathbf{N} \cup \{0\}$ *such that* $\lambda_{m+n} < \kappa\lambda_m \leq \lambda_{m+n+1}$. *Then, also for each* $m < j \leq m+n$, *there exists eigenvector* ω_j *of* $-\Delta$ *associated with* λ_j *satisfying*

$$(7) \qquad u(t) - \sum_{j=m}^{m+n} e^{-\lambda_j t}\omega_j = \begin{cases} O(e^{-\kappa\lambda_m t}) & if \quad \kappa\lambda_m < \lambda_{m+n+1} \\ O(te^{-\kappa\lambda_m t}) & if \quad \kappa\lambda_m = \lambda_{m+n+1} \end{cases} \quad in \quad H_0^1(\Omega).$$

To obtain (6), we apply an iteration argument with the important expression corresponding to (2) and the eigenfunction expansion associated with $-\Delta$. The calculations to establish (7) are based on [5]. The most difficult step in the proof of Theorem 1 is the following Proposition, which is obtained by deriving

$$\limsup_{t\to\infty} \frac{\|\nabla u(t)\|_2}{\|u(t)\|_2} < \infty.$$

Proposition 2. *(A lower estimate)* *We assume* (Λ). *Let* u *be the weak solution of* (II). *Then there exist constants* $C, \gamma > 0$ *depending on given data such that*

$$\|u(t)\|_2 \geq Ce^{-\gamma t} \quad for \quad t \geq 0.$$

We are concerned with a typical example. We see from the following proposition that the estimate (7) is optimal.

Proposition 3. *Let $\phi(r) = r + a|r|^\alpha r$ and $f(r) = b|r|^{p-1}r$, where $a, b, \alpha > 0$ and $p > 1$ are some constants. Assume that $u_0 \in L^\infty(\Omega)$ does not identically vanish in Ω. Let u be the weak solution of (II). Using the same notations as in the statement of Theorem 1, we have the following:*

(i) *Let $\kappa\lambda_m < \lambda_{m+n+1}$. Then,*

$$\lim_{t\to\infty} e^{\kappa\lambda_m t}\Big(u(t) - \sum_{j=m}^{m+n} e^{-\lambda_j t}\omega_j\Big)$$

$$= \begin{cases} \displaystyle\sum_{j=1}^{\infty} \frac{a\lambda_j}{(\alpha+1)\lambda_m - \lambda_j} P_j(|\omega_m|^\alpha \omega_m) & \text{if} \quad \alpha + 1 < p, \\ \displaystyle\sum_{j=1}^{\infty} \frac{a\lambda_j + b}{(\alpha+1)\lambda_m - \lambda_j} P_j(|\omega_m|^\alpha \omega_m) & \text{if} \quad \alpha + 1 = p, \\ \displaystyle\sum_{j=1}^{\infty} \frac{b}{p\lambda_m - \lambda_j} P_j(|\omega_m|^{p-1} \omega_m) & \text{if} \quad \alpha + 1 > p \end{cases}$$

in $L^2(\Omega)$.

(ii) *Let $\kappa\lambda_m = \lambda_{m+n+1}$. Then,*

$$\lim_{t\to\infty} \frac{1}{t} e^{\kappa\lambda_m t}\Big(u(t) - \sum_{j=m}^{m+n} e^{-\lambda_j t}\omega_j\Big)$$

$$= \begin{cases} -a\lambda_{m+n+1} P_{m+n+1}(|\omega_m|^\alpha \omega_m) & \text{if} \quad \alpha + 1 < p, \\ -(a\lambda_{m+n+1} + b) P_{m+n+1}(|\omega_m|^\alpha \omega_m) & \text{if} \quad \alpha + 1 = p, \\ -bP_{m+n+1}(|\omega_m|^{p-1} \omega_m) & \text{if} \quad \alpha + 1 > p \end{cases}$$

in $L^2(\Omega)$.

Remark. 1. Results similar to Theorem 1 hold for the Neumann problem and for the following type of equation: $u_t = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a^{ij}(x,u)\frac{\partial u}{\partial x_j}) - f(u)$.

2. In [2] Foias and Saut obtained some results parallel to ours for Navier-Stokes equation. We briefly mention the main difference between our work and [2]. In our

quasilinear case a lower estimate of $\|u(t)\|_2$ is essentially harder to obtain than in semilinear case. In [2] the result corresponding to (6) is established by proving that $\lambda(t) \equiv \|\nabla u(t)\|_2^2/\|u(t)\|_2^2 \to \lambda_j$ as $t \to \infty$ for some $j \in \mathrm{N}$ and that $\int_0^\infty |\lambda(t) - \lambda_j|dt < \infty$. On the other hand, we have a different simpler approach to (6); we use an iteration argument, as mentioned above.

For the proofs of the above results we refer to [4].

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臨界ソボレフ指数を持つ非線形高階楕円型方程式について

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§1.Introduction

臨界ソボレフ指数を持つ方程式は，Palais-Smale 条件が成立しないため，最近まであまり研究されなかった．しかし，Brezis と Nirenberg は λu という低階の項をつけることにより，解の存在証明に成功した．つまり，彼らは次の方程式を考えたのである．

$$\begin{cases} -\Delta u = |u|^{\frac{4}{n-2}}u + \lambda u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

彼らは $0 < \lambda < \lambda_1$ の場合の解の存在を示した．その後、Capozzi-Fortunato-Palmieri [4] と Zhang [7] により $\lambda_1 \leq \lambda$ の場合も解の存在が示された．

これらの事実に基づきここでは,次の方程式を考える．

(1)
$$\begin{cases} (-\Delta)^k u = |u|^{\frac{4k}{n-2k}}u + \lambda u & in\ \Omega \\ D^\alpha u = 0 \quad (|\alpha| \leq k-1) & on \quad \partial\Omega \end{cases}$$

ここで、Ω は $\mathbb{R}^n$ の有界領域であり，境界はなめらかと仮定する．この方程式の解を次の汎関数の臨界点として捜す．

$$\Phi_\lambda(u) = \frac{1}{2}\int_\Omega |Tu|^2 dx - \frac{\lambda}{2}\int_\Omega |u|^2 dx - \frac{n-2k}{2n}\int_\Omega |u|^{\frac{2n}{n-2k}} dx$$

但し、Tu は次のように定義する．

$$Tu = \begin{cases} \Delta^{\frac{k}{2}}u & k\text{は偶数}. \\ \nabla(\Delta^{\frac{k-1}{2}}u) & k\text{は奇数}. \end{cases}$$

簡単のため，以下では k に依存するものにも添え字 k を書かない．Palais-Smale 条件が成立しない汎関数を考えることになるが，その困難を Brezis -Nirenberg に従って解決する．

§2. 補助定理

まずは，$(-\Delta)^k u = u^{\frac{n+2k}{n-2k}}$ の全域解を求める．このために，次の関数を定める．

$$G_\alpha(x) = \frac{1}{2^\alpha \Gamma(\frac{\alpha}{2})\pi^{\frac{n}{2}}} \int_0^\infty e^{\frac{-|x|^2}{\delta}} e^{\frac{-\delta}{4}} \delta^{\frac{\alpha-n}{2}-1} d\delta \ (\alpha > 0) \tag{2}$$

ここで，$L^1(\mathbf{R}^n)$ に属する関数のフーリェ変換を次で定める．

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} e^{-ix\cdot\xi} f(x) dx$$

このとき，次の命題が使える．

命題 1 (Stein[6]).

$$G_\alpha(x) \in L^1(\mathbf{R}^n) \quad for \ \ \alpha > 0$$
$$\hat{G}_\alpha(\xi) = (1 + |\xi|^2)^{\frac{-\alpha}{2}}$$

証明は，Stein[6] を見よ．

このことから，

$$(|x|^2)^k G_{n-2k}(x) = const. G_{n+2k}(x)$$

がわかる．両辺をフーリェ変換すれば，

$$(-\Delta_\xi)^k \left(\frac{1}{1+|\xi|^2}\right)^{\frac{n-2k}{2}} = const. \left\{\left(\frac{1}{1+|\xi|^2}\right)^{\frac{n-2k}{2}}\right\}^{\frac{n+2k}{n-2k}} \tag{3}$$

となり，

$$U(x) = \frac{c_n}{(1+|\xi|^2)^{\frac{n-2k}{2}}} \tag{4}$$

は，$(-\Delta)^k u = u^{\frac{n+2k}{n-2k}}$ の全域解になることが分かる．ここで，c_n は normalizing constant である．Ambrosetti-Rabinowitz[1] によれば，求めるべき解は次の値の inf を与えるものである．

$$S_\lambda = \inf\{\|Tu\|_2^2 - \lambda\|u\|_2^2 \mid \quad u \in H_0^1(\Omega), \|u\|_{\frac{2n}{n-2k}} = 1\}$$

§3. 最小固有値までの場合

まず，つぎのような数と関数を定義する．

$$S = \inf\{\|Tu\|_2^2 \mid \|u\|_{\frac{2n}{n-2k}} = 1\}$$

$$u_\epsilon(x) = \frac{\varphi(x)}{(\epsilon + |x|^2)^{\frac{n-2k}{2}}}$$

ここで，$\varphi(x) \in C_0^\infty(\Omega), \varphi(x) \equiv 0, 0 \leq \varphi(x) \leq 1$, $\varphi(x)$ は固定する．ϵ は任意の正の数である．S は，$k = 1$ のとき決して達成されない（Pohozaev の恒等式により）が，$k \geq 2$ のときは分からない．しかし，S は scaling に関して不変なのでその値は，$\Omega = \mathbf{R}^n$ の場合と一致する．

注意．

ここでは，$H_0^k(\Omega)$-norm を $\|Tu\|_2$ととってよい．なぜならば，まず Ω が有界領域なのでポアンカレの不等式を使うと，$H_0^k(\Omega)$-norm は $(\Sigma_{|\alpha|=k}\|D^\alpha u\|_2^2)^{\frac{1}{2}}$ と同値である．さらに，プランシュレルの定理により，$\|D^\alpha u\|_2 = \|\xi^\alpha \hat{u}\|_2$ であり

$$(\xi^\alpha)^2 \leq |\xi|^{2k} \quad |\alpha| = k, k\text{は偶数}$$
$$(\xi_i \xi^\beta)^2 \leq \xi_i^2 |\xi|^{2(k-1)} \quad |\beta| = k-1, k\text{は奇数}, i = 1, \cdots, n$$

すると,$\|D^\alpha u\|_2 \leq \|Tu\|_2$を得る．

補題 2．

もし $n \geq 4k+1$, かつ, $0 < \lambda < \lambda_1$ ならば,

$$\|Tu_\epsilon\|_2^2 = \frac{C_1}{\epsilon^{\frac{n-2k}{2}}} + O(1)$$
$$\|u_\epsilon\|_{\frac{2n}{n-2k}}^2 = \frac{C_2}{\epsilon^{\frac{n-2k}{2}}} + O(1)$$
$$\|u_\epsilon\|_2^2 = \frac{C_3}{\epsilon^{\frac{n-4k}{2}}} + O(1)$$

