第６回発展方程式若手セミナー

報告集

１９８５

序

この報告集は，第６回発展方程式若手セミナーの講演をまとめたものである。本セミナーは，１９８４年８月２４日から８月２６日まで，前年度と同じく箱根の静雲荘で行なわれた。今回のセミナーでは，ひとつの特別講演と１０の一般講演があり，それぞれに興味ある新しい話題が提供された。特別講演は毎回恒例となっているものであるが，今回は広島大学の　俣野　博　氏にお願いし，

”非線型拡散方程式の解の漸近挙動について”

という題で，３時間にわたって最新の話題が提供された。講演および講演後の討論がきわめて活発に，しかも和気合い合いとした雰囲気のもとでなされたことにより，当セミナーの本来の目的は一応達成されたものと思う。

今回のセミナーを行なうに際し，様々な方にお世話になった。特に，セミナーには参加していただけなかったが，御茶ノ水女子大の高村幸男教授には前面的に御援助いただいた。ここに深く感謝の意を表します。また，姫路工業大学の　丸尾健二　氏からは，適切なアドバイスをいただき，おかげでセミナーがスムーズに運営されました。

セミナーの報告集の作成が大変遅れたことを深くお詫びいたします。なお，都合により俣野氏の講演記録がのせられなかったことは，大変残念でありますが，興味のある方は，氏の近著（ Pitman から出版予定）を御覧になってください。

１９８５年７月　　セミナー世話人

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第６回発展方程式若手セミナープログラム

８月２４日

18:00 ～19:30	食事及び自己紹介	
19:30 ～20:00	山崎昌男	非圧縮性流体の方程式に対する microlocal analysis

８月２５日

8:00～9:00	朝食	
9:30～10:00	川島秀一	ある非線型方程式の進行波解の安定性
10:15 ～10:45	岡本　久	Poincaré-Lyapunov の問題について
11:00 ～12:00	俣野　博	非線型拡散方程式の解の漸近挙動について
13:30 ～14:30	俣野　博（続き）	
14:45 ～15:45	俣野　博（続き）	
16:00 ～16:30	桔梗洋子	放物型方程式の解の漸近挙動
16:45 ～17:15	東海林まゆみ	完全流体の自由境界問題に対する数値シミュレーション
17:30 ～18:00	中桐信一	Partial Functional Differential Equations and Optimal Control

８月２６日

8:00～9:00	朝食	
9:00～9:30	大河内広子	非線型放物型方程式 $du/dt \in \varphi(u(t))+f(t)$ の周期解の存在について
9:45～10:15	重田多恵子	local surjectivity について
10:30 ～11:00	福田賢一	Second order semilinear equations
11:15 ～11:45	堤　誉志雄	Remarks on nonlinear Schrödinger equations

目次

非線型拡散方程式の解の漸近挙動について

俣野 博　　（広島大学　理学部　数学教室）

Microlocal Analysis for Nonlinear Equations for Incompressible Fluids

Masao Yamazaki

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We would like to study the microlocal properties of the equations of incompressible fluids, that is, the microlocal hypoellipticity of the Navier-Stokes equations

$$
\text{(NS)} \quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \displaystyle\sum_{k=1}^{n} \frac{\partial}{\partial x_k}(u_k \cdot u) + \nabla p = f & \text{in } (0,T) \times \Omega, \\ \nabla \cdot u = 0 & \text{in } (0,T) \times \Omega \end{cases}
$$

and the propagation of local and microlocal regularity of the Euler equations

$$
\text{(E)} \quad \begin{cases} \dfrac{\partial u}{\partial t} + \displaystyle\sum_{k=1}^{n} \frac{\partial}{\partial x_k}(u_k \cdot u) + \nabla p = f & \text{in } (0,T) \times \Omega, \\ \nabla \cdot u = 0 & \text{in } (0,T) \times \Omega. \end{cases}
$$

Here $0 < T < \infty$, Ω is an open set in $\mathbb{R}^n$ ($n \geq 2$), the external force $f = (f_1, \cdots, f_n)$ is a given real-valued function of $t \in I = [0,T]$ and $x \in \Omega$, and the velocity $u = (u_1, \cdots, u_n)$ and the pressure p are unknown real-valued functions of t and x.

Microlocal analysis for nonlinear equations has been recently studied by many authors. See Lascar [13], Beals [2], [3], [4], Rauch [15], Bony [6], Meyer [14], Beals-Reed [5], Rauch-Reed [16], [17], [18], [19], [20]. They supposed the existence of a solution with some regularity, or showed the short-time existence of such a solution; and then analyzed the solution microlocally. We work on a similar assumption; that is, we suppose the existence of a solution (u,p) which is 'strong' in some sense. Indeed, it seems too difficult to discuss on such properties of the 'weak' solutions at irregular points.

First we introduce a notation. For a function space $E \subset D'(\Omega)$ with a stronger topology than that of $D'(\Omega)$, let $B(I,E)$ denote the set of distributions $v(t,\cdot)$ with a parameter $t \in I$ such that

$$v(t,\cdot) \in E \quad \text{for all} \quad t \in I,$$

and that

$$\{ v(t,\cdot) : t \in I \} \text{ is bounded in } E.$$

Next we define some notions which will be used to describe our results.

Definition.

For a subset $C \subset I\times\Omega$ and a distribution $v(t,x) \in B(I,D'(\Omega))$, we say that $v(t,x)$ is locally in E on C if, for every compact subset K of C, there exists a function $\phi(t,x) \in C_0^\infty(I\times\Omega)$ such that $\phi(t,x) \equiv 1$ holds on some neighborhood of K and that $\phi(t,x)v(t,x) \in B(I,E)$ holds.

For a subset $\Gamma \subset I\times(T^*\Omega\backslash 0) = I\times\Omega\times(\mathbb{R}^n\backslash\{0\})$ and $v(t,x)$ as above, we say that $v(t,x)$ is microlocally in E on Γ if, for every compact subset K of Γ, there exist functions $\phi(t,x) \in C_0^\infty(\mathbb{R}^n)$ and $\phi(t,x,\xi) \in C^\infty(I,S^0)$ such that the following three conditions hold:

$$\begin{cases} \phi(t,x) \equiv 1 & \text{on some neighborhood of } \pi(K). \\ \phi(t,x,\xi) \equiv 1 & \text{on some conic neighborhood of } K. \\ \phi(t,x,D)(\phi u)(t,x) \in B(I,E). \end{cases}$$

Here S^0 denotes the class of the symbols of the zeroth order pseudodifferential operators on $\mathbb{R}^n_x$, π denotes the natural projection of $I\times(T^*\Omega\backslash 0)$ onto $I\times\Omega$, and we say that U is a conic neighborhood of K if there exists an open subset V of $I\times(T^*\Omega\backslash 0)$ such that $K \subset V \subset U$ and that $(t,x,\lambda\xi) \in V$ holds for $(t,x,\xi) \in V$ and $\lambda \geq 1$.

Now we can state the main theorem for (NS). We suppose that (u,p) is a solution of (NS), and that all u_j and p belong to the space $B(I,D'(\Omega))$.

<u>Theorem 1.</u> (Microlocal hypoellipticity of (NS))

— <u>Suppose</u> $0 < \mathring{t} \leq T$, $\mathring{x} \in \Omega$, $\mathring{\xi} \neq 0$ and $s > \max\{0, n/r-1\}$. <u>If each</u> u_j <u>is locally in</u> W^s_r <u>on the set</u> $\{(\mathring{t},\mathring{x})\}$ <u>and each</u> f_j <u>is microlocally in</u> $W^{2s-1-n/r}_r$ <u>on</u> $\{(\mathring{t},\mathring{x},\mathring{\xi})\}$, <u>then each</u> u_j <u>is microlocally in</u> $W^{2s+1-n/r-\delta}_r$ <u>on</u> $\{(\mathring{t},\mathring{x},\mathring{\xi})\}$ <u>for every positive number</u> δ.

Statements for p and $\frac{\partial u}{\partial t}$ will be given in Theorems 4 and 5 later.

To describe our results for (E), we must put further assumptions. Let σ be a number greater than 1. A function $v(t,x)$ on $I\times\Omega$ is said to belong to the class $C^{0,\sigma}(I\times\Omega)$ if the following two conditions (1) and (2) are satisfied:

(1) $\partial_x^\alpha u(t,x)$ exists and is bounded, continuous on $I \times \Omega$ for any $\alpha \in \mathbb{N}^n$ such that $|\alpha| < \sigma$.

(2) $|\partial_x^\alpha u(t,x) - \partial_x^\alpha u(t,y)| / |x-y|^{\sigma-k}$ is bounded on $I \times \Omega$ for any $\alpha \in \mathbb{N}^n$, where k is the greatest integer less than σ.

In the next definition and Theorems 2 and 3, we suppose that (u,p) is a solution of (E) such that $p \in B(I, \mathcal{D}'(\Omega))$ and that $u_j \in C^{0,\sigma}(I \times \Omega)$ for every $j = 1, \cdots, n$.

Remark 1.

For $n = 2$, Kato [10] proved the existence of the time-global solution of (E) satisfying the above assumptions. His results can be summarized as follows:

Suppose that Ω is a bounded domain with smooth boundary $\partial\Omega$ and that σ is a positive number such that $\sigma \notin \mathbb{N}$ and $\sigma > 1$. Put $f \equiv 0$, and let $u_0(x)$ be a function in $C^\sigma(\bar{\Omega})$ satisfying $u_0(x) \cdot n_x = 0$, where n_x is the normal vector of $\partial\Omega$ at x. Then there exists uniquely a pair (u,p) which satisfies

$$u(0,x) = u_0(x) \quad \text{on} \quad \Omega,$$

$$(0.1) \quad u(t,x) \cdot n_x = 0 \quad \text{on} \quad I \times \partial\Omega$$

and is a solution of (E) satisfying the above conditions.

For $n \geq 3$, the existence of the solutions satisfying the above conditions have been obtained by Ebin-Marsden [8], Swann [22], Kato [11], Bourguignon-Brezis [7], Temam [23] and Kato-Lai [12]. But, in this case, the number T depends on $u_0(x)$.

These results suggest that our assumption is not unnatural.

Next, to state our results, we introduce some notions.

Definition. We call a connected integral curve of the vector field

$$\frac{\partial}{\partial t} + \sum_{k=1}^{n} u_k \cdot \frac{\partial}{\partial x_k} \qquad \text{in} \quad I\times\Omega$$

a trajectory curve, and a connected integral curve of

$$\frac{\partial}{\partial t} + \sum_{k=1}^{n} u_k \cdot \frac{\partial}{\partial x_k} - \sum_{j,k=1}^{n} \xi_j \cdot \frac{\partial u_j}{\partial x_k} \cdot \frac{\partial}{\partial \xi_k} \qquad \text{in} \quad I\times(T^*\Omega\backslash 0)$$

a bicharacteristic. That is, a curve $\{(t,X(t))\} = C$ is a trajectory curve and $\{(t,X(t),\Xi(t))\} = \Gamma$ is a bicharacteristic if and only if $X(t)$ and $\Xi(t)$ satisfy the system

$$\begin{cases} (0.2)\quad \dfrac{\partial X_j}{\partial t} = u_j(t,X(t)), \\ (0.3)\quad \dfrac{\partial \Xi_j}{\partial t} = -\displaystyle\sum_{k=1}^{n} \Xi_k(t)\frac{\partial u_k}{\partial x_j}(t,X(t)). \end{cases}$$

Remark 2.

To solve the above system, we first solve the equations (0.2). Owing to the Lipschitz condition of $u_j(t,x)$ with respect to x, the system (0.2) can be solved uniquely, at least locally in time. Then the linear system (0.3) can be solved as long as $X(t)$, the solution of (0.2), exists, and the solution $\Xi(t)$ is homogeneous of degree 1 with respect to the initial value. Especially, if the initial value is not equal to zero, then $\Xi(t)$ never vanishes.

If $u(t,x)$ satisfies the boundary condition (0.1), then the system (0.2) can always be solved for whole $t \in I$.

Roughly speaking, our results for the equation (E) are as follows: let C be a trajectory curve and Γ be a bicharacteristic. Then the local regularity of the solution $u(t,\cdot)$, where $t \in I$ is regarded as a parameter, propagates along C, provided the external force f is sufficiently smooth along C. Similarly, if f is sufficiently smooth along Γ, then the microlocal regularity of $u(t,\cdot)$ propagates along Γ; that is, for two different times s and t, the wave front set (modulo an appropriate function space) of $u(s,\cdot)$ is mapped onto that of $v(t,\cdot)$ by the transformation of $T^*\Omega$ induced by the diffeomorphism of Ω determined by the trajectory curves.

More strictly, we have the following theorems.

Theorem 2. (Propagation of local regularity in (E))

Suppose that f_j is locally in W_r^s on a trajectory curve C for every j, where $s > 1$. Then, if there exists a point $(\mathring{t},\mathring{x}) \in C$ such that every $u_j(\mathring{t},\cdot)$ is locally in W_r^s at $\mathring{x}$, the solution $u_j(x)$ is locally in W_r^s on C for every j.

Theorem 3. (Propagation of microlocal regularity in (E))

Suppose that $\sigma > 2$, that every $f_j(x)$ is microlocally in W_r^s on a bicharacteristic Γ, and every $u_j(x)$ is locally in $W_r^{s+2-\sigma}$ on the trajectory curve $\pi(\Gamma)$.

Then, if there exists a point $(\mathring{t},\mathring{x},\mathring{\xi}) \in \Gamma$ such that every $u_j(\mathring{t},\cdot)$ is microlocally in W_r^s at $(\mathring{x},\mathring{\xi})$, the solution $u_j(t,\cdot)$ is microlocally in W_r^s on Γ.

Remark 3.

The trajectory curve and the bicharacteristic play the same roles as those of the bicharacteristic curve and the bicharacteristic strip respectively, in the theory of linear equations. Usually, for higher order differential equations or first order systems, local regularity does not propagate along bicharacteristic curves. But in this case, the equation is essentially first order, hence all bicharacteristics passing through the fiber of a base point are mapped onto the same trajectory curve by the projection π. Owing to this fact, our local propagation theorem is valid.

Finally we shall consider the regularity of $\frac{\partial u}{\partial t}(t,\cdot)$ and $p(t,\cdot)$. For this purpose, we put

$$(0.4)\qquad \tau = \begin{cases} \text{any number greater than } \max(0, n/r-n/2) \\ \qquad \text{if } s \leqq \max(-n/r, n/r-n)-1, \\ (s+n/r+1)/2 \quad \text{if } s > \max(-n/r, n/r-n)-1 \end{cases}$$

and

$$(0.5)\qquad \rho = \max(s, \tau)$$

for a real number s.

Then we have the following two theorems.

Theorem 4.

Let (u,p) be a solution of (NS) or (E) such that all u_j and p belong to the space $B(I,D'(\Omega))$, and suppose that C is a subset of $I\times\Omega$. If all f_j are locally in W_r^s on C and if all u_j are locally in W_r^ρ on C, where ρ is determined by (0.5), then p is locally in W_r^{s+1} on C and all $\frac{\partial u_j}{\partial t}$ is locally in W_r^{s-1} on C if (u,p) is the solution of (E), and is in W_r^{s-2} on C if (u,p) is the solution of (NS).

Theorem 5.

Let (u,p) be as in the previous theorem, and suppose that Γ is a subset of $I\times(T^*\Omega\backslash 0)$. If the conditions

$$\begin{cases} \text{Every } f_j \text{ is microlocally in } W_r^s \text{ on } \Gamma. \\ \text{Every } u_j \text{ is microlocally in } W_r^s \text{ on } \Gamma. \\ \text{Every } u_j \text{ is locally in } W_r^\tau \text{ on } \pi(\Gamma), \text{ where } \tau \text{ is determined by (0.4).} \end{cases}$$

are satisfied, then p is microlocally in W_r^{s+1} on Γ. Every $\frac{\partial u_j}{\partial t}$ is microlocally in W_r^{s-1} on Γ if (u,p) is a solution of (E), and is microlocally in W_r^{s-2} on Γ if (u,p) is a solution of (NS).