ここで，C_1, C_2, C_3 は定数で，$S = \frac{C_1}{C_2}$ を満たす．条件 $n \geq 4k+1$ は $U(x)$ が $L^2(\mathbf{R}^n)$ に属するためのものである．

証明

$D^\alpha \varphi(x) \equiv 0$ *near* 0 ($|\alpha| \geq 1$) を使い，$U(x)$ が $L^2(\mathbf{R}^n)$ に属することから，$\mathbf{R}^n$ での積分に直して直接計算すればよい．

なお，$n = 4k$ のときは

$$B_1 \subset \Omega \subset B_2$$

なる閉球 B_1, B_2 をとってその上で積分すると補題２を得る．　　　　（証明終わり）

このことを基に，次の補題を得る．

補題３．

もし $n \geq 4k$ ならば，

$$S_\lambda < S$$

証明

$$Q_\lambda(u) = \frac{\|Tu\|_2^2 - \lambda\|u\|_2^2}{\|u\|^2_{\frac{2n}{n-2k}}}$$

とおく．$n \geq 4k+1$ のとき

$$Q_\lambda(u_\epsilon) = S - \frac{C_3}{C_2}\epsilon^k + O(\epsilon^{\frac{n-2k}{2}})$$

が成立し，$n = 4k$のとき

$$Q_\lambda(u_\epsilon) = S - \frac{C_3}{C2}\epsilon^k|log\epsilon| + O(\epsilon^k)$$

が成立する．これらより

$$S_\lambda \leq Q_\lambda(u_\epsilon) < S$$

を得る．　　　（証明終わり）

補題４．

$S_\lambda < S$ のとき S_λ の inf が達成される．

証明

ここで，$v_j = u_j - u$ とおく．すると

$$\begin{cases} v_j \rightharpoonup 0 & weak\ in\ H_0^k(\Omega) \\ v_j \to 0 & strong\ in\ L^2(\Omega)\ \text{かつ}\ a.e.\ on\Omega \end{cases}$$

$u_j = u + v_j$ として，代入すれば

$$\|Tu\|_2^2 + \|Tv_j\|_2^2 - \lambda\|u\|_2^2 = S_\lambda + o(1)\ (j \to \infty)$$

を得る．一方 Brezis-Lieb [2] の補題を使うと

$$\|u+v_j\|_{\frac{2n}{n-2k}}^{\frac{2n}{n-2k}} = \|u\|_{\frac{2n}{n-2k}}^{\frac{2n}{n-2k}} + \|v_j\|_{\frac{2n}{n-2k}}^{\frac{2n}{n-2k}} + o(1) \quad (j \to \infty)$$

を得る．j が十分大きければ，$\|v_j\|_{\frac{2n}{n-2k}} \leq 1$ となるので，

$$1 \leq \|u\|^2_{\frac{2n}{n-2k}} + \|v_j\|^2_{\frac{2n}{n-2k}} + o(1)$$

を得る．S の定義から，

$$1 \leq \|u\|^2_{\frac{2n}{n-2k}} + \frac{1}{S}\|Tv_j\|_2^2 + o(1)$$

となり，両辺に S_λ をかけて

$$S_\lambda \leq S_\lambda \|u\|^2_{\frac{2n}{n-2k}} + \frac{S}{S_\lambda}\|Tv_j\|_2^2 + o(1)$$

また

$$S_\lambda = \|Tu\|_2^2 + \|Tv_j\|_2^2 - \lambda\|u\|_2^2 + o(1)$$

より

$$\|Tu\|_2^2 - \lambda\|u\|_2^2 \leq S_\lambda\|u\|^2_{\frac{2n}{n-2k}} + \frac{S_\lambda - S}{S}\|Tv_j\|_2^2 + o(1)$$

となり，これらから，

$$\|Tu\|_2^2 - \lambda\|u\|_2^2 \leq S_\lambda\|u\|^2_{\frac{2n}{n-2k}}$$

これは，S_λ の定義から u によって inf が達成されていることを示している． （証明終わり）

§4. 最小固有値を越えたとき

このときは，Zhang に従って考える．

$$M_\lambda^{(0)} = \{u \in H_0^k(\Omega) | u \not\equiv 0, <\Phi'_\lambda(u), u> = 0\}$$
$$M_\lambda^{(r)} = \{\Gamma \subset M_\lambda^{(0)} | \Gamma \text{は } S^r \text{ に奇関数により同相}\}$$

$$S_\lambda^{(r)} = \inf_{\Gamma \in M_\lambda^{(r)}} \sup_{u \in \Gamma} \Phi_\lambda(u)$$

とおく．ここで，$\lambda_l \le \lambda < \lambda_{l+1}$ とする．この場合は次のような空間で $\Phi_\lambda(u)$ を考えるとよい．

$$L = R \oplus < \frac{\varphi(x)}{(\epsilon + |x|^2)^{\frac{n-2k}{2}}} >$$

とおいたとき

$$L \cap M_0^{(s)}$$

で考えるのである．ここで，$s = \Sigma_{j=1}^l d_j$，

$$M = \oplus_{j=1}^{j=l} \oplus_{p=1}^{p=d_j} < (g_p^{(j)} + \alpha g_q^{(1)}) \phi_\delta >$$

$g_a^{(b)}$ は λ_a に対する固有関数，d_b はその次元とする．

$$\phi_\delta(x) = \begin{cases} 0 & |x| \le 2\delta \\ 1 & |x| \ge 3\delta \end{cases}$$

$\phi_\delta \in C^\infty$ である．

α は後でうまくとる．すると §3. と同様に考えられる．以上により次の定理を得る．

定理５．

問題（１）は，$H_0^k(\Omega)$ で$\lambda > 0$ のとき非自明な解を持つ．

なお，Ω が星型 (star-shaped) ならば，Pucci-Serrin[5] により
$\lambda < 0$ のとき解は自明解しかないことが知られている．

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界面反応を伴う拡散方程式の解の漸近挙動

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1　序

非線型境界条件を伴った拡散方程式に対する次の形の初期値境界値問題を考える。

$$
(1.1)\qquad
\begin{cases}
a(x)u_t = u_{xx} & (x,t) \in (0,1)\times(0,\infty),\\
b(x)v_t = v_{xx} & (x,t) \in (0,1)\times(0,\infty),\\
u_x(1,t) = v_x(1,t) = 0 & t \in (0,\infty),\\
u_x(0,t) = k_1 u^m(0,t)v^n(0,t) & t \in (0,\infty),\\
v_x(0,t) = k_2 u^m(0,t)v^n(0,t) & t \in (0,\infty),\\
u(x,0) = u_0(x) \ge 0 & x \in (0,1),\\
v(x,0) = v_0(x) \ge 0 & x \in (0,1).
\end{cases}
$$

この問題は、２種類の化学物質 U と V が平行な板の間を流れながら界面上で不可逆化学反応

$$(1.2)\qquad m\mathrm{U} + n\mathrm{V} \to \mathrm{U}_m\mathrm{V}_n$$

を起こすという現象の数学的な１モデルである。未知関数 u と v はそれぞれ U と V の水溶液の濃度を表す (図１)。このようなモデルは既に [3, 4, 5] などで解析され大域解の一意

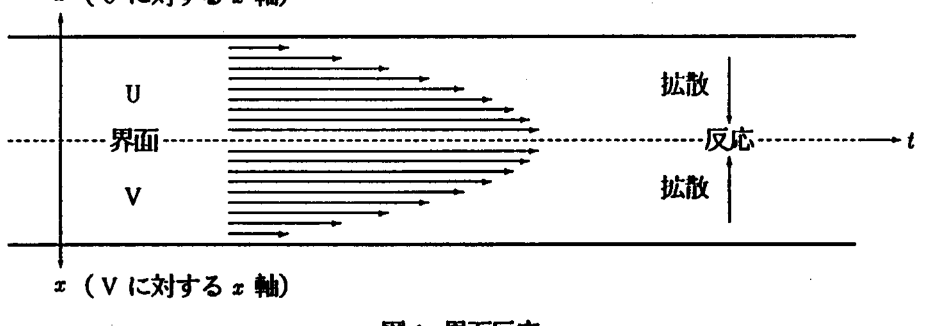

図 1: 界面反応

的な存在と定常状態へ収束する事が知られている。まず、これらの既知の事を紹介するとともに、これから我々が取り組む問題を整理しよう。

我々の問題では、$a(x)$ と $b(x)$ は $[0,1]$ 上の非負値 C^∞ 関数で $[0,1)$ 上では正の値を取るものとする。また、m , n は 1 以上の、k_1, k_2 は正の定数とする。このとき、非負な初期値 u_0, $v_0 \in L^\infty(0,1)$ に対し

$$
\begin{aligned}
(u,v) \in \{&C([0,\infty);L^2(0,1)) \cap C((0,\infty);H^2(0,1)) \cap H^1_{loc}(0,\infty;H^1(0,1)) \\
&\cap C^\infty([0,1]\times(0,\infty)) \cap L^\infty((0,1)\times(0,\infty))\}^2
\end{aligned}
$$

となる解が一意的に存在する。また、u, v も非負となる。

この解に対し

$$
A = \frac{1}{k_1}\int_0^1 au\,dx - \frac{1}{k_2}\int_0^1 bv\,dx \tag{1.3}
$$

は保存量となることに注意しよう。すなわち、t に依らない量となり、従って初期データによってきまる。

ここでは、解の定常状態への収束について考えよう。化学反応は可逆でないので、少なくとも一方の物質が無くなるまで反応し続け定常状態に落ち着くと考えられる。また、化学反応 (1.2) が起こるためには、m 個の U の分子と n 個の V の分子が界面上のある点で出会う必要がある。m や n が大きくなればそのような機会は起こりにくくなり、そのため、反応速度は遅くなると考えられる。

その結果、次の予想をたてることが出来る。

予想 1　解 (u,v) は、$t \to \infty$ とともにある定常状態 (u_∞, v_∞) に何らかの位相で収束する。

予想 2　収束の速さは、m や n が大きいほど遅い。

予想1が正しいとすると、(1.3) で与えられる保存量 A を考慮すれば、定常状態 (u_∞, v_∞) は、

$$
\begin{aligned}
u_\infty &= k_1 \max\{A,0\} \Big/ \int_0^1 a\,dx, \\
v_\infty &= k_2 \max\{-A,0\} \Big/ \int_0^1 b\,dx
\end{aligned}
$$

で与えられる。そして予想1に対しては、四宮 [3] によって sup ノルム に依る位相で正しいことが示された。そこで我々は予想2について考察しよう。但し、ノルムとして重み $a(x)$ 及び $b(x)$ を持つ L^1-ノルムで考える。その結果、予想2が成立することが確かめられた。