Remark 4.

Using the results of [25] and [26], we can replace the Sobolev space W_r^s by the Besov space B_{pq}^s and the Triebel-Lizorkin space F_{pq}^s, which are generalizations of the Hölder space and the Sobolev space respectively. For the definitions and the basic properties of these spaces, see Triebel [24]. The local propagation theorem for the equation (E) in the Hölder space was, as far as the author knows, first obtained by Giga [9], and the propagation of local analyticity was proved by Alinhac-Métivier [1].

Proof of these resuts are given in Yamazaki [28], [29].

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Asymptotic Stability of Traveling Wave Solutions for Scalar Conservation Laws with Viscosity

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This is a joint work with A. Matsumura (Kyoto University).

1. Introduction and main theorem

We study a scalar conservation law of the form

(1.1) $$u_t + f(u)_x = \mu u_{xx} ,$$

where f is a smooth function on an interval $[\underline{u},\overline{u}]$ such that

(1.2) $$f''(u) > 0 \quad \text{for all} \quad u \in [\underline{u},\overline{u}] ,$$

and μ is a positive constant. The equation (1.1) admits smooth traveling wave solutions with shock profile

(1.3) $$u(t,x) = U(\xi) , \qquad \xi = x - st ,$$

(1.4) $$U(\xi) \to u_\pm \quad \text{as} \quad \xi \to \pm\infty ,$$

where $u_\pm \in (\underline{u},\overline{u})$ and s (the shock speed) are constants satisfying the Rankine-Hugoniot condition

(1.5) $$s(u_+ - u_-) = f(u_+) - f(u_-) ,$$

and the shock condition

(1.6) $\quad f'(u_+) < s < f'(u_-)$, $\quad$ (or equivalently, $u_+ < u_-$) .

The function U can be determined by the ordinary differential equation

(1.7) $\quad \mu U_\xi = - sU + f(U) + a \equiv - M(U)$,

where $a = - su_\pm + f(u_\pm)$ is the integral constant. Note that the solution U of (1.7) with the condition (1.4) is unique up to a shift in ξ. Furthermore, from the inequality $M(u) = s(u - u_\pm) - (f(u) - f(u_\pm)) > 0$ (for $u \in (u_+, u_-)$), we know that U is a strictly decreasing function of $\xi \in \mathbb{R}$.

We consider the initial value problem for (1.1) with the initial condition

(1.8) $\quad u(0,x) = u_0(x)$,

where u_0 is a bounded measurable function such that

(1.9) $\quad u_0(x) \to u_\pm \quad$ as $\quad x \to \pm\infty$,

and the integrals

$$\int_{-\infty}^{0} (u_0(x) - u_-)dx \quad \text{and} \quad \int_0^{+\infty} (u_0(x) - u_+)dx$$

exist. Under these assumptions Il'in and Oleinik [2] proved that as $t \to \infty$, the solution $u(t,x)$ of the problem (1.1),(1.8) tends uniformly with respect to $x \in \mathbb{R}$ to the traveling wave solution $U(x - st)$ which is uniquely determined by the relation

(1.10) $$\int_{-\infty}^{+\infty} (u_0 - U)(x)dx = 0 .$$

They also showed that if the integrals

$$\int_{-\infty}^{x} (u_0(y) - u_-)dy \quad \text{and} \quad \int_x^{+\infty} (u_0(y) - u_+)dy$$

decay exponentially $e^{-\alpha|x|}$ $(\alpha > 0)$ as $x \to -\infty$ and $x \to +\infty$ respectively, then the convergence is of an exponential rate $e^{-\gamma t}$ $(\gamma > 0)$ as $t \to \infty$.

Our aim is to show an algebraic decay rate $t^{-\gamma}$ $(\gamma > 0)$ under suitable assumptions on the initial data. Let U be a traveling wave solution. We assume that

$$(1.11) \qquad u_0 - U \in H^1 ,$$

and the integral

$$(1.12) \qquad \Psi_0(x) = \int_{-\infty}^{x} (u_0 - U)(y)dy$$

exists for any $x \in \mathbb{R}$, and satisfies

$$(1.13) \qquad \Psi_0 \in L^2_\alpha \qquad \text{for some } \alpha \geq 0 .$$

Here H^ℓ denotes the L^2-Sobolev space of order ℓ, with the norm $\|\cdot\|_\ell$, and L^2_α $(\alpha \in \mathbb{R})$ is a weighted L^2-space defined by $L^2_\alpha = \{ f ; \langle x \rangle^{\alpha/2} f \in L^2 \}$, with the norm

$$|f|_\alpha = \left(\int \langle x \rangle^\alpha |f(x)|^2 dx \right)^{1/2} ,$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$. We also use the abbreviation $\|\cdot\|$ for L^2-norm $\|\cdot\|_0 = |\cdot|_0$. It should be noted that (1.11) and (1.13) imply (1.9) and (1.10).

Since f in (1.1) is defined only on $[\underline{u},\overline{u}]$, it is reasonable to assume

$$(1.14) \qquad u_0(x) \in [\underline{u},\overline{u}] \qquad \text{for any } x \in \mathbb{R} .$$

In the following we simply assume $\|u_0 - U\|_1 \leq \varepsilon_0$ instead of the condition (1.14), where ε_0 is a positive constant satisfying

$$\varepsilon_0 \le \min\{ u_+ - \underline{u}\, ,\, \bar{u} - u_- \} \, . \tag{1.15}$$

Our main theorem is the following

Theorem 1.1. ([3]) *Let* $u_\pm \in (\underline{u},\bar{u})$ *and* s *satisfy* (1.5) *and* (1.6), *and let* $U(x-st)$ *be a traveling wave solution which smoothly interpolates the asymptotic values* $u_\pm$ *with the speed* s. *Suppose that the initial data* u_0 *satisfy* (1.11) *and* (1.13) *for some* $\alpha \ge 0$. *Then there exists a positive constant* ε_1 $(\le \varepsilon_0)$ *such that if* $\|u_0 - U\|_1 + \|\Psi_0\| \le \varepsilon_1$, *the initial value problem* (1.1), (1.8) *has a unique global solution* $u(t,x)$ *with*

$$(1.16) \qquad \begin{aligned} & u - U \in C^0(0,\infty\,; H^1) \cap L^2(0,\infty\,; H^2)\, , \\ & u(t,x) \in [\underline{u},\bar{u}] \qquad \text{for any } \; t \ge 0,\; x \in \mathbb{R}. \end{aligned}$$

Moreover, the solution tends in the maximum norm to the traveling wave solution at the rate $t^{-\gamma/2}$ *with* $\gamma = [\alpha]$:

$$\sup_{x \in \mathbb{R}} |u(t,x) - U(x-st)| \le C_1 (1+t)^{-\gamma/2} (\|u_0 - U\|_1 + |\Psi_0|_\alpha) \tag{1.17}$$

for any $t \ge 0$, *where* C_1 *is a positive constant.*

Remark 1. Our theorem is motivated by the work of Nishihara [5], where a similar result was obtained for the Burgers equation (the equation (1.1) with $f(u) = u^2/2$) by using an explicit expression of solutions.

Remark 2. For resuls on the asymptotic stability (without decay rate) of traveling wave solutions for systems in gas dynamics, see [1], [3] and [4].

2. Reformation of the problem

Let us reset the problem (1.1),(1.8) on the moving coordinate $\xi = x - st$. Letting $U(\xi)$ be the traveling wave solution in Theorem 1.1, we put

(2.1) $$u(t,x) = U(\xi) + \psi(t,\xi) .$$

Then the problem is reduced to

(2.2) $$\psi_t - s\psi_\xi + \{f(U+\psi) - f(U)\}_\xi = \mu\psi_{\xi\xi} ,$$

(2.3) $$\psi(0,\xi) = \psi_0(\xi) \equiv (u_0 - U)(\xi) .$$

Inspired by the relation $\psi_0 = \Psi_{0,\xi}$ (see (1.12)) we seek the solution of (2.2) in the form

(2.4) $$\psi = \Psi_\xi .$$

Substituting it into (2.2) and integrating once with respect to ξ, we get

(2.5) $$\Psi_t - s\Psi_\xi + f(U+\Psi_\xi) - f(U) = \mu\Psi_{\xi\xi} ,$$

with the initial data

(2.6) $$\Psi(0,\xi) = \Psi_0(\xi) .$$

Let us define the solution space of (2.5) by

$$X(0,T) = \{ \Psi \in C^0(0,T; H^2) ; \Psi_\xi \in L^2(0,T; H^2) \} ,$$

with $0 < T \le +\infty$. Then the problem (2.5),(2.6) can be solved globally in time as follows.

<u>Theorem 2.1</u>. *Suppose* $\Psi_0 \in H^2 \cap L^2_\alpha$ *for some* $\alpha \ge 0$. *Then there*

exist positive constants ε_2 $(\le \varepsilon_0)$ *and* C_2 *such that if* $\|\Psi_0\|_2 \le \varepsilon_2$, *the problem* (2.5),(2.6) *has a unique global solution* $\Psi \in X(0,\infty)$ *satisfying*

$$(2.7) \qquad (1+t)^\gamma \|\Psi(t)\|_2^2 + \int_0^t (1+\tau)^\gamma \|\Psi_\xi(\tau)\|_2^2 \, d\tau \le C_2^2 (|\Psi_0|_\alpha^2 + \|\Psi_{0,\xi}\|_1^2)$$

for $t \ge 0$, *where* $0 \le \gamma \le [\alpha]$.

For the solution Ψ in Theorem 2.1, we set $\psi = \Psi_\xi$. Then ψ belongs to $C^0(0,\infty\,;H^1) \cap L^2(0,\infty\,;H^2)$ and is a global solution of the problem (2.2), (2.3). Therefore we get a desired solution of the original problem (1.1), (1.8) through the relation (2.1). On the other hand, the solution of (1.1) is unique in the function space $C^0(0,T\,;H^1) \cap L^2(0,T\,;H^2)$. Therefore Theorem 1.1 follows from Theorem 2.1.

To prove Theorem 2.1, we prepare a local existence result and a priori estimates of solutions for (2.5).

<u>Proposition 2.2</u> (local existence). *Suppose* $\Psi_0 \in H^2$ *and* $\|\Psi_0\|_2 \le \varepsilon_0/2$. *Then there is a positive constant* T_0 *depending on* ε_0 *such that the problem* (2.5),(2.6) *has a unique solution* $\Psi \in X(0,T_0)$ *satisfying*

$$(2.8) \qquad \|\Psi(t)\|_2^2 + \int_0^t \|\Psi_\xi(\tau)\|_2^2 \, d\tau \le 4\|\Psi_0\|_2^2$$

for $t \in [0,T_0]$. *Moreover, if* $\Psi_0 \in L_\alpha^2$ *for some* $\alpha \ge 0$, *then we have* $\Psi \in C^0(0,T_0\,;L_\alpha^2)$ *and* $\Psi_\xi \in L^2(0,T_0\,;L_\alpha^2)$.

<u>Proposition 2.3</u> (a priori estimate). *Let* T *be a positive constant. Suppose that the problem* (2.5),(2.6) *has a solution* $\Psi \in X(0,T)$ *satisfying* $\Psi \in C^0(0,T\,;L_\alpha^2)$ *and* $\Psi_\xi \in L^2(0,T\,;L_\alpha^2)$ *for some* $\alpha \ge 0$. *Then for each* $\beta \in [0,\alpha]$, *there exist positive constants* ε_3 $(\le \varepsilon_0)$ *and* C_3, *which are independent of* T *and* β, *such that if* $\sup_{0\le t\le T} \|\Psi(t)\|_2 \le \varepsilon_3$, *then*

$$(2.9) \qquad (1+t)^{\gamma}\|\Psi(t)\|_2^2 + \int_0^t (1+\tau)^{\gamma}\|\Psi_\xi(\tau)\|_2^2\,d\tau \le C_3^2(|\Psi_0|_\beta^2 + \|\Psi_{0,\xi}\|_1^2)$$

holds for $t \in [0,T]$, *where* $0 \le \gamma \le [\beta]$.

Proposition 2.2 can be proved in the standard way. So we omit its proof. Proposition 2.3 will be proved in the following two sections. Here we show Theorem 2.1 by the continuation arguments based on Propositions 2.2 and 2.3.

Proof of Theorem 2.1. Choose ε_2 and C_2 such that

$$\varepsilon_2 = \min\{\varepsilon_3/2,\ \varepsilon_3/2C_3\}\ , \quad C_2 = C_3\ .$$

Then the local solution of (2.5),(2.6) can be continued globally in time, provided the smallness condition $\|\Psi_0\|_2 \le \varepsilon_2$ is satisfied. In fact we have $\|\Psi_0\|_2 \le \varepsilon_2 \le \varepsilon_3/2$. Therefore, by Proposition 2.2, there is a positive constant $T_0 = T_0(\varepsilon_3)$ such that a solution exists on $[0,T_0]$ and satisfies $\|\Psi(t)\|_2 \le 2\|\Psi_0\|_2 \le \varepsilon_3$ for $t \in [0,T_0]$. Hence we can apply Proposition 2.3 with $T = T_0$, and get the estimate (2.9) for $t \in [0,T_0]$. In particular, putting $\beta = 0$, we have $\|\Psi(t)\|_2 \le C_3\|\Psi_0\|_2$ for $t \in [0,T_0]$. Noting that $\|\Psi(T_0)\|_2 \le C_3\varepsilon_2 \le \varepsilon_3/2$, we apply Proposition 2.2 by taking $t = T_0$ as the new initial time. Then we have a solution on $[T_0,2T_0]$ with the estimate $\|\Psi(t)\|_2 \le 2\|\Psi(T_0)\|_2 \le \varepsilon_3$ for $t \in [T_0,2T_0]$. Therefore $\|\Psi(t)\|_2 \le \varepsilon_3$ holds on $[0,2T_0]$. Hence Proposition 2.3 again gives the estimate (2.9) for $t \in [0,2T_0]$. In the same way we can extend the solution to the interval $[0,nT_0]$ succesively for $n = 1,2,\cdots$, and get a global solution. The estimate (2.7) is a consequence of (2.9) with $\beta = \alpha$. This completes the proof of Theorem 2.1.

3. Basic inequalities

Let $\Psi \in X(0,T)$ (for some $T > 0$) be a solution of (2.5),(2.6) satisfying $\Psi \in C^0(0,T; L^2_\alpha)$ and $\Psi_\xi \in L^2(0,T; L^2_\alpha)$ for some $\alpha \geq 0$. Put

$$N(t) = \sup_{0 \leq \tau \leq t} \|\Psi(\tau)\|_2 \quad \text{for} \quad t \in [0,T] ,$$

and assume that $N(T) \leq \varepsilon_0$, where ε_0 is a constant in (1.15). In order to estimate the solution, we rewrite the equation (2.5) in the form

$$\text{(3.1)} \qquad \Psi_t - (s - f'(U))\Psi_\xi - \mu\Psi_{\xi\xi} = F(U,\Psi_\xi) ,$$

where

$$\text{(3.2)} \qquad F(U,\psi) = -\{f(U+\psi) - f(U) - f'(U)\psi\} .$$

First we study the properties of the traveling wave solutions. Let $u_* \in (u_+,u_-)$ be a state uniquely determined by

$$s = (f(u_+) - f(u_-))/(u_+ - u_-) = f'(u_*) ,$$

and let $U = U(\xi)$ be the traveling wave solution in Theorem 1.1. Since U is strictly decreasing in $\xi \in \mathbb{R}$, there exists uniquely a number $\xi_* \in \mathbb{R}$ such that

$$\text{(3.3)} \qquad U(\xi_*) = u_* .$$

The following result on the coefficient in (3.1) plays an important role in deriving a priori estimates of weighted norm of solutions.

Lemma 3.1. *For any* $\beta \in [0,\alpha]$, *there is a positive constant* c_0 *independent of* β *such that*

(3.4) $$A_\beta(\xi) \equiv \frac{1}{2}\{\beta(\xi-\xi_*)<\xi-\xi_*>^{-1}(s-f'(U)) - <\xi-\xi_*>f'(U)_\xi\} \geq \beta c_0$$

for any $\xi \in \mathbb{R}$, *where* $<\xi> = (1+|\xi|^2)^{1/2}$.

Proof. From the inequality $U_\xi < 0$ (for $\xi \in \mathbb{R}$) and the conditions (1.2), (3.3) and (1.6) we can deduce that $g(\xi) \equiv s - f'(U(\xi))$ is an increasing function of $\xi \in \mathbb{R}$, and satisfies $g(\xi_*) = 0$, $g'(\xi_*) > 0$, and $g(\xi) \to g_+ > 0$ (resp. $g_- < 0$) as $\xi \to +\infty$ (resp. $-\infty$), where $g'(\xi_*) = \mu^{-1}f''(u_*)M(u_*)$ and $g_\pm = s - f'(u_\pm)$. Therefore,

$$(\xi-\xi_*)<\xi-\xi_*>^{-1}(s-f'(U)) \geq \begin{cases} \frac{1}{2}g'(\xi_*)(\xi-\xi_*)^2 & \text{for } \xi \text{ near } \xi_*, \\ c & \text{otherwise,} \end{cases}$$

where c is a positive constant. On the other hand, $-<\xi-\xi_*>f'(U)_\xi = <\xi-\xi_*>g'(\xi) > 0$ holds for $\xi \in \mathbb{R}$, and in particular, $-<\xi-\xi_*>f'(U)_\xi \geq g'(\xi_*)/2$ for ξ near ξ_*. These considerations prove the lemma.

Next, using Lemma 3.1, we get basic inequalities for weighted norm of solutions.

<u>Lemma 3.2</u>. *For any* $\beta, \gamma \in [0,\alpha]$, *there is a positive constant* C *independent of* T, β *and* γ *such that*

(3.5) $$\begin{aligned}(1+t)^\gamma|\Psi(t)|_\beta^2 &+ \beta\int_0^t(1+\tau)^\gamma|\Psi(\tau)|_{\beta-1}^2\,d\tau + \int_0^t(1+\tau)^\gamma|\Psi_\xi(\tau)|_\beta^2\,d\tau \\ &\leq C\{|\Psi_0|_\beta^2 + \gamma\int_0^t(1+\tau)^{\gamma-1}|\Psi(\tau)|_\beta^2\,d\tau + \beta\int_0^t(1+\tau)^\gamma\|\Psi_\xi(\tau)\|^2\,d\tau + \\ &+ \int_0^t\int(1+\tau)^\gamma<\xi>^\beta|\Psi||F(U,\Psi_\xi)|\,d\xi d\tau\}\end{aligned}$$

holds for $t \in [0,T]$.

Proof. Let ξ_* be the constant in (3.3). Multiplying (3.1) by

$(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta \Psi$, we have

$$\{\tfrac{1}{2}(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta \Psi^2\}_t - \tfrac{\gamma}{2}(1+t)^{\gamma-1} \langle \xi - \xi_* \rangle^\beta \Psi^2 + \tag{3.6}$$
$$+ (1+t)^\gamma \langle \xi - \xi_* \rangle^{\beta-1} A_\beta(\xi)\Psi^2 + \mu(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta \Psi_\xi^2 +$$
$$+ \beta\mu(1+t)^\gamma \langle \xi - \xi_* \rangle^{\beta-2}(\xi - \xi_*)\Psi\Psi_\xi + \{\cdots\}_\xi$$
$$= (1+t)^\gamma \langle \xi - \xi_* \rangle^\beta \Psi \cdot F(U,\Psi_\xi) \ ,$$

where $A_\beta(\xi)$ is defined in (3.4), and $\{\cdots\}_\xi$ denotes the term which disappears after integration with respect to $\xi \in \mathbb{R}$. Integrating (3.6) over $[0,t]\times\mathbb{R}$ and using the estimate (3.4), we have

$$(1+t)^\gamma |\Psi(t)|_\beta^2 + \beta\int_0^t (1+\tau)^\gamma |\Psi(\tau)|_{\beta-1}^2 d\tau + \int_0^t (1+\tau)^\gamma |\Psi_\xi(\tau)|_\beta^2 d\tau \tag{3.7}$$
$$\le C\{|\Psi_0|_\beta^2 + \gamma\int_0^t (1+\tau)^{\gamma-1} |\Psi(\tau)|_\beta^2 d\tau + \beta\int_0^t\!\!\int (1+\tau)^\gamma \langle \xi \rangle^{\beta-1} |\Psi\Psi_\xi|\, d\xi d\tau +$$
$$+ \int_0^t\!\!\int (1+\tau)^\gamma \langle \xi \rangle^\beta |\Psi|\,|F(U,\Psi_\xi)|\, d\xi d\tau \}$$

with some constant C. To get the desired estimate (3.5), we must estimate the third term on the right hand side of (3.7). Using Schwarz' inequality, we have

$$\beta C\int \langle \xi \rangle^{\beta-1} |\Psi\Psi_\xi|\, d\xi \le \frac{\beta}{2}|\Psi|_{\beta-1}^2 + \beta C\int \langle \xi \rangle^{\beta-1}\Psi_\xi^2\, d\xi$$

with a constant C. We choose a constnat R so large that $\alpha C \langle \xi \rangle^{-1} \le 1/2$ for any $|\xi| \ge R$, and divide the integral on the right hand side into two parts I_1 and I_2 according to the regions $|\xi| \ge R$ and $|\xi| \le R$. Then we have the estimates $I_1 \le \frac{1}{2}|\Psi_\xi|_\beta^2$ and $I_2 \le \beta C\|\Psi_\xi\|^2$ with some constant C. Substitution of these eatimates into (3.7) yields (3.5). This completes the proof of Lemma 3.2.

For derivatives of the solution, we have the following estimates.

Lemma 3.3. *Let* $\ell = 1$ *and* 2. *For any* $\gamma \in [0,\alpha]$, *there is a positive constant* C *independent of* T *and* γ *such that*

$$(3.8)_\ell \qquad (1+t)^\gamma \|\partial_\xi^\ell \Psi(t)\|^2 + \int_0^t (1+\tau)^\gamma \|\partial_\xi^{\ell+1}\Psi(\tau)\|^2 d\tau$$

$$\le C\{\|\partial_\xi^\ell \Psi_0\|^2 + \int_0^t (1+\tau)^\gamma \|\Psi_\xi(\tau)\|_{\ell-1}^2 d\tau +$$

$$+ \int_0^t\int (1+\tau)^\gamma |\partial_\xi^{\ell+1}\Psi| |\partial_\xi^{\ell-1} F(U,\Psi_\xi)| d\xi d\tau \}$$

holds for $t \in [0,T]$.

Proof. Let $\ell = 1$ and 2. Apply ∂_ξ^ℓ to (3.1) and multiply it by $(1+t)^\gamma \partial_\xi^\ell \Psi$. Integrate the resulting equation over $[0,t] \times \mathbb{R}$. Then we can get the desired estimate $(3.8)_\ell$ in the same way as in the previous lemma. The details are omitted.

4. A priori estimate

We proceed to estimate the solution of the problem (2.5),(2.6). Put

$$N_\alpha = |\Psi_0|_\alpha + \|\Psi_{0,\xi}\|_1 \quad \text{for} \quad \alpha \ge 0.$$

We first take $\beta = \gamma = 0$ in the inequalities (3.5), $(3.8)_1$ and $(3.8)_2$, and combine them successively. Then we have

$$\|\Psi(t)\|_2^2 + \int_0^t \|\Psi_\xi(\tau)\|_2^2 d\tau$$

$$\le C\{N_0^2 + \int_0^t\int (|\Psi| + |\Psi_{\xi\xi}|)|F(U,\Psi_\xi)| + |\Psi_{\xi\xi\xi}||F(U,\Psi_\xi)_\xi| d\xi d\tau \}$$

Since $F(U,\Psi_\xi) = O(|\Psi_\xi|^2)$ and $F(U,\Psi_\xi)_\xi = O(|\Psi_\xi|^2 + |\Psi_\xi||\Psi_{\xi\xi}|)$ for $|\Psi_\xi|$

$\to 0$ (see (3.2)), the integral on the right hand side is majorized by

$$CN(t)\int_0^t \|\Psi_\xi(\tau)\|_2^2\,d\tau$$

with some constant $C = C(\varepsilon_0)$, where we have used $N(T) \le \varepsilon_0$. Therefore we arrive at the following lemma.

Lemma 4.1. *There are positive constants* ε_4 *(* $\le \varepsilon_0$*) and* $C = C(\varepsilon_4)$ *independent of* T *such that if* $N(T) \le \varepsilon_4$*, the estimate* (2.9) *with* $\beta = \gamma = 0$ *holds for* $t \in [0,T]$ *:*

$$(4.1) \qquad \|\Psi(t)\|_2^2 + \int_0^t \|\Psi_\xi(\tau)\|_2^2\,d\tau \le CN_0^2 .$$

Next, combining (3.5) with (4.1), we derive the decay estimate for L^2-norm of the solution.

Lemma 4.2. *Let* $\gamma \in [0,\alpha] \cap \mathbb{Z}$. *There are positive constants* ε_5 *(* $\le \varepsilon_4$*) and* $C = C(\varepsilon_5)$ *independent of* T *and* γ *such that if* $N(T) \le \varepsilon_5$*, then*

$$(4.2) \qquad (1+t)^\gamma |\Psi(t)|_{\alpha-\gamma}^2 + (\alpha-\gamma)\int_0^t (1+\tau)^\gamma |\Psi(\tau)|_{\alpha-\gamma-1}^2\,d\tau + \int_0^t (1+\tau)^\gamma |\Psi_\xi(\tau)|_{\alpha-\gamma}^2\,d\tau \le CN_\alpha^2$$

holds for $t \in [0,T]$. *Consequently, for any* $0 \le \gamma \le [\alpha]$*, we have*

$$(4.3) \qquad (1+t)^\gamma \|\Psi(t)\|^2 + \int_0^t (1+\tau)^\gamma \|\Psi_\xi(\tau)\|^2\,d\tau \le CN_\alpha^2 .$$

Proof. We first estimate the last integral on the right hand side of (3.5). If $N(T) \le \varepsilon_0$, it is majorized by

$$CN(t)\int_0^t (1+\tau)^\gamma |\Psi_\xi(\tau)|_\beta^2\,d\tau$$

with a constnat $C = C(\varepsilon_0)$. Therefore, for suitable small $N(T)$, say $N(T) \le \varepsilon_5$, the inequality (3.5) becomes

$$(4.4) \qquad (1+t)^{\gamma}|\Psi(t)|_{\beta}^{2} + \beta\int_0^t (1+\tau)^{\gamma}|\Psi(\tau)|_{\beta-1}^{2}\,d\tau + \int_0^t (1+\tau)^{\gamma}|\Psi_{\xi}(\tau)|_{\beta}^{2}\,d\tau$$
$$\le C\{|\Psi_0|_{\beta}^{2} + \gamma\int_0^t (1+\tau)^{\gamma-1}|\Psi(\tau)|_{\beta}^{2}\,d\tau + \beta\int_0^t (1+\tau)^{\gamma}\|\Psi_{\xi}(\tau)\|^{2}\,d\tau\}$$

with a constant $C = C(\varepsilon_5)$.

Step 1 Letting $\beta = \alpha$ and $\gamma = 0$ in (4.4), we have (4.2) with $\gamma = 0$, where (4.1) was used. Therefore the lemma is proved for $\alpha < 1$.

Step 2 Firstly, letting $\beta = 0$ and $\gamma = 1$ in (4.4), and using (4.2) with $\gamma = 0$, we have (4.3) with $\gamma = 1$, where $\alpha \ge 1$ is assumed. Secondly, letting $\beta = \alpha - 1$ and $\gamma = 1$ in (4.4), and using the estimates (4.2) with $\gamma = 0$ and (4.3) with $\gamma = 1$, we have the desired estimate (4.2) with $\gamma = 1$. Therefore the proof is completed for $\alpha < 2$.

Step 3 We repeat the same procedure as in Step 2. The estimate (4.4) (with $\beta = 0$, $\gamma = 2$) together with (4.2) (with $\gamma = 1$) yields (4.3) (with $\gamma = 2$), where $\alpha \ge 2$ is assumed. Also, (4.4) (with $\beta = \alpha - 2$, $\gamma = 2$) together with (4.2) (with $\gamma = 1$) and (4.3) (with $\gamma = 2$) yields (4.2) (with $\gamma = 2$), which proves the lemma for $\alpha < 3$.

Repeating the same procedure, we can get the desired estimate (4.2) for any $\alpha \ge 0$. This completes the proof of Lemma 4.2.

Finally, we show the same decay rate $t^{-\gamma/2}$ for derivatives of the solution.

Lemma 4.3. *Let* $\ell = 1$ *and* 2. *For any* $0 \le \gamma \le [\alpha]$, *there are positive constants* ε_6 $(\le \varepsilon_5)$ *and* $C = C(\varepsilon_6)$ *independent of* T *and* γ *such that if* $N(T) \le \varepsilon_6$, *then the estimate*

$$(4.5)_\ell \qquad (1+t)^\gamma \|\partial_\xi^\ell \Psi(t)\|^2 + \int_0^t (1+\tau)^\gamma \|\partial_\xi^{\ell+1}\Psi(\tau)\|^2 d\tau \le CN_\alpha^2$$

holds for $t \in [0,T]$.

Proof. We combine $(3.8)_\ell$ ($\ell = 1, 2$) and (4.3). If $N(T) \le \varepsilon_0$, the last integral on the right hand side of $(3.8)_1$ is majorized by

$$CN(t)\int_0^t (1+\tau)^\gamma \|\Psi_\xi(\tau)\|_1^2 d\tau .$$

Therefore, for suitably small $N(T)$, we have

$$(1+t)^\gamma \|\Psi_\xi(t)\|^2 + \int_0^t (1+\tau)^\gamma \|\Psi_{\xi\xi}(\tau)\|^2 d\tau$$
$$\le C\{\|\Psi_{0,\xi}\|^2 + \int_0^t (1+\tau)^\gamma \|\Psi_\xi(\tau)\|^2 d\tau\} .$$

This inequality together with (4.3) gives the desired estimate $(4.5)_1$. Similarly, we can obtain $(4.5)_2$ using the estimates $(3.8)_2$, (4.3) and $(4.5)_1$. This completes the proof of Lemma 4.3.

Now, the estimate (2.9) follows directly from (4.3), $(4.5)_1$ and $(4.5)_2$. Therefore the proof of Proposition 2.3 is completed.

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On the equilibrium configuration of a rotating mass of fluid with self-gravitation.

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§1. Introduction.

In this note we state a very classical (but not so familiar to mathematicians) problem in fluid mechanics. It is stated as follows:

<u>PROBLEM.</u> Suppose that a mass of fluid lies in the three dimensional Euclidean space. By assumption the only force which acts on the fluid is that caused by the gravitation due to itself. We suppose that the body of fluid rotates with a fixed axis with a constant angular velocity. Then, find a surface which encloses the fluid in an equilibrium state.