定理1. 次の形の評価が成り立つ。

$$\int_0^1 a|u-u_\infty|dx \le C\rho_1(t), \quad \int_0^1 b|v-v_\infty|dx \le C\rho_2(t),$$

ここで、

$$\rho_j(t) = \begin{cases} e^{-\lambda t/(3-j)} \quad (\lambda > 0) & (A>0 \text{ かつ } n=1 \text{ のとき}), \\ (t+1)^{-1/\{(3-j)(n-1)\}} & (A>0 \text{ かつ } n>1 \text{ のとき}), \\ (t+1)^{-1/(m+n-1)} & (A=0 \text{ のとき}), \\ (t+1)^{-1/\{j(m-1)\}} & (A<0 \text{ かつ } m>1 \text{ のとき}), \\ e^{-\lambda t/j} \quad (\lambda>0) & (A<0 \text{ かつ } m=1 \text{ のとき}) \end{cases}$$

である。

証明は、基本的にはエネルギー法に依るが、細部はそれぞれの場合で異なる。その中でも最も技巧的であると思われる $A=0$ の場合について、その概略を第2節において紹介する。第3節において、他の場合に関する注意と、解の導関数 (u_x, v_x) の減衰の速さに関する結果を述べる。また、本稿は [2] の要約であるが、その後のこの研究に関する進展についても述べる。

2 $A=0$ のときの証明

定数 M を

$$M = \max\left\{\|u\|_{L^\infty((0,1)\times(0,\infty))}, \|v\|_{L^\infty((0,1)\times(0,\infty))}\right\} < \infty.$$

とおく。また、$[0, M]$ 上の関数 φ_{m+n} を

$$\varphi_{m+n}(u) = \begin{cases} \exp\left\{-\dfrac{c_{m+n}}{u^{m+n-1}}\right\} & (u>0) \\ 0 & (u=0) \end{cases}$$

で定義する。ここで、定数 c_{m+n} は

(2.1) $$0 \le u^{-2(m+n)}\varphi_{m+n}(u) \le C\varphi''_{m+n}(u) \quad (0 \le u \le M)$$

が成り立つように大きくとる。$c = c_{m+n}(m+n-1)$ とおくと、

(2.2) $$u^{m+n}\varphi'_{m+n}(u) = c\varphi_{m+n}(u)$$

が成り立つことに注意しよう。また今後 $k>0$ のとき $u^{-k}\varphi_{m+n}(u)\Big|_{u=0}=0$ であると解釈する。

証明に入ろう。(1.1) の第1の方程式の両辺に $\varphi'_{m+n}(u)$ を掛けて x について積分する。境界条件と (2.2) に注意すると、

$$\frac{d}{dt}\int_0^1 a\varphi_{m+n}(u)dx+\int_0^1\varphi''_{m+n}(u)u_x^2dx+ck_1u^{-n}(0,t)v^n(0,t)\varphi_{m+n}(u(0,t))=0 \tag{2.3}$$

を得る。もし、ある $t_0\geq 0$ において $\int_0^1 a\varphi_{m+n}(u(x,t_0))dx=0$ となったとする。(1.3) において $A=0$ である事を考慮すれば、$t\geq t_0$ において $u=v\equiv 0$ となり定理の主張は証明された事になる。従って、全ての $t\geq 0$ に対して $\int_0^1 a\varphi_{m+n}(u(x,t))dx>0$ としてよい。$(0,\infty)$ 上の集合 F_i を

$$\begin{aligned} F_1 &= \{t>0\,;\ u(0,t)\leq v(0,t)\},\\ F_2 &= \{t>0\,;\ u(0,t)\geq v(0,t)\} \end{aligned}$$

で定義する。

(2.2) 及び (2.1) より

$$\begin{aligned}\varphi_{m+n}(u) &\leq \varphi_{m+n}(u(0,t))+\int_0^1|\varphi'_{m+n}(u)u_x|dx\\ &= \varphi_{m+n}(u(0,t))+c\int_0^1|u^{-(m+n)}\varphi_{m+n}(u)v_x|dx\\ &\leq \varphi_{m+n}(u(0,t))+\epsilon\int_0^1\varphi_{m+n}(u)dx+C\int_0^1\varphi''_{m+n}(u)u_x^2dx,\end{aligned}$$

が成り立つ。よって、$t\in F_1$ ならば

$$\begin{aligned}\int_0^1 a\varphi_{m+n}(u(x,t))dx &\leq C\left(\varphi_{m+n}(u(0,t))+\int_0^1\varphi''_{m+n}(u)u_x^2dx\right)\\ &\leq C\left(u^{-n}(0,t)v^n(0,t)\varphi_{m+n}(u(0,t))+\int_0^1\varphi''_{m+n}(u)u_x^2dx\right)\end{aligned} \tag{2.4}$$

が成立する。

(2.3) と (2.4) より、ある正定数 λ が存在して微分不等式

$$\frac{d}{dt}\log\int_0^1 a\varphi_{m+n}(u)dx\leq-\lambda\chi_{F_1}(t)$$

が成り立つ事がわかる。ここで χ_{F_1} は集合 F_1 の特性関数である。この不等式より

$$\int_0^1 a\varphi_{m+n}(u)dx\leq C\exp\left\{-\lambda\int_0^t\chi_{F_1}(\tau)d\tau\right\}$$

という評価が得られる。

これに φ_{m+n} が凸関数であることを利用して Jensen の不等式を適用すると、

$$\varphi_{m+n}\left(\int_0^1 au\,dx \Big/ \int_0^1 a\,dx\right) \le C\exp\left\{-\lambda\int_0^t \chi_{F_1}(\tau)d\tau\right\}$$

が得られる。

同様に、

$$\varphi_{m+n}\left(\int_0^1 bv\,dx \Big/ \int_0^1 b\,dx\right) \le C\exp\left\{-\lambda\int_0^t \chi_{F_2}(\tau)d\tau\right\}$$

が成り立つ事がわかる。

$$\int_0^t (\chi_{F_1} + \chi_{F_2})\,d\tau \ge t$$

であるので

$$\varphi_{m+n}\left(\int_0^1 au\,dx \Big/ \int_0^1 a\,dx\right)\varphi_{m+n}\left(\int_0^1 bv\,dx \Big/ \int_0^1 b\,dx\right) \le Ce^{-\lambda t}.$$

がわかる。φ_{m+n} の定義式と (1.3) において $A = 0$ であることから、主張の評価式を得ることができる。 □

3 補足

第2節で述べた方法では、重み付き L^1-ノルムによる評価しか得られない。しかし、$A = 0$ で $m = n = 1$ の場合や $A \neq 0$ の場合には、他の重み付き L^p-ノルムによる評価も可能である。結果だけを記しておこう。

定理2. $A = 0$ かつ $m = n = 1$ とする。このとき、評価

$$\int_0^1 (au^2 + bv^2)dx \le C(t+1)^{-2}$$

が成り立つ。

定理3. $A \neq 0$ とする。このとき、次の形の評価が成り立つ。

1. $A>0$ かつ $n=1$ のとき、

$$\int_0^1 a(u-u_\infty)^2dx \le Ce^{-\lambda t}, \quad \int_0^1 bv^2dx \le Ce^{-2\lambda t}.$$

2. $A>0$ かつ $n>1$ のとき、

$$\int_0^1 a(u-u_\infty)^2dx \le C(t+1)^{-1/(n-1)}, \int_0^1 bv^pdx \le C(p)(t+1)^{-p/(n-1)} \ (1 \le p < \infty).$$

3. $A<0$ かつ $m>1$ のとき、

$$\int_0^1 au^pdx \le C(p)(t+1)^{-p/(m-1)} \ (1 \le p < \infty), \int_0^1 b(v-v_\infty)^2dx \le C(t+1)^{-1/(m-1)}.$$

4. $A<0$ かつ $m=1$ のとき、

$$\int_0^1 au^2dx \le Ce^{-2\lambda t}, \quad \int_0^1 b(v-v_\infty)^2dx \le Ce^{-\lambda t}.$$

また、$a(x)$, $b(x)$ より非負値関数 $\alpha(x)$, $\beta(x)$ をある方法で定義するとこれらを重みとする L^2-ノルムで u_x, v_x が $t \to \infty$ とともに 0 に減衰することが示され、その速さについて評価する事ができる。粗く言って、

$$(3.1) \qquad \int_0^1 (\alpha u_x^2 + \beta v_x^2)dx \le \begin{cases} C\rho_2^{2n}(t) & (A>0 \ のとき), \\ C\rho_1^{2m}(t)\rho_2^{2n}(t) & (A=0 \ のとき), \\ C\rho_1^{2m}(t) & (A<0 \ のとき) \end{cases}$$

である。詳細については、[2, Theorem 6.1] を見られたい。

本稿で扱った問題は龍谷大学の四ツ谷先生より教唆頂いたものである。[2] の原稿を先生に送ったところ、若手セミナーの直前に返事を頂いた。それによって、[2] の方法を改善することによって H^1-ノルム（重み無し）での収束の速さを計算できる事を教えて頂けた。すなわち、

$$\|(u-u_\infty, v-v_\infty)\|_{H^1(0,1)} \le C\min\{\rho_1(t), \rho_2(t)\}$$

が成立する。Sobolev の埋蔵定理によって、一様収束の意味でその速さがわかった事になる。しかし、導関数の収束の速さの評価は (3.1) より悪くなっている。

更に、10 月に京都大学数理解析研究所において発展方程式に関する研究集会が催された折りに、四ツ谷先生は導関数の収束の速さの評価も改善される事を知らせて下さった。

先生は、その方法を用いて可逆反応のモデルについて解析され、飯田（大阪大）- 山田（早稲田大）- 四ツ谷（龍谷大）の名前でその研究集会において公表された（講演者は飯田氏、内容は [1] 参照)。

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Nonlinear Ergodic Theorems for Commutative Semigroups in Banach Spaces

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1. Introduction.

Let C be a nonempty closed convex subset of a real Banach space X. A mapping $T : C \to C$ is said to be asymptotically nonexpansive if

$$\|T^n x - T^n y\| \le k_n \|x - y\|$$

for all $n \ge 1$ and $x, y \in C$, where $\lim_{n\to\infty} k_n = 1$. In particular if $k_n = 1$ for all $n \ge 1$, T is said to be nonexpansive.
Let $\mathcal{J} = \{T(t) : t \ge 0\}$ be a family of mappings from C into itself. $\mathcal{J}$ is called an asymptotically nonexpansive semigroup on C if $T(t+s) = T(t)T(s)$ for all $t, s \ge 0$, and if $T(t)x$ is continuous in $t \ge 0$ for each $x \in C$ and there exists a function $k(\cdot) : R^+ \to R^+$ with $\lim_{t\to\infty} k(t) = 1$ such that

$$\|T(t)x - T(t)y\| \le k(t)\|x - y\|$$

for all $t \ge 0$ and $x, y \in C$. In particular, if $k(t) = 1$ for all $t \ge 0$, then $\mathcal{J}$ is called a nonexpansive semigroup on C.

Recently, Hirano, Kido, and Takahashi [5] established nonlinear ergodic theorems for commutative semigroups of nonexpansive mappings in Banach spaces.

The purpose of this paper is to generalize their results to the case of commutative semigroups of asymptotically nonexpansive mappings. Using our results, we can simultaneously handle ergodic theorems for asymptotically nonexpansive mappings and semigroups, e.g., we can prove the weak convergence of $\{\frac{1}{n}\sum_{i=0}^{n-1} T^i x : n \geq 1\}$ as $n \to \infty$ and $\{\frac{1}{s}\int_0^s T(t)x\, dt : s > 0\}$ as $s \to \infty$ for each $x \in C$ in a unified way. See Section 4.

2. Preliminaries.

Throughout this paper X denotes a uniformly convex real Banach space, C a nonempty bounded closed convex subset of X, and G a commutative topological semigroup with the identity.
The value of $x^* \in X^*$ at $x \in X$ will be denoted by (x, x^*). The duality mapping J (multi-valued) from X into X^* will be defined by

$$J(x) = \{x^* \in X^* : (x, x^*) = \|x\|^2 = \|x^*\|^2\}$$

for $x \in X$. We say that X is (F) if the norm of X is Fréchet differentiable, i.e., for each $x \neq 0$,

$$\lim_{t \to \infty} \frac{\|x+ty\| - \|x\|}{t}$$

exists uniformly in $y \in B_1$, where $B_r = \{z \in X : \|z\| \leq r\}$ for $r > 0$. It is easy to see that X is (F) if and only if for any bounded set $B \subset X$ and any $x \in X$, $\lim_{t \to 0} (2t)^{-1}(\|x+ty\|^2 - \|x\|^2) = (y, J(x))$ uniformly in $y \in B$.

For a subset E of X, co E denotes the convex hull of E, clco E the closed convex hull of E.

Let $m(G)$ be the Banach space of all bounded real valued functions on G with the supremum norm. For each $s \in G$ and $f \in m(G)$, we define an element $r_s f$ in $m(G)$ by $(r_s f)(t) = f(t+s)$ for all $t \in G$. The mapping $r_s : f \mapsto r_s f$ is a continuous linear operator in $m(G)$ for all $s \in G$. Let D be a subspace of $m(G)$ and μ be an element of D^*, where D^* is the dual space of D. Then, we denote by $\mu(f)$ the value of μ at $f \in D$. To specify the variable t, we write the value $\mu(f)$ by $\int f(t)\, d\mu(t)$. When D contains a constant function 1, an element μ of D^* is called a mean on D if $\|\mu\| = \mu(1) = 1$. Further, let D be invariant under r_s for all $s \in G$. Then, a mean on D is said to be invariant if $\mu(r_s f) = \mu(f)$ for all $s \in G$ and $f \in D$.

Let $\mathcal{J} = \{T_t : t \in G\}$ be a family of mappings from C into itself. $\mathcal{J}$ is said to be a commutative semigroup of asymptotically nonexpansive mappings on C if the following conditions are satisfied :

(a) $T_{s+t}x = T_s T_t x$ for all s, $t \in G$ and $x \in C$;

(b) For each $x \in C$, the mapping $t \mapsto T_t x$ from G into C is continuous ;

(c) For each $t \in G$, there exists $k_t > 0$ such that $\|T_t x - T_t y\| \le k_t \|x - y\|$ for all x, $y \in C$ with $\lim_{t \in G} k_t = 1$, where $\lim_{t \in G} k_t$ denotes the limit of a net $k_{(\cdot)}$ on the directed system $(G, \le)$ and the binary relation $\le$ on G is defined by $a \le b$ if and only if there is $c \in G$ with $a + c = b$.

3. Nonlinear Ergodic Theorems.

Since X is uniformly convex and C is bounded, the set $F(\mathcal{J}) \equiv \bigcap_{s \in G} F(T_s)$ of common fixed points of $\{T_s : s \in G\}$ is nonempty bounded closed convex ; see [9].

Let D be a subspace of m(G) containing a constant function 1 and invariant under r_s for all $s \in G$. Assume that, for each $x \in C$ and $x^* \in X^*$, the function $s \mapsto (T_s x, x^*)$ is in D. Since X is reflexive, for any $\mu \in D^*$, we can consider a mapping $\mathcal{J}_\mu$ of C into such that

$$(\mathcal{J}_\mu x, x^*) = \int (T_t x, x^*) \, d\mu(t)$$

for every $x \in C$ and $x^* \in X^*$. Particularly, if μ is a mean on D, then $\mathcal{J}_\mu$ is a nonexpansive mapping of C into itself.

Theorem 1. Let D be a subspace of m(G) containing a constant function 1 and invariant under r_s for all $s \in G$. Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and let $\mathcal{J} = \{T_t : t \in G\}$ be a commutative semigroup of asymptotically nonexpansive mappings on C such that the function $s \mapsto (T_s x, x^*)$ is in D for each $x \in C$ and $x^* \in X^*$. Then, for every invariant mean μ on D, $\mathcal{J}_\mu$ is a nonexpansive retraction of C onto $F(\mathcal{J})$ such that $\mathcal{J}_\mu T_s = T_s \mathcal{J}_\mu = \mathcal{J}_\mu$ for each $s \in G$ and $\mathcal{J}_\mu x \in \overline{co}\,\{T_s x : s \in G\}$ for each $x \in C$.

Let D be a subspace of m(G) containing a constant function 1 and invariant under r_s for all $s \in G$. Then, a net $\{\mu_\alpha\}$ of continuous linear functionals on D is called strongly regular if it satisfies the following conditions :

(a) $\sup_{\alpha} \|\mu_\alpha\| < +\infty$;

(b) $\lim_{\alpha} \mu_\alpha(1) = 1$;

(c) $\lim_{\alpha} \|\mu_\alpha - r_s^* \mu_\alpha\| = 0$ for every $s \in G$,

where r_s^* is the conjugate operator of r_s for each $s \in G$.

Theorem 2. Let G, D, C, X and $\mathcal{J} = \{T_t : t \in G\}$ be as in theorem 1. Additionally, assume that X is (F). Then there is a unique nonexpansive retraction P of C onto $F(\mathcal{J})$ such that $PT_t = T_tP = P$ for each $t \in G$ and $Px \in \overline{co}\,\{T_tx : t \in G\}$ for each $x \in C$. Further, if $\{\mu_\alpha\}$ is a strongly regular net of continuous linear functionals on D, then for each $x \in C$, $\mathcal{J}_{\mu_\alpha} T_t x$ converges weakly to Px uniformly in $t \in G$.

For the proof of theorems 1 and 2, see [7].

4. Applications.

In this section, by using theorem 2, we provide nonlinear ergodic theorems for asymptotically nonexpansive mappings and semigroups in Banach spaces. Throughout this section, X is (F).

Let T be an asymptotically nonexpansive mapping from C into itself. Let $G = \{0, 1, 2, \cdots\}$, $\mathcal{J} = \{T^i : i \in G\}$ and $D = m(G)$ in theorem 2. We get the following theorems 3 - 5.

Theorem 3. For each $x \in C$, $\frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x$ converges weakly to some fixed point of T, as $n \to \infty$, uniformly in $k \geq 0$.

Proof. Put $\mu_n(f) = \frac{1}{n}\sum_{i=0}^{n-1} f(i)$ for each $n \geq 1$ and $f \in D$. Then, $\{\mu_n : n \geq 1\}$ is a strongly regular net on D. Q. E. D.

Let $N = \{0, 1, 2, \cdots\}$ and let $Q = (q_{n,m})_{n,m \in N}$ be a matrix satisfying the following conditions :

(a) $\sup_{n \geq 0} \sum_{m=0}^{\infty} |q_{n,m}| < +\infty$;

(b) $\lim_{n \to \infty} \sum_{m=0}^{\infty} q_{n,m} = 1$;

(c) $\lim_{n \to \infty} \sum_{m=0}^{\infty} |q_{n,m+1} - q_{n,m}| = 0.$

Then, Q is called a strongly regular matrix.

Theorem 4. If Q is a strongly regular matrix, then for each $x \in C$, $\sum_{m=0}^{\infty} q_{n,m} T^{m+k}x$ converges weakly to some fixed point of T, as $n \to \infty$, uniformly in $k \geq 0$.

Proof. $\mu_n(f) = \sum_{m=0}^{\infty} q_{n,m} f(m)$ for each $n \geq 1$ and $f \in D$. Then, $\{\mu_n : n \geq 1\}$ is a strongly regular net on D. Q. E. D.

Theorem 5. For each $x \in C$, $(1-r)\sum_{i=0}^{\infty} r^i T^{i+k} x$ converges weakly to some fixed point of T, as $r \uparrow 1$, uniformly on $k \geq 0$.

Proof. Put $\mu_r(f) = (1-r)\sum_{i=0}^{\infty} r^i f(i)$ for each $0 < r < 1$ and $f \in D$. Then, $\{\mu_r : 0 < r < 1\}$ is a strongly regular net on D. Q. E. D.

Let $\mathcal{J} = \{T(t) : t \geq 0\}$ be an asymptotically nonexpansive semigroup on C. Let $G = R^+$, $\mathcal{J} = \{T(t) : t \geq 0\}$ and D be the Banach space C(G) of bounded continuous functions on G in theorem 2. We get the following theorems 6 – 8.

Theorem 6. For each $x \in C$, $\frac{1}{s}\int_0^s T(t+h)x\,dt$ converges weakly to some fixed point of $\mathcal{J}$, as $s \to \infty$, uniformly in $h \geq 0$.

Proof. Put $\mu_s(f) = \frac{1}{s}\int_0^s f(t)\,dt$ for each $s > 0$ and $f \in D$. Then, $\{\mu_s : s > 0\}$ is a strongly regular net on D. Q. E. D.

Let $Q : R^+ \times R^+ \to R$ be a function satisfying the following conditions :

(a) $\sup_{s \geq 0} \int_0^{\infty} |Q(s,t)|\,dt < +\infty$;

(b) $\lim_{s \to \infty} \int_0^{\infty} Q(s,t)\,dt = 1$;

(c) $\lim_{s \to \infty} \int_0^{\infty} |Q(s,t+h) - Q(s,t)|\,dt = 0$ for all $h \geq 0$.

Then, $Q(\cdot,\cdot)$ is called a strongly regular kernel.

Theorem 7. If $Q(\cdot,\cdot)$ is a strongly regular kernel, then for each $x \in C$, $\int_0^\infty Q(s,t)T(t+h)x\,dt$ converges weakly to some fixed point of $\mathcal{J}$, as $s \to \infty$, uniformly in $h \geq 0$.