This problem was first proposed by Newton in his famous "Principia", and then developed by Maclaurin and Jacobi. Since that time, a large number of famous mathematicians made important contributions to this problem. But here we restrict ourselves to a most simple model considered by Maclaurin and Jacobi which we describe below. We first prepare some symbols. Let (x, y, z) be a coordinates system in $\mathbb{R}^3$. We take the z-axis as the axis of rotation. For a compact closed surface Γ, we denote by Ω a domain bounded by Γ. Now the problem which we consider in this note is described as follows:

<u>PROBLEM (P-S).</u> Given a constant ω, find a compact closed surface Γ such that

$$(1.1) \qquad V + \frac{\omega^2}{2}(x^2 + y^2) = \text{constant} \quad \text{on } \Gamma,$$

where V is a function on $\mathbb{R}^3$ given by

$$(1.2) \qquad V(x) = \int_\Omega \frac{dy}{|x-y|}.$$

<u>REMARK</u>. The function V is a C^1-function defind on $\mathbb{R}^3$ which represents a potential of the gravitation. The constant ω is the angular velocity.

This model is derived in the following way. We assume that the fluid is incompressible and inviscid. For simplicity we assume that the density of the fluid is the unity. We neglect the effect of the surface tension. Furthermore we assume (although this is a very restrictive assumption) that the fluid is at rest when it is viewed in a coordinates system which moves around the z-axis with the angular velocity ω. Then the Euler equation for the motion of the fluid is written as

$$- \omega^2 x = - \frac{\partial p}{\partial x} + \frac{\partial V}{\partial x},$$

$$- \omega^2 y = - \frac{\partial p}{\partial y} + \frac{\partial V}{\partial y},$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial V}{\partial z},$$

where p is the pressure of the fluid, V is a potential of the self-gravitation, whence it is given by (1.2). Consequently we obtain

$$p = V + \frac{\omega^2}{2}(x^2 + y^2) + \text{constant}.$$

Since the fluid is in an equilibrium state, the pressure must be constant at the boundary. Consequently we obtain the condition (1.1).

As is mentioned before, this problem has attracted a large number of mathematitians. But the most significant theory was developped independently by Poincaré [5] and by Lyapunov [6]. In the following section we review some of thier results and reconside: the problem from the viewpoint of modern mathematics.

§2. Explicit solutions.

We first consider the simplest case, i.e., the case where the angular velocity is zero. In this case the sphere is obviously a solution. The converse is also true. This was first proved by Lyapunov based on some physical principle. His "theorem" can be read as follows:if the equilibrium shape in the case of $\omega = 0$ is "stable", it must be a sphere. A slight generalization of this theorem has obtained recently in [4]. It is stated as follows.

THEOREM 1. If Γ is a solution to (1.1) with $\omega = 0$, then it is necessarily a sphere.

This theorem is a generalization of Lyapunov's in that THEOREM 1 does not require the stability in the assumption. The proof of THEOREM 1 is completely different from that by Lyapunov. Actually the proof is based on the " moving plane method ", which was used by Alexandrov, Serrin, Gidas-Ni-Nirenberg and recently Matano , where fascinating results were obtained (see [4]).

We now consider the case where $\omega > 0$. In this case the sphere is no longer a solution. But an ellipsoid becomes a solution. This might be quite natural when we see the term due to the centrifugal force in (1.1). To seek a solution we consider the ellipsoid described as follows:

$$(2.1) \qquad \Omega \; ; \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1.$$

We assume that $a \leq b \leq c$. In the special case where $a = b$, Maclaurin determined in 1742 a necessary and sufficient condition that the ellipsoid (2.1) becomes a solution. Later C. G. J. Jacobi solved the problem for the general ellipsoid in 1834. In what follows, following Lamb [2], we give a criterion that the ellipsoid (2.1) with $a = b$ become a solution.

We start with a formula that

$$(2.2) \qquad V(x) = \int_\Omega \frac{dy}{|x-y|} = -\alpha x^2 - \beta y^2 - \gamma z^2 + \delta \qquad (x \in \overline{\Omega}),$$

where Ω is given by (2.1). α, β, γ and δ are constants given by

$$\alpha = \pi abc\int_0^\infty \frac{d\lambda}{(a^2+\lambda)\Delta}, \quad \beta = \pi abc\int_0^\infty \frac{d\lambda}{(b^2+\lambda)\Delta}, \quad \alpha = \pi abc\int_0^\infty \frac{d\lambda}{(c^2+\lambda)\Delta},$$

$\delta = \pi abc \int_0^\infty \frac{d\lambda}{\Delta}$ where $\Delta = \{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}^{1/2}$. In the outside of the ellipsoid V is also represented explicitly, see [1, 2] for instance. By this formula the equilibrium condition (1.1) is now expressed as

$$(\tfrac{1}{2}\omega^2 - \alpha)x^2 + (\tfrac{1}{2}\omega^2 - \beta)y^2 - \gamma z^2 = \text{constant}$$

$$\text{on} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

This is equivalent to

$$(2.3) \qquad (\tfrac{1}{2}\omega^2 - \alpha)a^2 = (\tfrac{1}{2}\omega^2 - \beta)b^2 = -\gamma c^2.$$

For simplicity we restrict ourselves to the case where $a = b > c$, e.g., the Maclaurin ellipsoid. Since the fluid is rotating, it is natural that we assume the length c of the z-axis is the smallest But, actually, we can prove that there is no ellipsoidal figure of equilibrium with $c > a, b$ (see [2]).

Now we introduce a variable ζ such that

$$a = b = \frac{(\zeta^2+1)^{1/2}}{\zeta}c \qquad (0 < \zeta < \infty).$$

Then the constants are expressed as

$$\alpha = \beta = \pi(\zeta^2+1)\zeta\cot^{-1}\zeta - \zeta^2, \quad \gamma = 2\pi(\zeta^2+1)(1 - \zeta\cot^{-1}\zeta).$$

In the present case the condition (2.3) is expressed as

$$\frac{1}{2}\omega^2 = \alpha - \gamma\frac{c^2}{a^2} = (3\zeta^2+1)\zeta\cot^{-1}\zeta - 3\zeta^2. \tag{2.4}$$

This is the condition discovered by Maclaurin. The graph of this function is drawn in Fig. I.

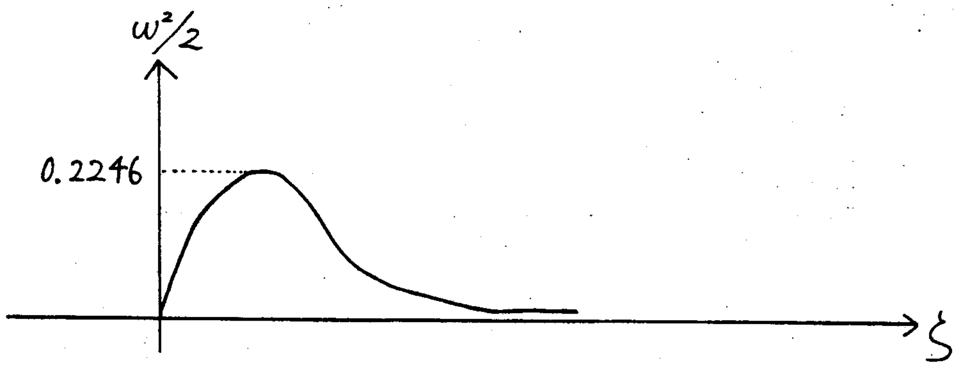

Fig. I

Consequently, for $0 < \omega^2/2 < 0.22466$, there is two roots of the equation (2.4). By means of these root ζ, the ellipsoid (2.1) with $a = b = \frac{(\zeta^2+1)^{1/2}}{\zeta}c$ become an equilibrium configuration of the fluid. These are called the Maclaurin spheroids.

§3. Bifurcation.

As we have observed in the preceding section, there exists an ellipsoid of revolution which is an equilibrium shape for some ω. On the other hand, there is an ellipsoid with three different length of the axis, which was discovered by Jacobi in 1834. For the proof, see Lamb [2] or Hagiwara [7]. Is is worthy of notice that for ω^2 smaller than 0.1871 there are four ellipsoids of equilibrium, two of which are Maclaurin spheroid, and the remaining two are the

Jacobi ellipsoids. Actually the two Jacobi ellipsoid is mutually congruent and one is obtained from the other by changing the x-axis and the y-axis. On the other hand, the two Maclaurin spheroids are not congruent. These series of solutions exhibits a concrete example of bifurcation phenomena. Namely the Jacobi sequence bifurcates from the Maclaurin sequence. The famous discovery of Poincaré and Lyapunov is that there is a countable number of sequence of non-ellipsoidal figures of equilibrium which bifurcates from the Jacobi ellipsoid. Consequently this is an example of secondary bifurcation. Although the configuration of the solution emanating from the Jacobi ellipsoid can be approximately seen by using the Lamé functions, the configuration of the solution which are far from the bifurcation point was not known until very recently. But, according to the progress of computers, various kinds of equilibrium solutions are computed very accurately. Among others, Eriguchi and Hachisu have made a great contribution (see, e.g., [8, 9]). According to there computations, the bifurcating solution becomes distorted and in some cases it becomes a double star.

The papers by Poincaré or Lyapunov was written about 100 years ago. Hence we think that to reconstruct their "proof" has some significance from the viewpoint of modern mathematics. Further, it is an interest open problem to give a mathematical description of the bifurcating solutions which are far from the ellipsoids.

§4. Nonstationary problem.

So far we haved discussed the stationary problem. In this section we consider the system of equations which describes the nonstationary motion of the self-gravitating perfect fluid. We first make a definition. For a fixed time t, the domain occupied by the fluid is denoted by $\Omega(t)$. The boundary of $\Omega(t)$ is denoted by $\gamma(t)$. We assume that the boundary $\gamma(t)$ is given by

$$F(t,x,y,z) = 0.$$

Then the problem is to find $\gamma(t)$, $\vec{u} = (u_1, u_2, u_3)$ and p satisfying the following equations:

$$(4.1) \qquad \frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\nabla p + \nabla V \qquad \text{in } \bigcup_{0<t} \Omega(t),$$

$$(4.2) \qquad \operatorname{div} u = 0 \qquad \text{in } \bigcup_{0<t} \Omega(t),$$

$$(4.3) \qquad \frac{\partial F}{\partial t} + u\cdot\nabla F = 0 \qquad \text{on } \gamma(t),$$

$$(4.4) \qquad p = 0 \qquad \text{on } \gamma(t),$$

$$(4.5) \qquad \vec{u}(0,x,y,z) = \vec{u}_0(x,y,z) \text{ and } \gamma(0) \text{ are given.}$$

Well-posedness of this system of equations are not known until now. If this system can be solved, then we will be able to study the stability of equilibrium states. So this is also an interesting open question.

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放物型方程式の解の漸近挙動

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1. 序

ここでは、時間に依存した、放物型混合問題の解の漸近挙動について考える。

Ω を $\mathbb{R}^n$ 上の $C^{2+\nu}$－級 $(0<\nu<1)$ 有界領域とし、以下の様な記号を定義する。

$P=(x,t)$, $Q=(y,s)\in\bar{\Omega}\times\bar{\mathbb{R}}^+$　$(\mathbb{R}^+=(0,\infty))$

に対して、

$$d(P,Q)=(|x-y|^2+|t-s|)^{1/2}$$

$$|u|_0^D=\sup_{P\in D}|u(P)|$$

$$H_\nu^D(u)=\sup_{P,Q\in D}\frac{|u(P)-u(Q)|}{d(P,Q)^\nu}$$

$$|u|_\nu^D=|u|_0^D+H_\nu^D(u)$$

$$|u|_{2+\nu}^D=\sum_{|\alpha|+2i\leq 2,\ i\leq 1}|D_t^i D_x^\alpha u|_\nu^D$$

又、$C^{2+\nu,(2+\nu)/2}(D)$ に属していて、かつ $|\cdot|_{2+\nu}^D<\infty$ なる関数の全体を $C^{(2+\nu)}(D)$ で表わす事とする。

2. 定理

次の様な混合問題を考える。

$$
\text{(E1)}\quad
\begin{cases}
D_t u = \sum_{i,j=1}^{n} a_{ij}(x,t) D_{ij} u + \sum_{i=1}^{n} b_i(x,t) D_i u \\
\qquad + c(x,t) u + f(x,t,u) & \text{in } \Omega \times \mathbb{R}^+ \\
u = h(x,t) & \text{on } \partial\Omega \times \mathbb{R}^+ \\
u(\cdot, 0) = u_0(x) & \text{in } \Omega
\end{cases}
$$

(E1) に対して、次を仮定する。

<u>仮定 (A)</u>

- $a_{ij} = a_{ji} \quad (i, j = 1, \cdots, n)$
- $\exists \lambda > 0$ s.t. $\forall (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}^+}$, $\forall \xi \in \mathbb{R}^n$;

  $$\sum a_{ij}(x,t) \xi_i \xi_j \geq \lambda |\xi|^2$$

- $a_{ij}, b_i, c \in C^{(\nu)}(\overline{\Omega} \times \overline{\mathbb{R}^+})$

  $h \in C^{(2+\nu)}(\partial\Omega \times \overline{\mathbb{R}^+})$, $u_0 \in C^{2+\nu}(\overline{\Omega})$

- $f(\cdot, \cdot, \cdot) : \overline{\Omega} \times \overline{\mathbb{R}^+} \times \mathbb{R} \longrightarrow \mathbb{R}$: conti.
- $f(\cdot, \cdot, p) : \overline{\Omega} \times \overline{\mathbb{R}^+} \longrightarrow \mathbb{R} \quad \in C^{(2+\nu)}(\overline{\Omega} \times \overline{\mathbb{R}^+})$

  $\forall \rho > 0$; $\exists$ const. $K > 0$ s.t.

  $$\sup_{p \in [-\rho, \rho]} |f(x,t,p)|_{2+\nu}^{\overline{\Omega} \times \overline{\mathbb{R}^+}} \leq K$$

- $f(x, t, \cdot) : \mathbb{R} \longrightarrow \mathbb{R} \quad \in C^{2+\nu}(\mathbb{R})$

  $\exists$ const. $K > 0$ s.t.

  $$\sup_{x \in \overline{\Omega},\, t \in \overline{\mathbb{R}^+}} \ \sup_{p, q \in \mathbb{R}} \frac{|f(x,t,p) - f(x,t,q)|}{|p-q|^{\nu}} < K$$

$\partial f/\partial p$; conti. in $\overline{\Omega} \times \overline{\mathbb{R}}^+ \times \mathbb{R}$

- $\forall \rho > 0$;

$$\varlimsup_{t\to\infty} \sup_{x\in\overline{\Omega},\, p\in[-\rho,\rho]} \{c(x,t) + \partial f/\partial p(x,t,p)\} \leq 0 \qquad (1)$$

以上の仮定の下に、次の命題が成り立つ.

<u>命　題</u>　[Amann 1]

(E1)に対して、(A)を仮定する。さらに、(E1)を満たす全ての関数 $\varphi(x,t)$ について、a priori estimate

$$\sup_{x\in\overline{\Omega},\, t\in\overline{\mathbb{R}}^+} |\varphi(x,t)| \leq {}^{\exists}K_0 \qquad (2)$$

が成り立っていれば、(E1)は unique global classical solution $u(x,t)$ を持つ.

注意）　ここで、$u(x,t) \in C^{(2+\nu)}(\overline{\Omega} \times \mathbb{R}^+)$ である。又、(2)は、一般には、supersolution と subsolution の存在によって示す事ができる。逆に、もし、(E1)が global solution を持てば、(1)を次の様に強めることによって、(2)が自然に得られる.

$${}^{\exists}K \leq \varliminf_{t\to\infty} \inf_{x\in\overline{\Omega},\, p\in\mathbb{R}} \{c(x,t) + \partial f/\partial p(x,t,p)\}$$

$$\varlimsup_{t\to\infty} \sup_{x\in\overline{\Omega},\, p\in\mathbb{R}} \{c(x,t) + \partial f/\partial p(x,t,p)\} \leq 0$$

補題

(E1)に対して(A)と a priori estimate (1)を仮定すると、解 $u(x,t)$ に関して、次の不等式が成り立つ。

$$\sup_{x,y\in\overline{\Omega},\, t\in\overline{R}^+} \frac{|D_x^\alpha u(x,t)-D_x^\alpha u(y,t)|}{|x-y|^{\nu_2}} \leq {}^\exists K \quad (3)$$

$$\sup_{x\in\overline{\Omega},\, t\in\overline{R}^+} |D_x^\alpha u(x,t)| \leq {}^\exists K \quad (|\alpha|\leq 2) \quad (4)$$

この補題は、(1)を用いないで証明できるが、計算が繁雑なので省略する。証明の方法は、[1], [2], [3], [5] と組み合わせる事によって得られる。

(E1)の解の漸近挙動を考える時、まず始めに、方程式自体が t に依存しない方程式に収束する場合について考察する。

仮定(B)

- $D_t a_{ij}$, $D_t b_i$, $D_t c$, $D_t h \longrightarrow 0$ $(t\to\infty)$ uniformly for $x\in\overline{\Omega}$
  $a_{ij}\longrightarrow a_{ij}^0(x)$, $b_i \longrightarrow b_i^0(x)$, $c\longrightarrow c^0(x)$
  $h\longrightarrow h^0(x)$ $(t\to\infty)$ uniformly for $x\in\overline{\Omega}$
- $\forall\rho>0$;
  $$\lim_{t\to\infty} \sup_{x\in\overline{\Omega},\, p\in[-\rho,\rho]} |D_t f(x,t,p)|=0$$
  $$\lim_{t\to\infty} \sup_{x\in\overline{\Omega},\, p\in[-\rho,\rho]} |D_x f(x,t,p)-D_x f^0(x,p)|=0$$

仮定(B)で、各dataが収束した極限から、新しい楕円型境界値問題を作る。

$$(E2)\quad \begin{cases} 0=\sum_{i,j=1}^{n} a_{ij}^{\circ}(x) D_{ij} v + \sum_{i=1}^{n} b_i^{\circ}(x) D_i v + c^{\circ}(x) v \\ \quad + f^{\circ}(x, v) & \text{in } \Omega \\ v = h^{\circ}(x) & \text{on } \partial\Omega \end{cases}$$

この時、次の定理が成り立つ

定　理　1

(E1)に対して、(A)，(B) 及び a priori estimate (2) を仮定すれば，次の性質を持つ関数 $v(x)$ が唯一つ存在する。すなわち

$$\begin{cases} \text{(i)} & \lim_{t\to\infty} \sup_{x\in\overline{\Omega}} |u(x,t) - v(x)| = 0 \\ \text{(ii)} & v(x) \text{ は (E2) の classical solution である.} \end{cases}$$

注意）　$v(x) \in C^{2+\nu}(\overline{\Omega})$

証明）

補題の(3)，(4)より．$u(x,t)$ の sup-norm による ω-limit set と，C^2-norm による ω-limit set は，一致する。ここで，$D_t u(x,t)$ が $t\to\infty$ の時，x に関して，一様に0に収束する事を示す。　$u^h(x,t) = (u(x,t+h) - u(x,t))/h$ とおくと，

$$D_t u^h = L(x, t+h) u^h \quad \text{in } \Omega$$
$$+ \{ c(x, t+h) + f_p(x, t+h, \lambda u(x, t+h) + (1-\lambda) u(x, t)) \}$$
$$\times u^h + [\tfrac{1}{h} \{ L(x, t+h) - L(x, t) \} u(x, t)$$
$$+ c^h u(x, t) + \tfrac{1}{h} \{ f(x, t+h, u(x, t)) - f(x, t, u(x, t)) \}]$$
$$u^h = h^h \quad \text{on } \partial\Omega$$

が成り立つ。但し、$L(x, t) = \Sigma a_{ij}(x, t) D_{ij} + \Sigma b_i(x, t) D_i$ である。補題より、$D_x^\alpha u$ $(|\alpha| \leq 2)$ は有界であるので、$\forall \sigma < \forall t$ に対して、次の不等式が成り立つ。

$$\sup_{x \in \bar{\Omega}} |u^h(x, t)| \leq C \Big[\sup_{x \in \bar{\Omega}, t \in [\sigma, \infty)} \Big| (L^h(x, t) + c^h(x, t)) u(x, t)$$
$$+ \frac{1}{h} (f(x, t+h, u(x, t)) - f(x, t, u(x, t))) \Big| + \sup_{x \in \partial\Omega, t \in [\sigma, \infty)} | h^h(x, t)$$
$$+ c' e^{-\gamma(t-\sigma)} \Big] \qquad ([3. P160. Cor. 1] より)$$

ここで、C, C', γ は h, σ に依存しない正定数である。よって、$h \to 0$, $t \to \infty$ とすれば、$D_t u(x, t) \to 0$ が得られる。以上より、u の sup-norm による ω-limit set の元は、全て、(E2) の classical solution である事がわかる。又、(1) より、$f(x, p) + c(x) p$ は、p に関して、単調減少である事が容易に示されるので、(E2) の解は唯一つであり、結論を得る。

系

定理1の仮定の下である定数 $K_1, K_2, \gamma > 0$ が存在して、任意の $\sigma > 0$ と $t > \sigma$ に対して、次の不等式が成り立つ。

$$|D_x u(x,t)| \leq K_1 \Big[\sup_{x\in\bar{\Omega},\, t\in[\sigma,\infty)} \Big\{ \sum |D_x a_{ij}(x,t)|$$

$$+ \sum |D_x b_i(x,t)| + |D_x c(x,t)|$$

$$+ \sup_{p\in[-K_0,K_0]} |D_x f(x,t,p)| \Big\} + \sup_{x\in\partial\Omega,\, t\in[\sigma,\infty)} |D_x h(x,t)|$$

$$+ K_2 e^{-\gamma(t-\sigma)} \Big]$$

証明)

[3, Ch.6, P160, Cor 1] よりすぐに得られる。

次に、(E1) が periodic な問題に収束する時も、(1) を仮定すれば、同様な結論が得られる事を示す。極限方程式として、次の境界値問題を考える。

$$(E3)\begin{cases} D_t w = \sum_{i,j=1}^{n} \tilde{a}_{ij}(x,t) D_{ij} w + \sum_{i=1}^{n} \tilde{b}_i(x,t) D_i w \\ \qquad + \tilde{c}(x,t) w + \tilde{f}(x,t,p) & \text{in } \Omega\times R^+ \\ w = \tilde{h}(x,t) & \text{on } \partial\Omega\times R^+ \end{cases}$$

仮定 (C)

- (E3) は (A) と同程度のなめらかさを持つとする。
- $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{c}$, $\tilde{h}$, $\tilde{f}$ は、それぞれ、t について、T-periodic である。

・ $\lim_{t\to\infty} \sup_{x\in\bar{\Omega}} |a_{ij}(x,t) - \tilde{a}_{ij}(x,t)| = 0$

b_i, c, h についても同様

・ $\forall \rho > 0$;

$\lim_{t\to\infty} \sup_{x\in\bar{\Omega},\, p\in[-\rho,\rho]} | f(x,t,p) - \tilde{f}(x,t,p) | = 0$

この時、次が成り立つ。

<u>定 理 2</u>

(E1)に対して、(2)と(A)，(E1)と(E3)に対して、(C)を仮定する。又、(E3)は、T−periodic な classical solution $w(x,t)$をもつとする。この時、

$$\lim_{t\to\infty} \sup_{x\in\bar{\Omega}} | u(x,t) - w(x,t) | = 0$$

証明）

$\varphi(x,t) \equiv u(x,t) - w(x,t)$ とおく。この時、

$$D_t\varphi = L(x,t)\varphi + c\varphi + (L(x,t) - \tilde{L}(x,t))w + (c-\tilde{c})w + \{f(x,t,u) - f(x,t,w)\} + \{f(x,t,w) - \tilde{f}(x,t,w)\} \quad \text{in } \Omega\times R^+$$

$$\varphi = \tilde{h} \quad \text{on } \partial\Omega\times R^+$$

が成り立つ。

但し，

$$L(x,t)=\sum a_{ij}(x,t)D_{ij}+\sum b_i(x,t)D_i$$
$$\tilde{L}(x,t)=\sum \tilde{a}_{ij}(x,t)D_{ij}+\sum \tilde{b}_i(x,t)D_i$$

である。

$$(L-\tilde{L})w\ ,\ (c-\tilde{c})w\ ,\ f(x,t,w)-\tilde{f}(x,t,w)$$

はそれぞれ $t\to\infty$ の時，xに関して一様に，0に収束する。（補題より）又，ある $\lambda=\lambda(x,t)$ について，

$$f(x,t,u)-f(x,t,w)$$
$$=D_p f(x,t,\lambda u+(1-\lambda)w)\varphi$$

であるが，$\lambda u+(1-\lambda)w$ が有界である事に注意して，[3]．Ch.6．Th.1 の証明を見れば，この Th.1 がそのまま使える事がわかり，(1) の仮定の下で，$\varphi(x,t)$ が $t\to\infty$ の時，xに関して，0に一様収束する事がわかる。

つまり，$\displaystyle\lim_{t\to\infty}\sup_{x\in\bar{\Omega}}|u(x,t)-w(x,t)|=0$

が成り立つ。

注意）定理2の収束の速さも，[3] を用いると，容易に評価できる。

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完全流体の自由境界問題に対する数値シミュレーション

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1．

真空な空間内に在る星の周囲を，完全渦無し流体が循環しているものとする。このとき，その流体の外側の自由境界の運動を２次元モデルとして定式化し（［１］），数値シミュレーションによりその運動の様子を調べてみた。

このときの定常問題および非定常問題は，流れの関数　V　を用いて各々問題１，２の如く表わされる。

［問題１］　次の（１．１）－（１．４）を満足する関数　V　及び　Γ　の外側にある閉曲線　γ　を求めよ。

(1.1) $$\Delta V = 0 \qquad \text{in } \Omega_\gamma \ ,$$

(1.2) $$V|_\Gamma = 0 \ , \quad V|_\gamma = a \ ,$$

(1.3) $$\frac{1}{2}|\nabla V|^2 + Q + \sigma K_\gamma = \text{constant} \quad \text{on } \gamma \ ,$$

(1.4) $$|\Omega_\gamma| = \omega_0 \ .$$

ここで，　Γ　は原点を中心とする単位円周，　Ω_γ　は　Γ　と　γ　で囲まれた２重連結領域であり，　a，σ，ω_0　は，　与えられた定数である。　a　は循環量パラメータ，　σ　は表面張力係数である。　Q　は　Γ　の外側で定義された関数で，　例えば，　$Q=Q_0(r)=-g/r$　ととる。（g　は正定数，r　は原点からの距離　）。　$|\Omega_\gamma|$　は　Ω_γ　の面積である。

［問題２］　次の（２．１）－（２．８）を満足する関数　Ｖ，Ｐ　及び　Γ　の外側にある閉曲線　$\tau(t)=\{\ (r,\theta)\mid r=\tau(t,\theta),\ 0\leq\theta<2\pi\ \}$
を求めよ。

(2.1) $$\Delta V = 0 \qquad \text{in } \Omega_{T,\tau} \equiv \bigcup_{0<t<T}\Omega_{\tau(t)} ,$$

(2.2) $$V(t,1,\theta) = 0 \qquad \text{for } 0 < t < T,\ 0 \leq \theta < 2\pi$$

(2.3) $$\frac{\partial}{\partial\theta}V(t,\tau(t,\theta),\theta) = \tau(t,\theta)\frac{\partial\tau}{\partial t}(t,\theta) \qquad (\ 0 < t < T\ )$$

(2.4) $$\frac{1}{r}\frac{\partial^2 V}{\partial t\partial\theta} + \frac{\partial}{\partial r}\left(\ \frac{1}{2}|\nabla V|^2 + P - \frac{g}{r}\ \right) = 0 \qquad \text{in } \Omega_{T,\tau} ,$$

(2.5) $$-\frac{\partial^2 V}{\partial t\partial r} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\ \frac{1}{2}|\nabla V|^2 + P - \frac{g}{r}\ \right) = 0 \qquad \text{in } \Omega_{T,\tau} ,$$

(2.6) $$P = \sigma K_{\tau(t)} \qquad \text{on } \tau(t) ,$$

(2.7) $$V(0,r,\theta) = V_0(r,\theta) , \quad \tau(0,\theta) = \tau_0(\theta) ,$$

(2.8) $$|\Omega_{\tau(t)}| \equiv \omega_0 .$$

２．　計算結果．

２．１　定常問題．

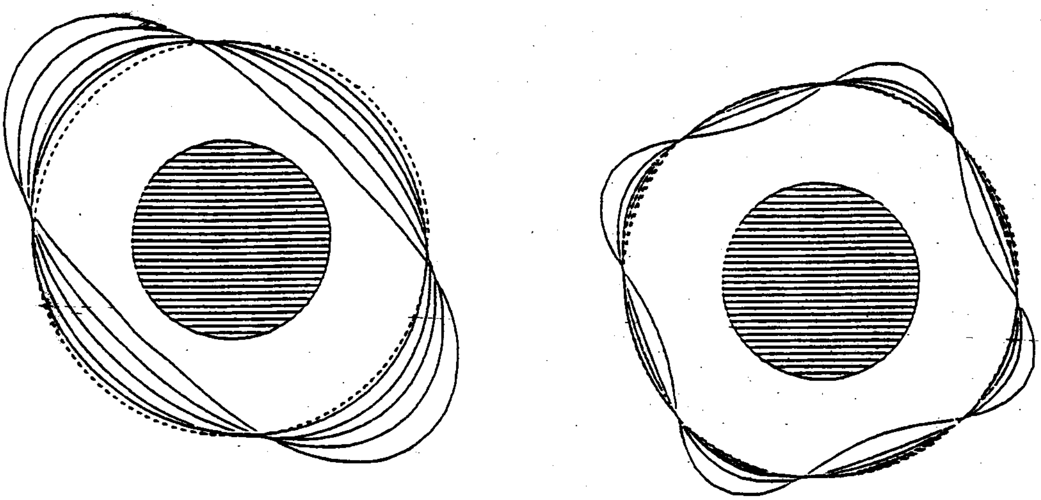
$Q_o(r)=-1/g$　とおく。　$Q=Q_o$　のときには，［問題１］は任意のａに対し　$\tau=r_o$　（　$r_o>1$　は　$\pi(r_o^2-1)=\omega_o$　となるもの　）　――　原点を中心とした半径ｒ。の円周　――　が自明解として存在する。（図中の点線部分）

（１）　摂動現象．

ａを小さいところに固定してＱをＱ。に少し摂動を与えたものとしたとき，　τ　が自明解τ。からどのようにずれるかを計算した結果を　図－１　に示す。
表面張力係数σが小さくなるに従って，自明解からのずれは大きくなっている。

$Q = -1/r(1+\frac{1}{2}\sin(2\theta))$
partition = 64
parameter a = 0.1
surface tension σ = 0.2 0.5 1 2 5

$Q = -1/r(1+\frac{1}{2}\sin(4\theta))$
partition = 64
parameter a = 0.1
surface tension σ = 0.2 0.5 1 2 5



図－1

（２）　分岐解。

自然数ｎに対し，　ａｎを次式で定義する。

$$a_n = \left(\frac{\dfrac{\sigma(n^2-1)}{r_0^2} + \dfrac{\partial Q_0}{\partial r}(r_0)}{\dfrac{1}{r_0} + \dfrac{n}{r_0}\,\dfrac{r_0^n + r_0^{-n}}{r^n - r_0^{-n}}} \right)^{\frac{1}{2}} r_0 \log r_0$$