Proof. Put $\mu_s(f) = \int_0^\infty Q(s,t)f(t)\,dt$ for each $s > 0$ and $f \in D$. Then, $\{\mu_s : s > 0\}$ is a strongly regular net on D. Q.E.D.

Theorem 8. For each $x \in C$, $\lambda\int_0^\infty e^{-\lambda t}T(t+h)x\,dt$ converges weakly to some fixed point of $\mathcal{J}$, as $\lambda \downarrow 0$, uniformly in $h \geq 0$.

Proof. Put $\mu_\lambda(f) = \lambda\int_0^\infty e^{-\lambda t}f(t)\,dt$ for each $\lambda > 0$ and $f \in D$. Then, $\{\mu_\lambda : \lambda > 0\}$ is a strongly regular net on D. Q.E.D.

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Periodic Stability for Stefan Problems with Nonlinear Flux Conditions

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0. Introduction

This paper is concerned with a degenerate parabolic equation

$$u_t - \Delta\beta(u) = 0 \quad \text{in } Q = I \times \Omega \tag{0.1}$$

with nonlinear flux condition

$$\frac{\partial\beta(u)}{\partial n} + g(t, x, \beta(u)) = 0 \quad \text{on } \Sigma = I \times \Gamma, \tag{0.2}$$

where I is an interval in $\mathbf{R}$ of the form $[t_0, \infty)$ or $\mathbf{R}$; Ω is a bounded domain in $\mathbf{R}^N (N \geq 1)$ with smooth boundary $\Gamma = \partial\Omega; \beta : \mathbf{R} \to \mathbf{R}$ is a given non-decreasing function; $g = g(t, x, \xi) : \mathbf{R} \times \Gamma \times \mathbf{R} \to \mathbf{R}$ is a given function which is non-decreasing in $\xi \in \mathbf{R}$ for a.e. $(t, x) \in \mathbf{R} \times \Gamma$; $(\partial/\partial n)$ denotes the outward normal derivative on Γ.

In this paper we denote by "P on I" the system $\{(0.1),(0.2)\}$, and refer to the papers Visintin [8], Niezgodka-Pawlow [6] and Niezgodka-Pawlow-Visintin [7] for the existence and uniqueness of a solution to the Cauchy problem for P in a generalized sense. In the same framework of generalized solutions, under periodicity condition $g(t+T, x, \xi) = g(t, x, \xi)$ on $\mathbf{R} \times \Gamma \times \mathbf{R}$ for a given number $T > 0$, we will discuss some properties on the structure of periodic solutions with period T.

1. Statements of main results

Throughout this paper, let $\beta : \mathbf{R} \to \mathbf{R}$ be a function and suppose that

(β1) β is non-decreasing and Lipschitz continuous on $\mathbf{R}$ with Lipschitz constant C_β;

(β2) $\beta(0) = 0$ and $\liminf_{|r| \to \infty} \dfrac{\beta(r)}{r} =: L_\beta > 0$.

Also, let $g = g(t, x, \xi) : \mathbf{R} \times \Gamma \times \mathbf{R} \to \mathbf{R}$ be a function satisfying the Carathéodory condition, i.e. $g(t, x, \xi)$ is continuous in $\xi \in \mathbf{R}$ for a.e. $(t, x) \in \mathbf{R} \times \Gamma$ and is measurable in $(t, x) \in \mathbf{R} \times \Gamma$ for all $\xi \in \mathbf{R}$.

Moreover, suppose that

(g1) $g(t, x, \xi)$ is locally Lipschitz continuous in ξ uniformly with respect to $(t, x) \in \mathbf{R} \times \Gamma$, that is, for each $M > 0$ there is a constant $C_g(M) \geq 0$ such that

$$| g(t, x, \xi) - g(t, x, \xi') | \leq C_g(M) | \xi - \xi' |$$

for all ξ, ξ' with $| \xi | \leq M$, $| \xi' | \leq M$ and for a.e. $(t, x) \in \mathbf{R} \times \Gamma$;

(g2) $g(t, x, \xi)$ is non-decreasing in $\xi \in \mathbf{R}$ for a.e. $(t, x) \in \mathbf{R} \times \Gamma$;

($g3$) for any $\xi \in \mathbf{R}$, $g(\cdot,\cdot,\xi) \in L^2_{loc}(\mathbf{R}; L^2(\Gamma))$.

For the sake of simplicity of notations we put

$H = L^2(\Omega)$ with inner product $(\cdot,\cdot)$ and norm $|\cdot|_H$

and

$V = H^1(\Omega)$ with duality pairing $\langle\cdot,\cdot\rangle$ between its dual space V' and V, and with norm $|\cdot|_V$.

Also, we define a bilinear form $a(\cdot,\cdot)$ on $V \times V$ by

$$a(v,w) = \int_\Omega \nabla v \cdot \nabla w \, dx \qquad \text{for } v,\ w \in V,$$

We now formulate problem P in the variational sense.

Definition 1. *Let J be a compact interval of the form $[t_0,t_1]$. Then a function $u : J \to H$ is called a weak solution of P on J, if it satisfies the following ($w1$) and ($w2$):*

($w1$) $u \in C_w(J;H)$, $u \in L^\infty(J \times \Omega)$ and $\beta(u) \in L^2(J;V)$;

($w2$) $u' \in L^2(J;V')$ and

$$\langle u'(t), z\rangle + a(\beta(u), z) + \int_\Gamma g(t,\cdot,\beta(u(t,\cdot)))z d\Gamma = 0$$

for any $z \in V$ and a.e. $t \in J$.

Definition 2. *Let J' be the whole line or any interval of the form $[t_0',\infty)$. Then a function $u : J' \to H$ is called a weak solution of P on J', if it is a weak solution of P on J for every compact subinterval J of J' in the sense of Definition 1.*

Next we formulate the Cauchy problem and the problem with the periodic condition in time for P.

Definition 3. *(i) Let $J' = [t_0,t_1]$ or $[t_0,\infty)$, and let $u_0 \in H$. Then $u : J' \to H$ is a weak solution of the Cauchy problem with initial condition $u(t_0) = u_0$, denoted by $CP(u_0)$ on J', for problem P on J', if u is a weak solution of P on J' with $u(t_0) = u_0$.*

(ii) Let T be a positive number, and let $u : \mathbf{R} \to H$ be a weak solution of P on $\mathbf{R}$ such that $u(t+T) = u(t)$ for all $t \in \mathbf{R}$. Then u is called a T-periodic weak solution of P on $\mathbf{R}$.

We now recall an existence-uniqueness result for $CP(u_0)$.

Theorem 1 (cf.[7,8]). *Suppose further that there are two constants M_1, M_2 with $M_1 \le M_2$ such that*

$$g(t,x,\beta(M_1)) \le 0,\ g(t,x,\beta(M_2)) \ge 0 \quad \text{for a.e. } (t,x) \in \mathbf{R} \times \Gamma. \tag{1.1}$$

Let t_0 be any number in $\mathbf{R}$, and let $\hat{M}_1$ and $\hat{M}_2$ be constants such that $\hat{M}_1 \le M_1$ and $\hat{M}_2 \ge M_2$. Then, for any function u_0 in $L^\infty(\Omega)$ satisfying

$$\hat{M}_1 \le u_0 \le \hat{M}_2 \quad \text{a.e. on } \Omega, \tag{1.2}$$

there exists one and only one weak solution u of $CP(u_0)$ on $J' = [t_0, \infty)$ such that

$$\hat{M}_1 \leq u \leq \hat{M}_2 \quad \text{a.e. on } J' \times \Omega. \tag{1.3}$$

Denoting by $\mathcal{P}_T$ the set of all T-periodic weak solutions of P on $\mathbf{R}$, the main result of this paper are stated in the following theorem.

Theorem 2. *Let T be a positive number, and suppose*

(g4) $g(t+T, x, \xi) = g(t, x, \xi)$ for all $\xi \in \mathbf{R}$ and a.e. $(t, x) \in \mathbf{R} \times \Gamma$.

Further, suppose that there are two constants M_1, M_2 with $M_1 \leq M_2$, for which (1.1) holds. Then the following statements $(A) \sim (E)$ hold.

(A) There exists $u \in \mathcal{P}_T$ such that

$$M_1 \leq u \leq M_2 \quad \text{a.e. on } \mathbf{R} \times \Omega. \tag{1.4}$$

(B) If $\omega_1, \omega_2 \in \mathcal{P}_T$, then

$$g(\cdot,\cdot,\beta(\omega_1)) = g(\cdot,\cdot,\beta(\omega_2)) \quad \text{a.e. on } \mathbf{R} \times \Gamma. \tag{1.5}$$

(C) If $\omega_1, \omega_2 \in \mathcal{P}_T$ and $\int_\Omega \omega_1(0,x)dx \leq \int_\Omega \omega_2(0,x)dx$, then

$$\beta(\omega_1) \leq \beta(\omega_2) \quad \text{a.e. on } \mathbf{R} \times \Omega. \tag{1.6}$$

(D) If $\omega_1, \omega_2 \in \mathcal{P}_T$ and $\int_\Omega \omega_1(0,x)dx < \int_\Omega \omega_2(0,x)dx$, then for any $a_0 \in \mathbf{R}$ with $\int_\Omega \omega_1(0,x)dx < a_0 < \int_\Omega \omega_2(0,x)dx$ there exists a T-periodic weak solution ω to P on $\mathbf{R}$ such that

$$a_0 = \int_\Omega \omega(0,x)dx.$$

(E) Let t_0 be any number and let u be any weak solution of P on $[t_0, \infty)$. Then there is $\omega \in \mathcal{P}_T$ such that

$$\beta(u(nT+\cdot)) \to \beta(\omega) \text{ in } L^2(0,T;V) \text{ as } n \to \infty. \tag{1.7}$$

In Theorem 2 it is mentioned that $\{\ \beta(\omega)\ ;\ \omega \in \mathcal{P}_T\ \}$ is a totally ordered set with respect to the usual order of functions on $\mathbf{R} \times \Omega$. However, as is seen from the following example, $\{\omega \in \mathcal{P}_T\}$ is no longer a totally ordered set.

Example. Let

$$\beta(r) = \begin{cases} r-1 & \text{for } r \geq 1, \\ 0 & \text{for } 0 < r < 1, \\ r & \text{for } r \leq 0, \end{cases}$$

and $g(t,x,\xi) \equiv 0$. Then all the conditions of Theorem 2 are satisfied, and clearly every measurable function $u(t,x)$, which is independent of $t \in \mathbf{R}$ and satisfies $u(t,x) \equiv u(x) \in [0,1]$ for a.e. $x \in \Omega$, is a T-periodic weak solution of P on $\mathbf{R}$.

2. Some auxiliary results

In place of (0.2) we consider the non-homogeneous flux condition

$$\frac{\partial \beta(u)}{\partial n} = h \quad \text{on } \Sigma = I \times \Gamma. \tag{2.1}$$

and denote "$\hat{P}$ on I" by the system {(0.1),(2.1)}.

Definition 4. *Let $J = [t_0, t_1]$ be a compact interval. For $h \in L^2_{loc}(\mathbf{R}; L^2(\Gamma))$, we say that u is a weak solution of problem $\hat{P}$, if $u \in C_w(J; H) \cap W^{1,2}(J; V')$, $\beta(u) \in L^2(J; V)$ and*

$$\langle u'(t), z\rangle + a(\beta(u), z) = \int_\Gamma h(t, \cdot) z d\Gamma \quad \text{for any } z \in V \text{ and a.e. } t \in J.$$

It should be noted here that u is not required to be bounded on $J \times \Omega$ in the definition.

For a general interval $J' \subset \mathbf{R}$, weak solutions of $\hat{P}$ on J' are defined in a manner similar to Definition 2. Also, weak solutions of the Cauchy problem and the problem with T-periodic condition are defined just as Definition 3.

The following results are due to Haraux-Kenmochi [5].

Theorem 3 *Let T be a positive number, and assume that $h(t+T, x) = h(t, x)$ a.e. on $\mathbf{R} \times \Gamma$, and*

$$\int_0^T \int_\Gamma h(t, x) d\Gamma dt = 0$$

Then the following statements (a) $\sim$ (d) hold:

(a) For each $a_0 \in \mathbf{R}$ there exists a T-periodic weak solution u of $\hat{P}$ on $\mathbf{R}$ such that

$$\int_\Omega u(0, x) dx = a_0.$$

(b) Let u be a weak solution of $\hat{P}$ on $\mathbf{R}$. Then u is T-periodic on $\mathbf{R}$ if and only if $u \in L^\infty(\mathbf{R}; H)$.

(c) Let u_1, u_2 be T-periodic weak solutions of $\hat{P}$ on $\mathbf{R}$ such that

$$\int_\Omega u_1(0, x) dx \le \int_\Omega u_2(0, x) dx.$$

Then

$$\beta(u_1) \le \beta(u_2) \quad \text{a.e. on } \mathbf{R} \times \Omega,$$

(d) For any weak solution u of $\hat{P}$ on $[t_0, \infty)$, there exists a T-periodic weak solution ω of $\hat{P}$ on $\mathbf{R}$ such that

$$\int_\Omega u(t, x) dx = \int_\Omega \omega(t, x) dx \quad \text{for all } t \ge t_0,$$

and

$$u(t) - \omega(t) \to 0 \text{ weakly in } H \text{ as } t \to \infty.$$

Remark. In [5], the statement (c) in the case of $\int_\Omega \omega_1(0, x) dx < \int_\Omega \omega_2(0, x) dx$ was not discussed. In the appendix we will give the outline of the proof.

Next we state the following two propositions about comparison and convergence of weak solutions.

Proposition 1. *Let $u_{0,i}$ be any function in $L^\infty(\Omega)$, and u_i be the weak solution of $CP(u_{0,i})$ on a compact interval $J = [t_0, t_1] \subset \mathbf{R}$ for $i = 1, 2$. Then*

$$| (u_1(t) - u_2(t))^+ |_{L^1(\Omega)} \le | (u_1(s) - u_2(s))^+ |_{L^1(\Omega)}, \tag{2.2}$$

$$| u_1(t) - u_2(t) |_{L^1(\Omega)} \le | u_1(s) - u_2(s) |_{L^1(\Omega)}, \tag{2.3}$$

for any $s, t \in J$ with $s \le t$. In particular, if $u_{0,1} \le u_{0,2}$ a.e. on Ω, then

$$u_1 \le u_2 \quad \text{a.e. on } J \times \Omega.$$

Proposition 2. *Let $\{u_{0,n}\}$ be a bounded sequence in $L^\infty(\Omega)$ such that $u_{0,n} \to u_0$ weakly in H (as $n \to \infty$), and let $J = [t_0, t_1] \subset \mathbf{R}$. Then the weak solution u_n of $CP(u_{0,n})$ on J converges to the weak solution u of $CP(u_0)$ on J in the sense that*

(i) $u_n \to u$ weakly in $L^\infty(J \times \Omega)$ and weakly in $W^{1,2}(J; V')$, hence $u_n \to u$ in $C_w(J; H)$;*

(ii) $\beta(u_n) \to \beta(u)$ in $L^2(J; H) \cap L^2_{loc}((t_0, t_1]; V)$ and weakly in $L^2(J; V)$;

(iii) $g(\cdot, \cdot, \beta(u_n)) \to g(\cdot, \cdot, \beta(u))$ in $L^2(J; L^2(\Gamma))$.

In particular, if $\hat{\beta}(u_{0,n}) \to \hat{\beta}(u_0)$ in $L^2(J; V)$, then $\beta(u_{0,n}) \to \beta(u)$ in $L^2(J; V)$, where $\hat{\beta}(z) = \int_\Omega \int_0^{z(x)} \beta(r) dr dx$.

3. Sketch of the proof of the main theorem

Now we consider Cauchy problem $CP(M_1)$ on $[0, T]$. Then, by Theorem 1, we get $M_1 \le u(\cdot\,; M_1)$ where $u(\cdot\,; z)$ is a weak solution of $CP(z)$ on $[0, T]$. So we have $u(\cdot\,; M_1) \le u(\cdot\,; u(T; M_1))$ owing to Proposition 1. Iterating this procedure, we see that $\omega(t) \equiv \lim_n u(t; u(nT; M_1))$ exists. Moreover, by Proposition 1 we have $\omega(0) = \omega(T)$. Since it is easily derived that

$$M_1 \le \omega \le M_2 \quad \text{a.e. on } [0, T] \times \Omega,$$

the periodic extension of ω is a desired T-periodic weak solution of P on $\mathbf{R}$. Therefore (A) is proved.

Next let $\omega_1, \omega_2 \in \mathcal{P}_T$. We can easily construct $\omega \in \mathcal{P}_T$ such that $\omega \le \min\{\omega_1, \omega_2\}$. Then, from $(w2)$ of Definition 1 it follows that

$$\langle \omega'(t) - \omega_i'(t), 1 \rangle + \int_\Gamma (g(t, x, \beta(\omega(t)) - g(t, x, \beta(\omega_i(t))) d\Gamma = 0$$

for a.e. $t \in [0, T], i = 1, 2$. By the monotonicity of β and $g(t, x, \cdot)$, and the T-periodicity of ω, ω_i, we have (B).

By the above result, there exists $h \in L^2_{loc}(\mathbf{R}; L^2(\Gamma))$ such that $h = -g(\cdot, \cdot, \beta(\omega))$ for all $\omega \in \mathcal{P}_T$. So the proof of (C) is clear from (c) of Theorem 3.

We proceed to the proof of (D). Now, let $\omega_1, \omega_2 \in \mathcal{P}_T$. Then they are T-periodic weak solutions of $\hat{P}$ on $\mathbf{R}$. Next let $a_0 \in \mathbf{R}$ such that

$$\int_\Omega \omega_1(0,x)dx < a_0 < \int_\Omega \omega_2(0,x)dx.$$

Then, by (a) and (c) of Theorem 3, there exists a T-periodic weak solution of $\hat{P}$ on $\mathbf{R}$ such that

$$\beta(\omega_1) \le \beta(\omega) \le \beta(\omega_2) \quad \text{a.e. on } \mathbf{R} \times \Omega.$$

By (A) we get $g(\cdot,\cdot,\beta(\omega)) = -h$. Thus ω is the T-periodic weak solution of P on $\mathbf{R}$.

Finally we prove (E). Put $u_0 := u(t_0)$. Then we see that $a_0 \equiv \lim_{n\to\infty} \int_\Omega u(nT,x)dx$ exists. Since $\{\beta(u(nT+\cdot))\}$ is compact in $L^2(0,T;V)$ there exist a subsequence $\{n_k\}$ and a T-periodic weak solution ω of P on $\mathbf{R}$ such that $\beta(u(n_kT+\cdot))$ converges to $\beta(\omega)$ in $L^2(0,T;V)$. Moreover we see that $a_0 = \int_\Omega \omega(0,x)dx$. $\beta(\omega)$ is uniquely determined by a_0 because of (C), and ω satisfies the properties in (E). For the detailed proofs we refer to the paper Aiki-Kenmochi-Shinoda [1].

Appendix

First let u be any weak solution of P on $J = [t_0, t_1]$. For each $\varepsilon \in (0,1]$, we consider the approximate Cauchy problem, denoted by $\hat{P}_\varepsilon$, of $\hat{P}$ on J:

$$\begin{cases} u_{\varepsilon t} - \Delta\beta_\varepsilon(u_\varepsilon) = 0 & \text{in } (t_0,t_1)\times\Omega, \\ \dfrac{\partial\beta_\varepsilon(u_\varepsilon)}{\partial n} = h_\varepsilon & \text{on } (t_0,t_1)\times\Gamma, \\ u_\varepsilon(t_0) = z_\varepsilon & \text{in } \Omega, \end{cases}$$

where $\{\beta_\varepsilon\}, \{h_\varepsilon\}$ and $\{z_\varepsilon\}$ are smooth approximations of β, h and $u(t_0,\cdot)$, respectively, such that

$$\varepsilon \le \frac{d}{dr}\beta_\varepsilon(r) \le C_\beta + 1, \ \beta_\varepsilon(0) = 0,$$
$$\beta_\varepsilon \to \beta \text{ uniformly on each compact interval of } \mathbf{R} \text{ as } \varepsilon \to 0,$$
$$h_\varepsilon \to h \text{ in } L^2(J; L^2(\Gamma)) \text{ as } \varepsilon \to 0,$$

and

$$\frac{\partial\beta_\varepsilon(z_\varepsilon)}{\partial n} = h_\varepsilon(t_0,\cdot) \text{ on } \Gamma, \ z_\varepsilon \to u(t_0,\cdot) \text{ in } H \text{ as } \varepsilon \to 0.$$

By the classical theory, $\hat{P}_\varepsilon$ has a smooth solution u_ε for each $\varepsilon \in (0,1]$. We know that the weak solution u of $\hat{P}$ is unique. By the usual estimates we have the following convergences:

$$\begin{cases} u_\varepsilon \to u & \text{in } C_w(J;H), \text{ and weakly in } W^{1,2}(J;V'), \\ u_\varepsilon(t_0) \to u(t_0) & \text{in } H, \\ \beta_\varepsilon(u_\varepsilon) \to \beta(u) & \text{weakly in } L^2(J;V) \end{cases}$$

as $\varepsilon \to 0$ without extracting any subsequence $\{\varepsilon_n\}$, $n \to \infty$.