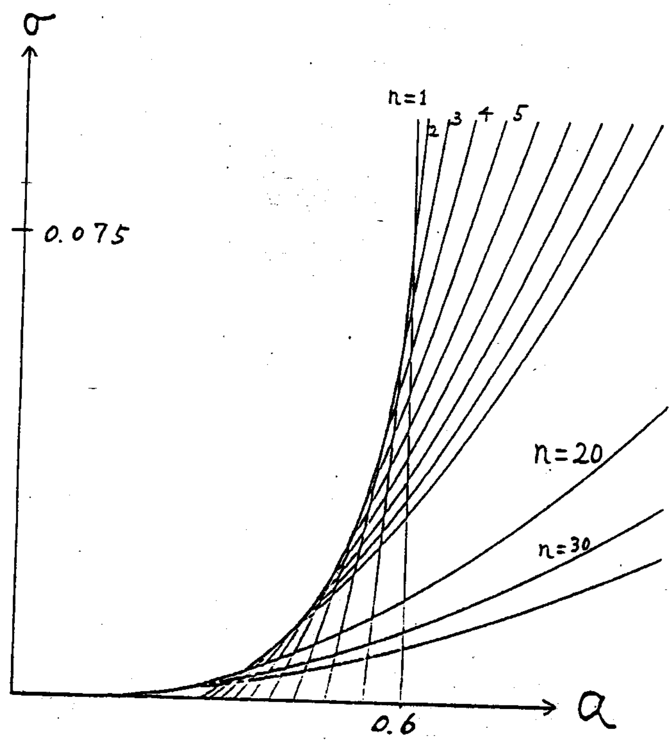
σ　と　ａｎ　との関係は　図－２　のようになる。

このとき，　ある　ｎ　に対し，　ａｍ≠ａｎ　（ｍ≠ｎ）　と仮定すれば，ａｎは分岐点である。　すなわち，　ａｎ　の近くにはｎケのうねりを持つ解が存在することがわかっている。

そこで、 a1 から出る分岐枝 及び その分岐枝上の点a，b，c，dにおける分岐解の形状をシミュレーションた結果を 図3 に示す。

σ の値を変えたとき，枝の末端部分での分岐解を比較したものが 図－4 である。 σ を増化させるに従って， 分岐解の形状は曲率一定の方向に変化し，σ が小さいところでは， 非線型性が顕著に現われてきている。

図－5は同様にして求めた a2 からの分岐解である。



図－2

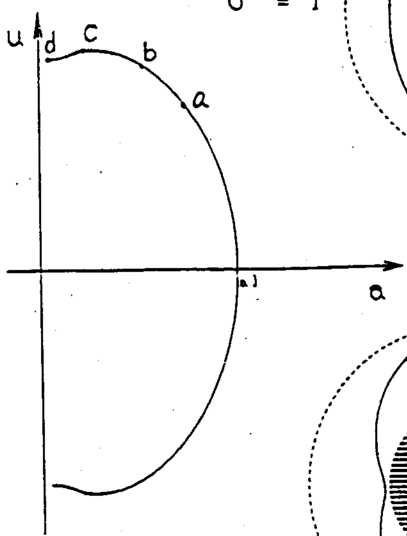
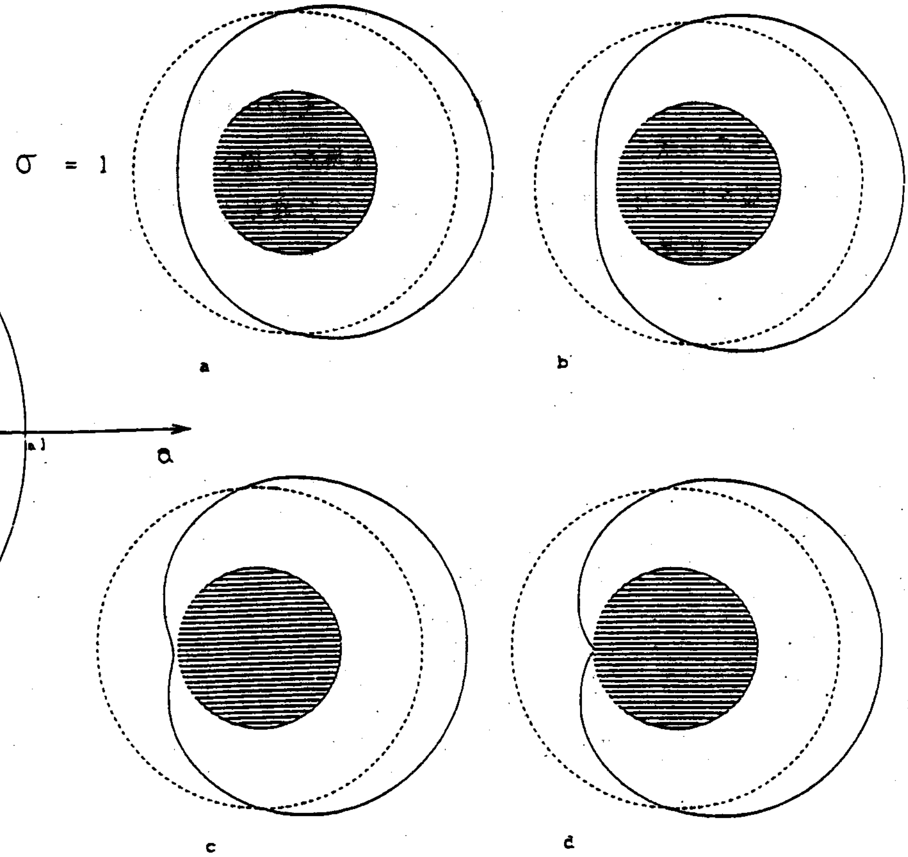


図-3



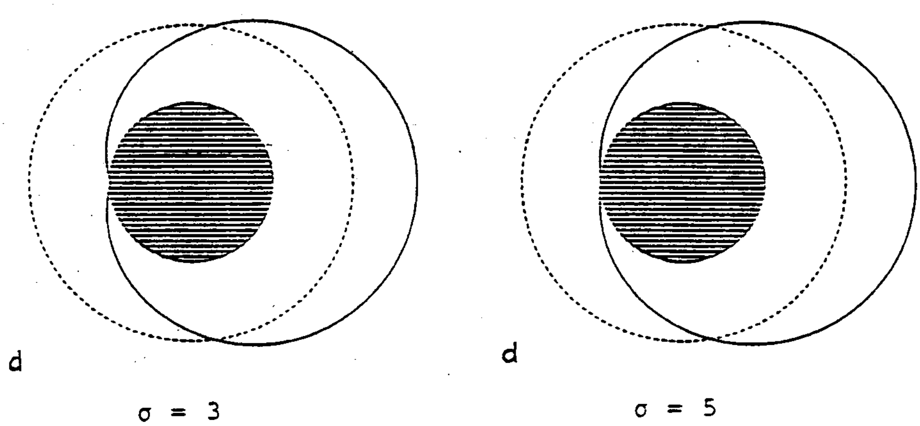
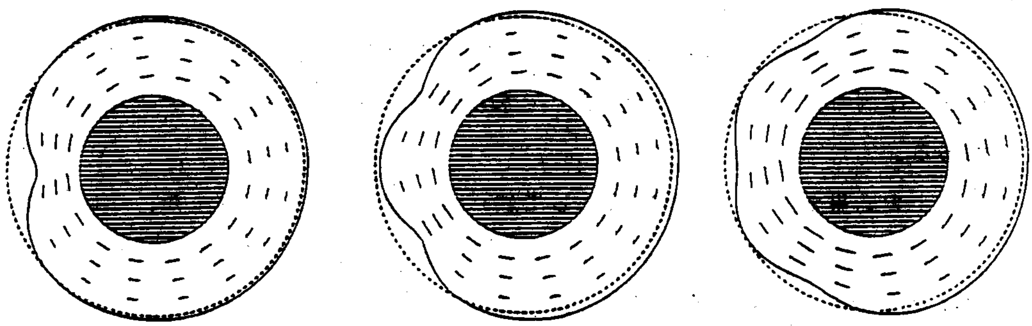


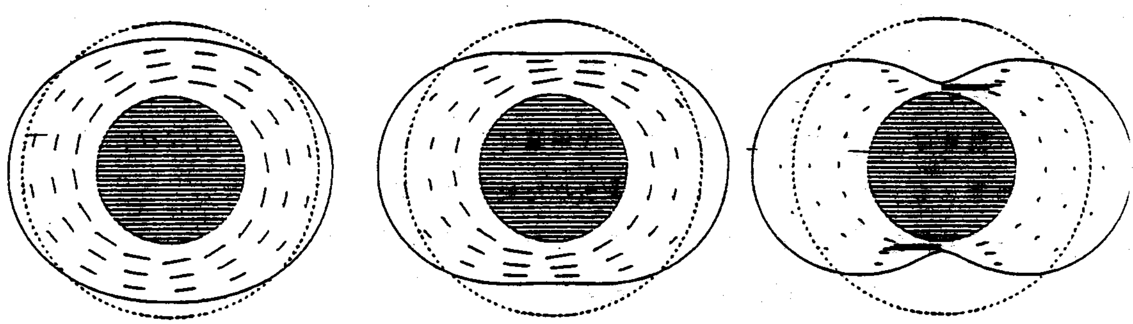
図-4

parameter a = 0.4956 parameter a = 0.5349 parameter a = 0.5760
surf.tension = .08 surf.tension = .07 surf.tension = .063



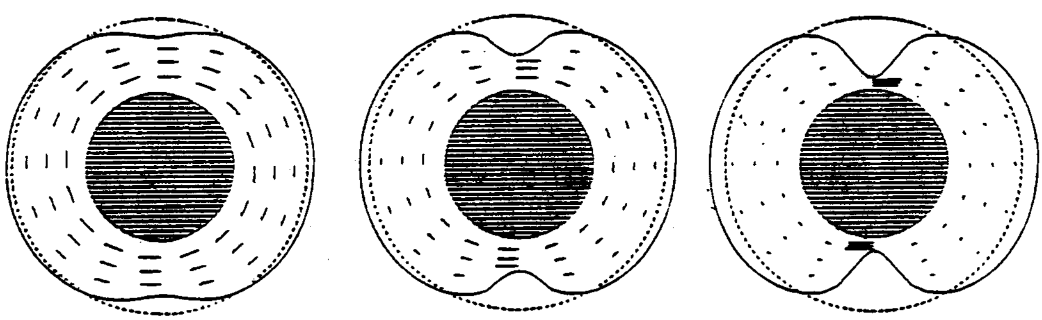
σが小さいところでのa1からの分岐解
図－4のツヅキ

σ=5 a2=2.1694



a = 2.0617 a = 1.7485 a = 0.6871

σ=0.07 , a2=0.5966



a = 0.5403 a = 0.4018 a = 0.1991

図－5

２．２　非定常問題。

自明解　及び　上で求めた分岐解の安定性を予測すべく，　［問題２］を計算した結果が　図６－８　である。

図６，７　は初期形状を自明解の近くにとり，　ａ　の値をそれぞれ最初の分岐点ａ１　の前と後にとった場合を比較したものである。

図８　は同様に，　２．１の（２）で求めた分岐解の近くから出発した場合の動きである。

以上の結果から，　まだ結論として述べる段階ではないと思うが，　いずれにしても定常波が漸近安定になっているという状況はなさそうである。　ただし，パラメーターが図２の包絡線の左側の範囲内にあるときは　シミュレーション結果が周期的な運動を示しつつ，一応充分と考え得る時間まで計算できているところから，ここでの安定な進行波解の存在が暗示されているように思う。　今後こちらの方の計算も試みるつもりである。

３．　計算手法．

計算は，　境界要素法　及び　代用電化法　の２手法を用いて行なった。
両者とも一長一短はあるが，　得られた結果はほぼ同じであった。

定常問題における計算については［２］に示されている。

ここでは非定常問題に対する計算スキームの概略を示しておく。

［非定常問題の計算スキーム］

ある時刻ｔでの　$\tau(t,\theta)$　と　$\partial\tau(t,\theta)/\partial t \equiv \zeta(t,\theta)$
が与えられているものとする。

ｓｔｅｐ１　　次の（１）（２）を解いて　$V = V_1 + f(t) V_2$　とする。
ただし，　ｆ（ｔ）は（３）の条件により定まる。

(1) $$\begin{cases} \Delta V_1 = 0 & \text{in } \Omega(t) = \{ (r,\theta) \ ; \ 1 < r < \tau(t,\theta) \} \\ V_1|_\Gamma = 0, & V_1(\tau(t,\theta),\theta) = \int_0^\theta \tau(t,\phi)\frac{\partial \tau}{\partial t}(t,\phi)d\phi. \end{cases}$$

(2) $$\begin{cases} \Delta V_2 = 0 & \text{in } \Omega(t), \\ V_2|_\Gamma = 0, & V_2(\tau(t,\theta),\theta) = 1. \end{cases}$$

(3) $$\int_0^{2\pi}\frac{\partial V}{\partial r}(t,1,\theta)d\theta = \int_0^{2\pi}\frac{\partial V}{\partial r}(t-\tau,1,\theta)d\theta.$$

ｓｔｅｐ２　次を解く。

$$\begin{cases} \Delta Q = 0 \\ \frac{\partial Q}{\partial r} = 0 \quad \text{on } \Gamma, \qquad Q|_{\tau(t)} = \frac{1}{2}|\nabla V|^2 + \sigma K_{\tau(t)} - \frac{g}{\tau}. \end{cases}$$

ｓｔｅｐ３　時刻ｔ＋τでの　τ　と　ζ　を次により求める。

$$\begin{cases} \frac{\partial}{\partial t}\tau^2 = 2\tau\tau_t \ , \\ \frac{\partial}{\partial t}\tau\tau_t = -\tau Q_\tau + \frac{1}{\tau}Q_\theta\tau_\theta + \frac{\partial}{\partial\theta}(V_\tau\tau_t) \end{cases}$$

これは　（２．３）の両辺を微分し，それに関係式（２．４）を用いることにより得られる。

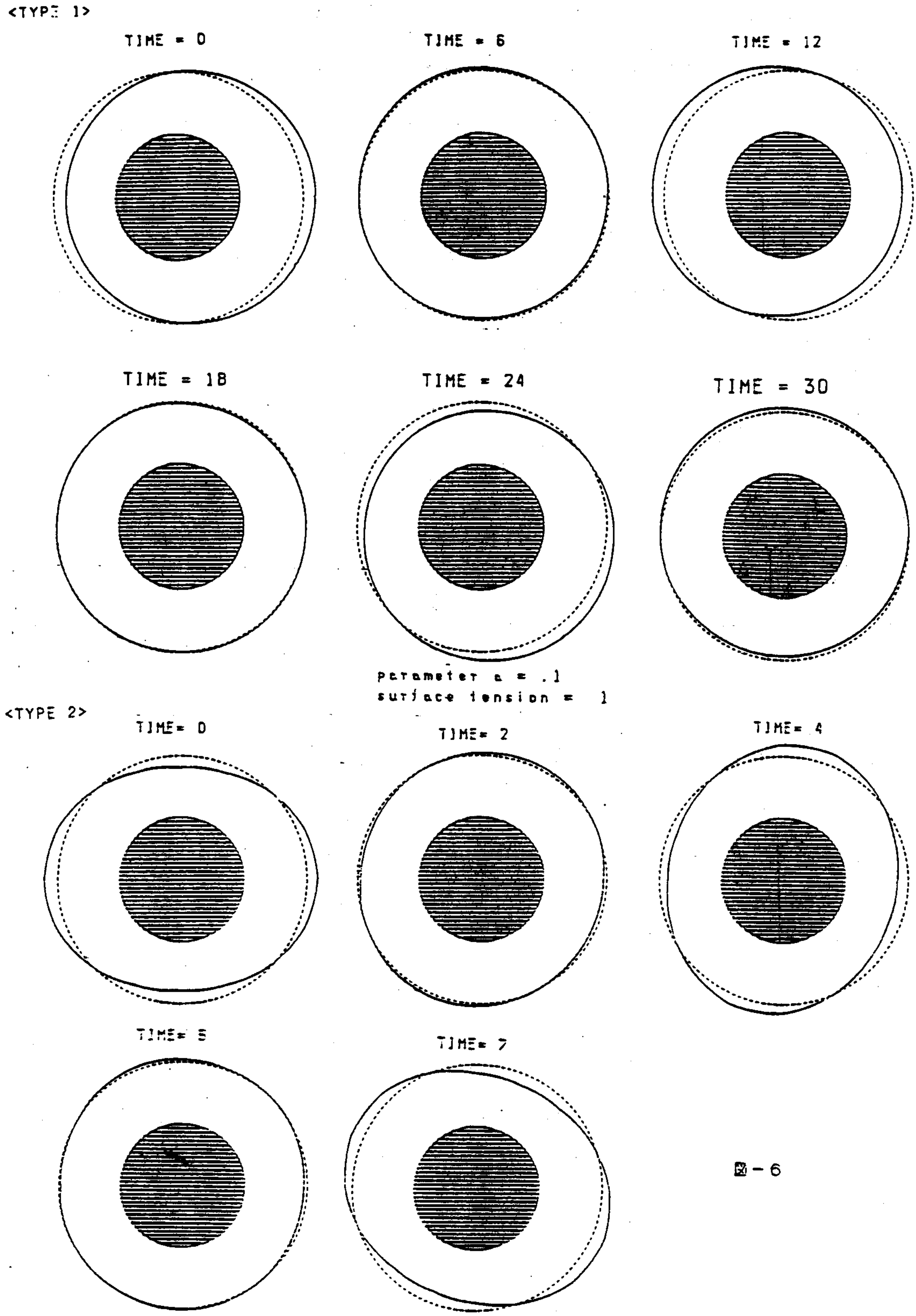

<TYPE 1>
TIME = 0
TIME = 6
TIME = 12
TIME = 18
TIME = 24
TIME = 30
parameter a = .1
surface tension = 1
<TYPE 2>
TIME= 0
TIME= 2
TIME= 4
TIME= 5
TIME= 7