Next, let u_1, u_2 be two T-periodic weak solutions of $\hat{P}$ on $\mathbf{R}$ such that $\int_\Omega u_1(0,x)dx < \int_\Omega u_2(0,x)dx$ and let $u_{1\varepsilon}, u_{2\varepsilon}$ be the approximate solutions associated with u_1, u_2, respectively. Then, we have, for any $s, t \in \mathbf{R}$ with $s \le t$,

$$| (u_{1\varepsilon}(t) - u_{2\varepsilon}(t))^+ |_{L^1(\Omega)} \le | (u_{1\varepsilon}(s) - u_{2\varepsilon}(s))^+ |_{L^1(\Omega)}$$

Therefore, letting $\varepsilon \to 0$ gives

$$| (u_1(t) - u_2(t))^+ |_{L^1(\Omega)} \le | (u_1(s) - u_2(s))^+ |_{L^1(\Omega)} \tag{A.1}$$

for any $s, t \in \mathbf{R}$ with $s \le t$.

Now choose z_i $(i = 1, 2)$ in H such that $z_1 \le z_2$ a.e. on $J \times \Omega$, $\int_\Omega u_i(0,x)dx = \int_\Omega z_i(x)dx$, and denote by $\tilde{u}_i$ the weak solution of $\hat{P}$ on $J = [t_0, \infty)$ with initial condition $\tilde{u}_i(0) = z_i$. Then, by $(A.1)$, $\tilde{u}_1 \le \tilde{u}_2$ a.e. on $J \times \Omega$. Now applying (d) of Theorem 3, we see that there is a T-periodic weak solution ω_i, $i = 1, 2$, of $\hat{P}$ on $\mathbf{R}$ such that

$$\tilde{u}_i(t) - \omega_i(t) \to 0 \text{ weakly in } H \text{ as } t \to \infty$$

and

$$\int_\Omega u_i(0,x)dx = \int_\Omega \omega_i(0,x)dx. \tag{A.2}$$

Therefore we have $\omega_1 \le \omega_2$, hence $\beta(\omega_1) \le \beta(\omega_2)$ a.e. on $\mathbf{R} \times \Omega$. On the other hand, $(A.2)$ implies that

$$\beta(u_i) = \beta(\omega_i) \quad \text{a.e. on } \mathbf{R} \times \Omega.$$

Therefore we obtain

$$\beta(u_1) \le \beta(u_2) \quad \text{a.e. on } \mathbf{R} \times \Omega.$$

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The Cauchy problem of Hartree type Schrödinger equation in H^s

§1 Introduction

ここでは 次の 非線型 Schrödinger 方程式

$$(1)\begin{cases} i\partial_t u = -\Delta u + F(u) & (t,x)\in[0,\infty)\times\mathbb{R}^n \\ u(0)=\phi & x\in\mathbb{R}^n \end{cases}$$

を考える。 ここに 非線型項 F は、"Hartree type"

$F(u) = u(W * |u|^2)$, $*$ は $\mathbb{R}^n$ の convolution

の形とする。

このtypeの方程式の Cauchy 問題については、
Cadam and Glassey [1], Ginibre and Velo [2], Hayashi and Ozawa [3], [4], Hayashi, Nakamitsu and M. Tsutsumi [5] など多数の結果がある。 特に、Strichartz [6] による線型作用素の評価は、より一般の Schrödinger 方程式の解の構成に、非常に有用なものとなった。

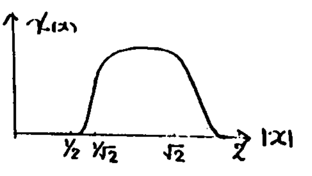
上に並べた結果は、いずれも初期値空間が Sobolev spc. $W^{k,2}$ あるいは Weighted Sobolev spc. $W^{k,2}_m$ の話である。 そこで、ここでは初期値空間が H^s ($s\in[0,\infty)$) の場合の Cauchy 問題を考えてみる。 この話は、非線型項 F が "巾 type" $F(u)=|u|^{p-1}u$ の時に、Cazenave and Weissler [7] によって初めて述べられたもので、以下の内容は、この結果を modify して用いたものである。

§2 Besov spc.

(1) の局所解を構成するには、線型部分"$-\Delta u$"と非線型部分"$F(u)$"を別々に評価することが必要である。H^s の s が自然数の場合 (i.e. $W^{k,2}$ の場合) は、Sobolev spc. $W^{k,p}$ の中で両者とも取り扱う事が出来る。しかし、s が一般の正数の時は、Fourier 変換 $\mathcal{F}$ を用いた $H^{s,p}$ の定義

$H^{s,p} = \mathcal{F}^{-1}\langle\xi\rangle^{-s}\mathcal{F}L^p$ が非線型項とmatchせず、これでは評価が通らない。そこで、Besov spc. $B^s_{p,q}$ 上で取り扱う事にする。

○ Besov spc. の定義 (Triebel [8] による)



$\psi(x) \in \mathcal{D}(\mathbb{R}^n)$, $\psi \geq 0$ を、

$\text{supp}\,\psi \subset \{x \in \mathbb{R}^n : \frac{1}{2} \leq |x| \leq 2\}$

$\psi > 0$ if $\frac{1}{\sqrt{2}} \leq |x| \leq \sqrt{2}$

なるもの固定し、

$\varphi_j(x) = \psi(2^{-j}x)\left(\sum_{k=-\infty}^{\infty}\psi(2^{-k}x)\right)$ if $j = 1, 2, 3, \cdots$

$\varphi_0(x) = 1 - \sum_{j=1}^{\infty}\varphi_j(x)$ と、函数列 $\{\varphi_j\}_{j=0}^{\infty}$ を定める。

この時、$-\infty < s < \infty$, $1 \leq p, q \leq \infty$ に対し

$$B^s_{p,q}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B^s_{p,q}} = \|\{2^{sj}\mathcal{F}^{-1}\varphi_j\mathcal{F}f\} \,|\, \ell^q(L^p(\mathbb{R}^n))\| < \infty\}$$

ここに右辺は、函数として L^p ノルムを取った数列の ℓ^q ノルムを取ること。

Besov spc 相互と、他の空間 特に Sobolev spc. との関係としては、次が成立する。

- $B^s_{p,q_0} \subset B^s_{p,q_1}$ if $1 \leq q_0 \leq q_1 \leq \infty$.
- $B^{s+\varepsilon}_{p,q_0} \subset B^s_{p,q_1}$ if $\varepsilon > 0$, $1 \leq q_0, q_1 \leq \infty$
- $B^s_{p,p\wedge 2} \subset H^{s,p} \subset B^s_{p,p\vee 2}$ if $p < \infty$
- $B^{s_0}_{p_0,q}(\mathbb{R}^n) \subset B^{s_1}_{p_1,q}(\mathbb{R}^n)$ if $p_0 \leq p_1$, $s_1 \leq s_0$, $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$

いずれも埋め込みは連続である。

また、線型作用素の評価には、次の実補間関係が重要である。

$$(B^{\Delta_0}_{p,q_0}, B^{\Delta_1}_{p,q_1})_{\theta,q} = B^{\Delta}_{p,q} \qquad \text{if } \Delta = (1-\theta)\Delta_0 + \theta\Delta_1$$

一方、非線型項の評価には、次の関係が重要となる。

L'a 1

$0 \le k < \Delta < k+1$、 kは整数 とする。このとき

$$\|f\|_{\dot{B}^{\Delta}_{p,q}} \equiv \sum_{|\alpha|=k} \left[\int_0^\infty \{t^{k-\Delta} \sup_{|h|\le t} \|\tau_h(\partial^\alpha f) - \partial^\alpha f\|_p\}^q \frac{dt}{t}\right]^{1/q}$$

とおけば、 $\|\cdot\|_{L^p} + \|\cdot\|_{\dot{B}^{\Delta}_{p,q}}$ は $\|\cdot\|_{B^{\Delta}_{p,q}}$ と同値な $B^{\Delta}_{p,q}$ のノルムとなる。 ここに τ_hは $\mathbb{R}^n$での hだけの並進である。

すなわち、Besov spc. は、上の補間関係によって線型の評価にmatchする一方、L'a1 によって非線型項の評価にもmatchした、都合の良い函数空間である。

§3. Main results

以下、微分方程式(1)を、次の積分方程式

(2) $\quad u = U\phi - iSF(u)$

に直して考える。 ここに $\phi(x)$、$f(t,x)$に対し、U、Sは

$$U\phi = e^{it\Delta}\phi, \qquad Sf = \int_0^t e^{i(t-s)\Delta} f(s)\,ds$$

と定義される線型作用素である。

非線型項 Fの中の "Potential" W に、次の仮定を置く。

仮定 A1

$$W = W_1 + W_2. \qquad W_1 \in L^\infty,\ W_2 \in L^q$$

$$\text{ここで} \quad q > \frac{n}{2\Delta+2} \ge 1$$

仮定A2

$$W = W_1 + W_2, \qquad W_1 \in L^{\gamma_1}, \qquad W_2 \in L^{\gamma_2}_w$$

ここで　$\infty > \gamma_1 > \gamma_2 = \frac{n}{2\Delta+2} > 1$

（注： L^{γ}_w は weak L^{γ} spc.で、だいたい $W = |x|^{-\frac{n}{\gamma}}$ のようなものが入る。）

この時、(2)の局所解の存在について、次が成立する。

Th. 2

$\phi \in H^{\Delta}$ とする。この時 仮定A1、又は A2 の下で、

(2)の Local Solution

$$u \in C([0,T]; H^{\Delta}) \cap L^{\theta}(0,T; B^{\Delta}_{p,2})$$

が存在する。

ここで、(θ, p)は admissible pair. すなわち、

$$\frac{1}{2} \geq \frac{1}{p} > \frac{1}{2} - \frac{1}{n}$$

$$\frac{1}{\theta} = \frac{n}{2}\left(\frac{1}{2} - \frac{1}{p}\right)$$

なる関係を満す 任意の pair である。

また、解の最大存在時間を T^*とすると、$T^* < \infty$なら、$\|u(t)\|_{H^{\Delta}} \nearrow \infty$ as $t \nearrow T^*$.

さらに、次の意味で、初期値への連続依存性が成立する。

$\phi_n \to \phi$ in H^{Δ} の時、対応する(2)の解 u_n, u は、

任意の $T < T^*$ と $\varepsilon > 0$ に対し、

$$u_n \to u \quad \text{in } C([0,T]; H^{\Delta-\varepsilon})$$

大域解の存在については、次の仮定を置く。

仮定A3

W は Real で, $W = W_1 + W_2$, $W_1 \in L^\infty$, $W_2 \in L^q$

$q > n/2$, $q > 1$

仮定A4

W は Real, even で, $W = W_1 + W_2$. $W_1 \in L^\infty$, $W_2 \in L^q$

$q > n/4$, $q > 1$

さらに $W_- \equiv (-W) \vee 0 \in L^r$

$r \geq 1 \ (n=1)$, $r > 1 \ (n=2)$, $r \geq n/2 \ (n \geq 3)$

定理3

$0 \leq \Delta < 1$ のときは A3

$1 \leq \Delta$ のときは A4

を仮定する。

この時、積分方程式(2)の、定理2で存在を保証される局所解は、大域解になる。つまり、最大存在時間 $T^* = \infty$

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Hysteresisのある制御項をもつ変分不等式について

広島工大　　小山哲也

0. 目的.

$\Omega \subset \mathbb{R}^N$, $(N \geq 1)$ を Lipschitz 連続な境界 Γ をもつ有界領域とするとき，次の発展変分不等式の初期値境界値問題を考える.

$$
(VI) \quad
\begin{array}{ll}
u_t(t,x) - \Delta_p u(t,x) - G(u)(t,x) \geq 0 & \text{in } Q_{T_0} := (0,T_0) \times \Omega, \\
u(t,x) \geq h(t,x) & \text{in } Q_{T_0} \\
(u_t - \Delta_p u - G(u))(u-h) = 0 & \text{in } Q_{T_0}, \\
u(t,x) = 0 & \text{for } t \in [0,T_0],\ x \in \Gamma, \\
u(0,x) = u_0(x) & \text{in } \Omega. \\
(p \geq 2.) &
\end{array}
$$

ここで G は hysteron operator $\mathcal{H}$ (後述) により

$$G(u)(t,x) := \mathcal{H}(u(\cdot,x), w_0(x))(t)$$

で定められる operator とする.

h, u_0 に対する適当な条件のもとで，$\mathcal{H}$ が "Lipschitz 連続な関数によって定められる hysteron operator" であるとき，(VI) の解が一意的に存在する事を示す.

1. Hysteron operators.

Rectangular hysteron.

$0 < T \leq T_0$ とし，$\rho_1 < \rho_2$ に対し，

$$\mathcal{H}_\rho : \mathcal{D}(\subset C([0,T]) \times \mathbb{R}) \to BV(0,T)$$

を

$$\mathcal{H}_\rho(\xi, w_0)(t) = \begin{cases} w_0, & \text{if } A_t := \{\tau \in (0,t]\ ;\ v(\tau) = \rho_1 \text{ または } \rho_2\} = \phi \\ 0, & \text{if } A_t \neq \phi,\ v(\max A_t) = \rho_1, \\ 1, & \text{if } A_t \neq \phi,\ v(\max A_t) = \rho_2. \end{cases}$$

でさだめる. これを Rectangular hysteron とよぶことにする. 但し w_0 は初期 switch 状態であり，$\mathcal{D}$ は

$$(\xi, w_0) \in \mathcal{D} \Longleftrightarrow w_0 \begin{cases} = 0, & \text{if } \xi(0) \leq \rho_1, \\ \in [0,1], & \text{if } \rho_1 < \xi(0) < \rho_2, \\ = 1, & \text{if } \rho_2 \leq \xi(0). \end{cases}$$

で定める. これは hysteresis 性 のある switch の model である.

しかしこの $\mathcal{H}_\rho$ は demiclosed ではなく，demiclosure をとると多価になる. (しかし この場合でも 適当に拡張された $\mathcal{H}_\rho$ を用いて G (多価) を作り，これを含む (VI) に対して解の

存在が示されている．(Visintin [5].))

Lipschitz hysteron. そこでより扱い易い次のような hysteron を考える．

$$f_d, f_a \in Lip(\mathbb{R}) \text{ は単調増加, } \mathbb{R} \text{ 上 } f_a \le f_d, \text{ かつ } \|f_d\|_{Lip(\mathbb{R})}, \|f_a\|_{Lip(\mathbb{R})} \le L$$
$$\mathcal{D} := \{ (\xi, w_0) \in C([0,T]) \times \mathbb{R} \; ; \; f_a(\xi(0)) \le w_0 \le f_d(\xi(0)) \}$$

とする．このとき

$$\mathcal{H} : \mathcal{D} \to C([0,T])$$

を次のようにさだめる．

$$\xi \in C([0,T]) \; ; 0 = t_0 < t_1 < \cdots < t_n = T \text{ があって各 } [t_{i-1}, t_i] \text{ 上 1 次関数.}$$
$$(\xi, w_0) \in \mathcal{D}, \; t \in [0,T]$$

にたいして

$$\mathcal{H}(\xi, w_0)(t) = \begin{cases} w_0, \\ \quad \text{if } t = 0, \\ \min\{f_d(\xi(t)), \max\{f_a(\xi(t)), \mathcal{H}(\xi, w_0)(t_{i-1})\}\}, \\ \quad \text{if } t \in [t_{i-1}, t_i], \; i = 1, 2, \cdots, n. \end{cases}$$

と定める．

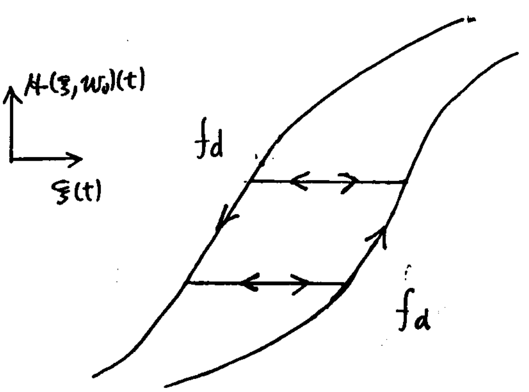
次に，任意の $(\xi_1, w_{0,1}), (\xi_2, w_{0,2}) \in \mathcal{D}$, $t \in [0,T]$ に対し

$$|\mathcal{H}(\xi_1, w_{0,1})(t) - \mathcal{H}(\xi_2, w_{0,2})(t)|$$
$$\le \max\{\|(f_a(\xi_1) - f_a(\xi_2))\|_{L^\infty(0,t)}, \|(f_d(\xi_1) - f_d(\xi_2))\|_{L^\infty(0,t)}, |w_{0,1} - w_{0,2}|\}$$

がなりたつから $\mathcal{H}$ は $\mathcal{D}$ 上に一意的に拡張され，

$$\|\mathcal{H}(\xi_1, w_{0,1}) - \mathcal{H}(\xi_2, w_{0,2})\|_{C([0,T])} \le \max\{L\|\xi_1 - \xi_2\|_{C([0,T])}, |w_{0,1} - w_{0,2}|\}$$

となる．



2. (VI) の解の一意存在.

$$\varphi^t(z) := \begin{cases} \frac{1}{p}\int_\Omega \sum_i^N |\frac{\partial z}{\partial x_i}|^p dx, & z \in K(t) := \{z \in W_0^{1,p}(\Omega); z \geq h(\cdot, t) \text{ a.e. on } \Omega\}, \\ \infty, & \text{otherwise} \end{cases}$$

とすると φ^t は $L^2(\Omega)$ 上の適正下半連続凸関数となり (VI) は次と同等になる：

$$(CP) \qquad \frac{du}{dt} + \partial\varphi^t(u) - G(u) \ni 0, \quad \text{for a.e. } 0 < t < T_0, \qquad u(0) = u_0,$$

定理. G を Lipshitz hysteron $\mathcal{H}$ により

$$G(v)(t,x) = \mathcal{H}(v(x,\cdot), w_0)(t)$$

で定められる operator とし，

$$f \in L^\infty(Q_{T_0}),$$
$$h \in W^{1,2}(0,T_0; L^2(\Omega)) \cap W^{1,1}(0,T_0; W^{1,p}(\Omega)) \cap L^\infty(Q_{T_0}),$$
$$h(t,x) \leq 0 \text{ a.e. on } [0,T_0] \times \Gamma,$$
$$u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),\ u_0 \geq h(0,\cdot) \text{ a.e. on } \Omega,$$
$$w_0 \in L^2(\Omega), f_a(u_0) \leq w_0 \leq f_d(u_0) \text{ a.e.on}\Omega,$$

とするとき (VI) は $W^{1,2}(0,T_0; L^2(\Omega)) \cap L^\infty(0,T_0; W_0^{1,p}(\Omega)) \cap L^\infty(Q_{T_0})$ において唯一解をもつ.

証明の概略.(i)

$$X := L^2(\Omega; C([0,T_0])), Y := W^{1,2}(0,T_0; L^2(\Omega)) \cap L^\infty(0,T_0; W^{1,p}(\Omega))$$

とおく.

$$G; X \cap L^\infty(Q_{T_0}) \to X \cap L^\infty(Q_{T_0})$$

は well-defined であり 任意の $v_1, v_2 \in X \cap L^\infty(Q_{T_0})$ $(v_1(0) = v_2(0) = u_0)$ に対して

$$\|G(v_1) - G(v_2)\|_{L^\infty(Q_{T_0})} \leq L\|v_1 - v_2\|_{L^\infty(Q_{T_0})},$$

がなりたつ.

(ii) $\omega \in L^\infty(Q_{T_0})$ に対して 問題

$$(CP)_\omega \qquad \frac{\partial z}{\partial t} + \partial\varphi^t z - \omega \ni 0, \quad \text{for a.e. } 0 < t < T_0, \qquad z(0) = u_0,$$

は一意な強解 $z =: k(\omega) \in Y \cap L^\infty(Q_{T_0})$ をもち，ω, T_0 によらない定数 $C_1 \sim C_3 > 0$ があって

$$\|k(\omega)\|_Y \leq C_1 T_0 \|\omega\|_{L^2(Q_{T_0})} + C_2,$$
$$\|k(\omega)\|_{L^\infty(Q_{T_0})} \leq T_0 \|\omega\|_{L^\infty(Q_{T_0})} + C_3,$$
$$\|k(\omega_1)(t) - k(\omega_2)(t)\|_{L^\infty(\Omega)} \leq \int_0^t \|\omega_1 - \omega_2\|_{L^\infty(Q_\tau)} d\tau$$

が成り立つ．(Kenmochi [2], Kenmochi-Koyama [6].)

(iii) $Y \hookrightarrow X$ （連続）であり

$$k \circ G : X \cap L^\infty(Q_{T_0}) \to X \cap L^\infty(Q_{T_0})$$

である．

(iv) 任意の $v, v_1, v_2 \in X \cap L^\infty(Q_{T_0})$ に対して

$$\|k \circ G(v)\|_{X \cap L^\infty(Q_{T_0})} \leq \exists M,$$
$$\|k \circ G(v_1) - k \circ G(v_2)\|_{L^\infty(Q_t)} \leq \int_0^t L \|v_1 - v_2\|_{L^\infty(Q_\tau)} d\tau$$

がなりたつので，$k \circ G$ は $L^\infty(Q_{T_0})$ で 唯一の不動点をもつ．これが (VI) の唯一解をあたえる．□

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