図-6

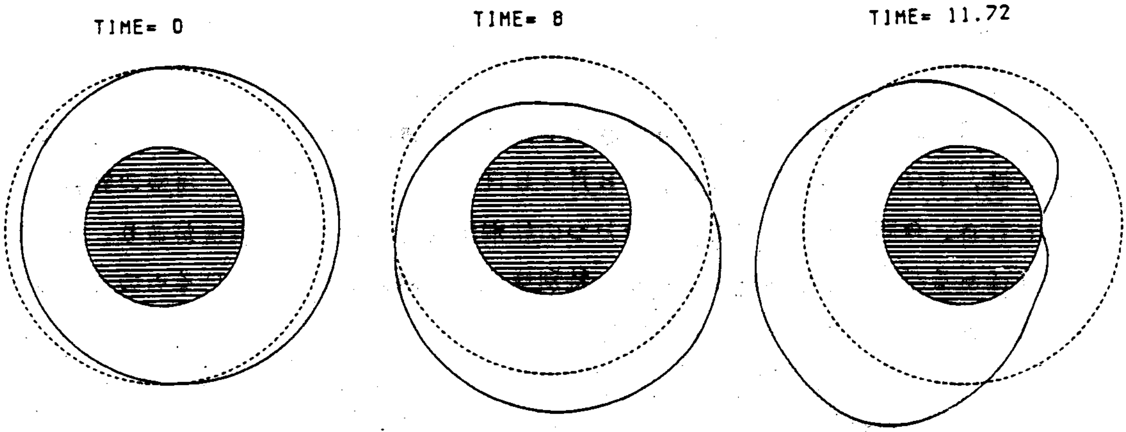


図-7　parameter a = .95
surf.tension= 1

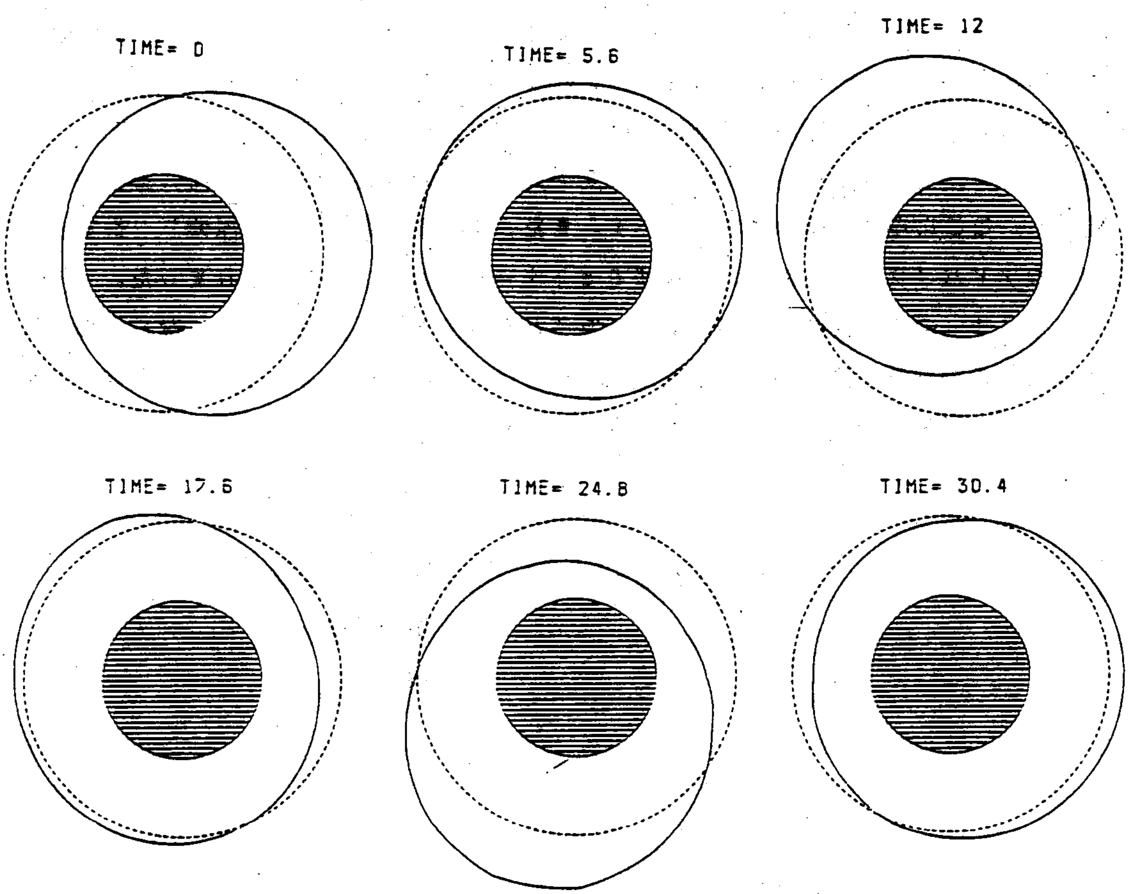


図-8　parameter a =0.1500
surf.tension= 1

序

この報告集は，第6回発展方程式若手セミナーの講演をまとめたものである。本セミナーは，1984年8月24日から8月26日まで，前年度と同じく箱根の静雲荘で行なわれた。今回のセミナーでは，ひとつの特別講演と10の一般講演があり，それぞれに興味ある新しい話題が提供された。特別講演は毎回恒例となっているものであるが今回は広島大学の　俣野　博　氏にお願いし，

”非線型拡散方程式の解の漸近挙動について”

という題で，3時間にわたって最新の話題が提供された。講演および講演後の討論がきわめて活発に，しかも和気合い合いとした雰囲気のもとでなされたことにより，当セミナーの本来の目的は一応達成されたものと思う。

今回のセミナーを行なうに際し，様々な方にお世話になった。特に，セミナーには参加していただけなかったが，御茶ノ水女子大の高村幸男教授には前面的に御援助いただいた。ここに深く感謝の意を表します。また，姫路工業大学の　丸尾健二　氏からは，適切なアドバイスをいただき，おかげでセミナーがスムーズに運営されました。

セミナーの報告集の作成が大変遅れたことを深くお詫びいたします。なお，都合により俣野氏の講演記録がのせられなかったことは，大変残念でありますが，興味のある方は，氏の近著（ Pitman から出版予定）を御覧になってください。

1985年7月　　　セミナー世話人

伊藤　達夫

岡本　久

文　献

[１]　岡本　久：完全流体の自由境界問題における摂動と分岐
第５回発展方程式若手セミナー報告集（’８４）

[２]　H.Fujita,H.Okamoto,M.Shoji:
Numerical analysis of a free boundary problem of a perfect fluid by the boundary element method,
to appear in Japan Journal

[３]　Brebbia,C.A.著，神谷紀生他訳：境界要素法入門，培風館，１９８０

[４]　村島定行著：代用電化法とその応用，森北出版，１９８３

Partial Functional Differential Equations and Optimal Control

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1. Introduction

There exists a great number of literatures which study optimal control problems of abstract control systems in Banach and Hilbert spaces (see books [1,2,3] and the references cited therein). The most studies have been done for the systems without delay, and the papers treating the systems with retardation are not many [4,5,6,7,8]. Furthermore in the above literatures the continuous retardation effect is not in consideration and the concept of fundamental solution (or Green function) is not used so that the calculations are complicated.

— In this paper we study some typical optimal control problem, a kind of integral convex cost problem for general linear retarded systems in Banach spaces. Our approach to solve the problem is based on the representation formula of the (mild) solution in terms of the fundamental solution and the variational method developed in [1] and [9].

The content of this paper is as follows: After system descriptions and formulation of the control problems are given, the retarded adjoint system is introduced and the representation of the adjoint state is given in Section 2. In Section 3, two existence theorems of optimal controls are given, one is for bounded control set and the other is for unbounded control set. In Sectio 4, the necessary conditions for optimality are described by the adjoint state and integral inequality. Some examples of necessary optimality conditions for technologically important costs are also given in Section 4. In Section 5, th maximum principle for Lagrange problem is established with some examples. In Section 6, the bang-bang principle for terminal value problem with time varying control domain and its applications to uniqueness and expression of the optimal control are given under some regularity condition of the adjoint system. All proofs of the results in this paper are sketched or omitted. Detailed proofs will appear in [10].

2. System Description, Control Problem and Adjoint System

First we give the notations and terminology used in this paper. Let X and Y be real (separable) Banach spaces with norms $|\cdot|$ and $|\cdot|_Y$, respectively. The adjoint spaces of X, Y are denoted by X^*, Y^* and their norms are denoted by $|\cdot|_{X^*}$ and $|\cdot|_{Y^*}$. We write the duality pairing between X and X^* by $\langle\ ,\ \rangle$ and the pairing between Y and Y^* by $\langle\ ,\ \rangle_{Y,Y^*}$. Let $L(X,Y)$ be the Banach space of bounded linear operators from X into Y. When $X = Y$, $L(X,Y)$ is denoted by $L(X)$. Their operator norms are denoted by $\|\cdot\|$. Given an interval $I \subset R$, we denote by $L_p(I; X)$ and $C(I; X)$ the usual Banach spaces of measurable functions which are p-Bochner integrable ($1 \le p < \infty$) or essentially bounded ($p = \infty$) on I and strongly continuous on I, respectively. The norm of $L_p(I; X)$ is denoted by $\|\cdot\|_{L_p}$. The function χ_I means the characteristic function of the interval I.

Let $T > 0$, $h > 0$ be fixed and let $I = [0, T]$, $I_h = [-h, 0]$. We consider the following linear hereditary control system on X:

$$
(CS)\begin{cases}
\dfrac{dx(t)}{dt} = A_0x(t) + \displaystyle\int_{-h}^{0} d\eta(s)x(s+t) + f(t) + B(t)u(t) \quad \text{a.e. } t \in I, & (2.1)\\
x(0) = g^0, \quad x(s) = g^1(s) \quad \text{a.e. } s \in [-h, 0), & (2.2)\\
u \in U_{ad},
\end{cases}
$$

where $f \in L_p(I; X)$, $g = (g^0, g^1) \in X \times L_{p'}(I_h; X)$, $U_{ad} \subset L_p(I; Y)$, p, $p' \in [1,\infty]$,

$B \in L(I; \mathcal{L}(Y,X))$ and A_0 generates a strongly continuous semigroup $T(t)$, $t \geq 0$ on X. As for the retardation term in (2.1) we suppose that the Stieltjes measure η is given by

$$\eta(s) = -\sum_{r=1}^{m} \chi_{(-\infty,\ -h_r]}(s) A_r - \int_s^0 D(\xi)\,d\xi \qquad s \in I_h, \tag{2.3}$$

where $0 \leq h_1 < \cdots < h_m \leq h$ are non-negative constants, A_r $(r=1,\cdots,m)$ are bounded linear operators on X and $D \in L_1(I_h; \mathcal{L}(X))$.

The quantities $x(t)$, $u(t)$, $B(t)$ and U_{ad} in (CS) denote a system state (or a trajectory), a control, a controller and a class of admissible controls, respectively.

Let $G(t)$ be the fundamental solution of (CS) which is a unique solution of

$$G(t) = \begin{cases} T(t) + \int_0^t T(t-s) \int_{-h}^0 d\eta(\xi) G(\xi+s)\,ds, & t \geq 0 \\ O, \quad t < 0, \end{cases} \tag{2.4}$$

where O is the null operator on X. We know that $G(t)$ is strongly continuous on R^+. If the condition

$$D \in L_q(I_h; \mathcal{L}(X)),\ 1/p' + 1/q = 1 \tag{2.5}$$

is satisfied, then for each $t \in R^+$ the operator valued function U_t on I_h defined by

$$U_t(s) = \sum_{r=1}^{m} G(t-s-h_r) A_r \chi_{[-h_r,\ 0]}(s) + \int_{-h}^{s} G(t-s+\xi) D(\xi)\,d\xi, \qquad s \in I_h \tag{2.6}$$

belongs to $L_q(I_h; \mathcal{L}(X))$. Hence the function

$$x(t) = x(t;\ f,g) + \int_0^t G(t-s) B(s) u(s)\,ds \tag{2.7}$$

is well-defined and is a member of $C(I;\ X)$, where

$$x(t;\ f,g) = \int_0^t G(t-s) f(s)\,ds + \left(G(t) g^0 + \int_{-h}^0 U_t(s) g^1(s)\,ds \right), \quad t \in I. \tag{2.8}$$

It is proved in [10] that the function $x(t)$ in (2.7) satisfies the integrated form of (2.1), (2.2) in terms of $T(t)$ if (2.5) is satisfied. In this sense we shall call this $x(t)$ the mild (or weak) solution of (CS). Since we use the class of mild solutions (2.7) to investigate the control problems for (CS), the condition (2.5) is always assumed.

In what follows the admissible set U_{ad} is assumed to be closed and convex in $L_p(I; Y)$. We sometimes denote $x(t)$ in (2.7) by $x_u(t)$ to express the dependence on $u \in U_{ad}$. The function x_u is called the trajectory corresponding to u.

Let $J = J(u,x)$ be the integral convex cost given by

$$J = \phi_0(x(T)) + \int_I (f_0(x(t),t) + k_0(u(t),t))dt, \tag{2.9}$$

where $\phi_0 : X \to R$, $f_0 : X \times I \to R$, $k_0 : Y \times I \to R$. We study the following control problems P_1 and P_2 on the finite interval $I = [0, T]$.

P_1. Find a control $u \in U_{ad}$ which minimizes the cost J subject to the constraint (CS).

P_2. Find optimality conditions for $(\bar{u}, x_{\bar{u}})$ such that

$$\inf_{u \in U_{ad}} J(u,x) = J(\bar{u}, x_{\bar{u}}), \quad \bar{u} \in U_{ad}. \tag{2.10}$$

In P_1 such as $u \in U_{ad}$ is called an optimal control for the cost J. In P_2 the pair $(\bar{u}, x_{\bar{u}})$ is called the optimal solution for J. We will solve P_1 partly by showing the existence of optimal controls in Section 3 and solve P_2 by deriving necessary optimality conditions of both integral and pointwise types in Section 4. More further properties such as maximum principle and bang-bang principle are studied in Section 5 and Section 6. To give a difinite form of those optimality conditions it is required some knowledge on the adjoint system.

Now we introduce the retarded adjoint system in the case where X is reflexive. Let X be reflexive and $q_0^* \in X^*$, $q_1^* \in L_1(I; X^*)$. The retarded adjoint system (AS) on X^* is defined by

$$(AS) \begin{cases} \dfrac{dp(t)}{dt} + A_0^* p(t) + \displaystyle\int_{-h}^{0} d\eta^*(s) p(t-s) - q_1^*(t) = 0, \quad \text{a.e. } t \in I \\ p(T) = -q_0^*, \quad p(s) = 0 \quad s \in (T, T+h], \end{cases} \tag{2.11}$$

where A_0^*, $\eta^*(s)$ denote the duals of A_0, $\eta(s)$, respectively. Since X is reflexive, it is known [11] that the adjoint operator A_0^* generates a c_0-semigroup $T^*(t)$ on X^* which is the adjoint of $T(t)$, $t \geq 0$. Hence we can construct the fundamental solution $G_*(t)$ as in [10]. That is, $G_*(t)$ is characterized as the (unique) solution of

$$G_*(t) = \begin{cases} T^*(t) + \displaystyle\int_0^t T^*(t-s) \int_{-h}^{0} d\eta^*(\xi) G_*(\xi+s) ds, \quad t \geq 0 \\ 0, \quad t < 0. \end{cases} \tag{2.12}$$

We denote by $G^*(t)$ the adjoint of $G(t)$. Then it is verified that $G^*(t) = G_*(t)$. By changing time direction in (AS), we consider the following system on X^*:

$$(CS)^* \begin{cases} \dfrac{dw(t)}{dt} = A_0^* w(t) + \displaystyle\int_{-h}^{0} d\eta^*(s) w(t+s) + q_1^*(T-t) \quad \text{a.e. } t \in I \\ w(0) = -q_0^*, \quad w(s) = 0 \quad s \in [-h, 0). \end{cases} \tag{2.13}$$

The mild solution $w(t)$ of $(CS)^*$ is represented by

$$w(t) = G^*(t)(-q_0^*) + \int_0^t G^*(t-s) q_1^*(T-s) ds \tag{2.14}$$

It is easily seen that the system $(CS)^*$ is transformed to the system (AS) by a change of variable $t \to T-t$. Hence by (2.14) the function $p(t)$ given by

$$p(t) = w(T-t) = G^*(T-t)(-q_0^*) + \int_t^T G^*(s-t)(-q_1^*(s))ds, \quad t \in I \tag{2.15}$$

may be called the mild (or weak) solution of (AS). We often call that $p(t)$ in (2.15) solves (AS) in the weak sense.

Remark 2.1. Even if X is not reflexive, the adjoint system can be constructed by the adjoint theory in [11].

3. Existence of Optimal Control

In what follows we assume that Y is reflexive and $1 < p < \infty$. We consider two cases to solve the problem P_1, one is the case where U_{ad} is bounded and the other is where U_{ad} is unbounded in $L_p(I; Y)$. For a bounded U_{ad} we suppose the following assumption H_1 on ϕ_0, f_0 and k_0.

H_1: (1) $\phi_0 : X \to R$ is continuous and convex;

(2) $f_0 : X \times I \to R$ is measurable in $t \in I$ for each $x \in X$ and continuous and convex in $x \in X$ for a.e. $t \in I$ and further for each bounded set $K \subset X$ there exists a measurable function $m_K \in L_1(I; R)$ such that

$$\sup_{x \in K} |f_0(x,t)| \le m_K(t) \quad \text{a.e. } t \in I;$$

(3) $k_0 : Y \times I \to R$ satisfies that for any $u \in U_{ad}$, $k_0(u(t),t)$ is integrable on I and the functional $\xi_0 : L_p(I; Y) \to R$ defined by

$$\xi_0(u) = \int_I k_0(u(t),t)dt \tag{3.1}$$

is weakly lower semi-continuous.

THEOREM 3.1. Let U_{ad} be bounded and H_1 be satisfied. Then there exists a control $u_0 \in U_{ad}$ that minimizes the cost J in (2.9).

(Proof) Let $\{u_n\}$ be a minimizing sequence of J such that

$$\inf_{u \in U_{ad}} J = \lim_{n \to \infty} J(u_n, x_n),$$

where x_n is the trajectory corresponding to u_n. Since U_{ad} is bounded and weakly closed, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ and an $u_0 \in U_{ad}$ such that

$$u_{n_k} \to u_0 \quad \text{weakly in } L_p(I;\ Y). \tag{3.2}$$

Using (3.2), H_1 and Legesgue-Fatou's lemma, u_0 is shown to be an optimal control for J.

Next, we consider the case where U_{ad} is unbounded. In this case we suppose H_1 and the following additional assumption H_2.

H_2: (1) there exists a constant c_0 such that $\phi_0(x) \geq c_0$ on X;

(2) there exists a constant $c_1 > 0$ such that $f_0(x,t) \geq -c_1$ on $X \times I$;

(3) there exists a monotone increasing function $\theta_0 \in C(R^+;\ R)$ such that $\lim_{r \to \infty} \theta_0(r) = \infty$ and

$$\xi_0(u) = \int_I k_0(u(t),t)dt \geq \theta_0(\| u \|_{L_p}) \quad \text{for } u \in U_{ad}.$$

<u>THEOREM 3.2.</u> Let H_1 and H_2 be satisfied. Then there exists a control $u_0 \in U_{ad}$ which minimizes the cost J in (2.9).

(Proof) Note that

$$J \geq \theta_0(\| u \|_{L_p}) + c_0 - c_1 T \quad \text{for } u \in U_{ad}.$$

4. Optimality Condition

In this section we study the problem P_2, or we seek necessary optimality conditions of the optimal solution (u,x) for J in (2.9). The existence of optimal solutions is assumed in this section. To give two types of optimality conditions we introduce the following two assumptions H_3 and H_3^w.

H_3: (1) $\phi_0: X \to R$ is continuous and Gateau differentiable, and the Gateau derivative $d\phi_0(x) \in X^*$ for each $x \in X$;

(2) $f_0: X \times I \to R$ is measurable in $t \in I$ for each $x \in X$ and continuous and convex on X for a.e. $t \in I$ and further there exist functions $\partial_1 f_0 : X \times I \to X^*$, $\theta_1 \in L_1(I; R)$, $\theta_2 \in C(R^+; R)$ such that

a) $\partial_1 f_0$ is measurable in $t \in I$ for each $x \in X$ and continuous in $x \in X$ for a.e. $t \in I$ and the value $\partial_1 f_0(x,t)$ is the Gateau derivative of $f_0(x,t)$ in the first argument for (x,t) in $X \times I$, and

b) $|\partial_1 f_0(x,t)|_{X^*} \leq \theta_1(t) + \theta_2(|x|)$ for $(x,t) \in X \times I$;

(3) $k_0: Y \times I \to R$ is measurable in $t \in I$ for each $u \in Y$ and continuous and convex on Y for a.e. $t \in I$ and further there exist functions $\partial_1 k_0: Y \times I \to Y^*$, $\theta_3 \in L_q(I; R)$ and $M_4 > 0$ such that

a) $\partial_1 k_0$ is measurable in $t \in I$ for each $u \in Y$ and continuous in $u \in Y$ for a.e. $t \in I$ and the value $\partial_1 k_0(u,t)$ is the Gateau derivative of $k_0(u,t)$ in the first argument for (u,t) in $Y \times I$, and

b) $|\partial_1 k_0(u,t)|_{Y^*} \leq \theta_3(t) + M_4 |u|_Y^{p/q}$ for $(u,t) \in Y \times I$.

Next we give the condition $(3)^w$ which is different from $H_3(3)$.

(s)^w $k_0: Y \times I \to R$ is measurable in $t \in I$ for each $u \in Y$ and continuous and convex on Y for a.e. $t \in I$ and further there exist a function $\theta_5 \in L_1(I; R)$ and $M_6 > 0$ such that

$$|k_0(u,t)| \le \theta_5(t) + M_6|u|_Y^p \quad \text{for} \quad (u,t) \in Y \times I.$$

The assumption H_3^w is the set of conditions $H_3(1)$, $H_3(2)$ and $(3)^w$. The assumption H_3 is for the differentiable costs and H_3^w is for non-differentiable costs. The following is the main theorem which gives the necessary conditions of optimality for the problem P_2.

<u>THEOREM 4.1.</u> Let H_3 (resp. H_3^w) be satisfied and let $(u,x) \in U_{ad} \times C(I; X)$ be an optimal solution for J in (2.9). Then the integral inequality

$$\int_I \langle v(t)-u(t),\ \partial_1 k_0(u(t),t)-B^*(t)p(t)\rangle_{Y,Y^*}dt \ge 0 \quad \text{for all} \quad v \in U_{ad} \tag{4.1}$$

$$\text{(resp. } \int_I \langle v(t)-u(t), -B^*(t)p(t)\rangle_{Y,Y^*}dt + \int_I (k_0(v(t),t)-k_0(u(t),t))dt \ge 0 \quad \text{for all} \quad v \in U_{ad} \quad (4.2))$$

holds, where

$$p(t) = -G^*(T-t)d\phi_0(x(T)) - \int_t^T G^*(s-t)\partial_1 f_0(x(s),s)ds. \tag{4.3}$$

If $U_{ad} = L_p(I; X)$, then the condition (4.1) is reduced to that

$$\partial_1 k_0(u(t),t) - B^*(t)p(t) = 0 \quad \text{a.e.} \quad t \in I. \tag{4.4}$$

Furthermore if X is reflexive, $p \in C(I; X^*)$ satisfies

$$\text{(AS)} \begin{cases} \dfrac{dp(t)}{dt} + A_0^* p(t) + \displaystyle\int_{-h}^{0} d\eta^*(s)p(t-s) - \partial_1 f_0(x(t),t) = 0 \quad \text{a.e.} \quad t \in I, \\ p(T) = -d\phi_0(x(T)), \quad p(s) = 0 \quad s \in (T, T+h] \end{cases}$$

in the weak sense.

(Proof) Let H_3 be satisfied. Then the cost J given in (2.9) is Gateau differentiable. The inequality (4.1) follows from the necessary optimality condition

$$J'(u)(v - u) \geq 0 \quad \text{for all} \quad v \in U_{ad}$$

in [1,p.11] and the representation (2.15). Next, let H_3^w be satisfied. Then we can use the optimality condition

$$(J - \xi_0)'(u)(v - u) + (\xi_0(v) - \xi_0(u)) \geq 0 \quad \text{for all} \quad v \in U_{ad}$$

in [1,p.13] to obtain (4.2), where ξ_0 is given in (3.1). The condition (4.4) is obvious from (4.1) and $U_{ad} = L_p(I; X)$.

<u>Remark 4.1.</u> Consider the special case where Y is a Hilbert space, $p = 2$ and $U_{ad} = \{ u \in L_2(I; Y) : \| u \|_{L_2} \leq \alpha \}$. In this case the optimal control $\bar{u}$ is characterized by the relation

$$u = -\alpha \frac{\Lambda^{-1}K(u)}{\| \Lambda^{-1}K(u) \|_{L_2(I;\, Y)}},$$

where Λ is the cannonical isomorphism of $L_2(I; Y)$ into $L_2(I; Y^*)$ and $K(u)(t) = \partial_1 k_0(u(t),t) - B^*(t)p(t)$ a.e. $t \in I$.

Now we give pointwise necessary conditions for optimality. Let U be a closed convex set in Y and the admissible set U_{ad} be given by

$$U_{ad} = \{ u \in L_p(I; Y) : u(t) \in U \text{ a.e. } t \in I \}. \tag{4.5}$$

The next corollary follows from the Lebesgue density theorem.

<u>COROLLARY 4.1.</u> Let the assumptions in Theorem 4.1 be satisfied and U_{ad} be given by (4.5). Then the condition (4.1) (resp. (4.2)) is reduced to the pointwise optimality condition that for a.e. $t \in I$,

$$\langle v - u(t), \partial_1 k_0(u(t),t) - B^*(t)p(t) \rangle_{Y,Y^*} \geq 0 \quad \text{for all} \quad v \in U$$

(resp. $\langle v - u(t), - B^*(t)p(t)\rangle_{Y,Y^*} + (k_0(v,t) - k_0(u(t),t)) \geq 0$

for all $v \in U$).

(Proof) The proof is similar to that given in [2,p.290-291]. Remark that $\partial_1 k_0(u(t),t) - B^*(t)p(t)$ and $k_0(u(t),t)$ are measurable and integrable on I by H_3 and H_3^W.

Example 4.1. (Regulator problem) Let X and Y be Hilbert spaces with inner products $(\, , \,)$ and $\langle \, , \, \rangle_Y$, respectively. We suppose $U_{ad} = L_2(I; Y)$. The spaces X and X^* are identified. The cost J_1 is given by

$$J_1 = (x(T),Nx(T)) + \int_I (x(t),W(t)x(t))dt + \xi_Q(u), \tag{4.6}$$

where

$$\xi_Q(u) = \frac{1}{2}\int_I \langle u(t),Q(t)u(t)\rangle_Y dt. \tag{4.7}$$

In (4.6), (4.7) we assume that $N \in \mathcal{L}(X)$, $W(\cdot) \in L_\infty(I; \mathcal{L}(X))$, $Q(\cdot) \in L_\infty(I;\mathcal{L}(Y))$; N, $W(s)$, $Q(s)$ are positive and symmetric for each $s \in I$; there exists a constant $c > 0$ such that

$$\langle u,Q(t)u\rangle_Y \geq c|u|_Y^2 \quad \text{for a.e. } t \in I.$$

Under the above conditions it is verified that $\xi_Q(u)$ is strongly continuous and strictly convex in $L_2(I; Y)$ ([1,Chapter 3]). Since J_1 is also strictly convex, there exists a unique optimal control for J_1. Then we have

COROLLARY 4.2. Let the cost J_1 be given by (4.6), (4.7). Then there exists a unique optimal solution $(u,x) \in L_2(I; Y) \times C(I; X)$ for J_1. The optimal control $u(t)$ is given by

$$u(t) = Q^{-1}(t)B^*(t)p(t) \quad \text{a.e. } t \in I,$$

where the pair $(x,p) \in C(I; X) \times C(I: X)$ satisfies the system of equations

$$\begin{cases} \dfrac{dx(t)}{dt} = A_0 x(t) + \displaystyle\int_{-h}^{0} d\eta(s)x(t+s) + B(t)Q^{-1}(t)B^*(t)p(t) + f(t) \quad \text{a.e. } t \in I, \\ x(0) = g^0, \quad x(s) = g^1(s) \quad \text{a.e. } s \in [-h, 0), \\ \dfrac{dp(t)}{dt} + A_0^* p(t) + \displaystyle\int_{-h}^{0} d\eta^*(s)P(t-s) - W(t)x(t) = 0 \quad \text{a.e. } t \in I, \\ p(T) = -Nx(T), \quad p(s) = 0 \quad s \in (T, T+h], \end{cases}$$

in the weak sense.

The regulator problem is very important in system design and is investigated in many references. We refer to the books [1,2,3] for infinite dimensional systems without delay and [12,13] for finite dimensional retarded systems. The literature dealing infinite dimensional retarded systems are few [4,5].

5. Maximum Principle

The purpose of this section is to establish the maximum principle for the time varying control domain with the convex integral cost

$$J = \phi_0(x(T)) + \int_I (f_0(x(t),t) + k_0(u(t),t))dt. \tag{5.1}$$

We assume the existence of optimal solutions for J and the assumption H_3^w in this and next sections. Let the admissible set U_{ad} be

$$U_{ad} = \{ u \in L_p(I; Y) : u(t) \in U(t) \quad \text{a.e. } t \in I \}, \tag{5.2}$$

where the (time varying) control domain $U(t) \subset Y$, $t \in I$ satisfies

H_4: (1) $U(t)$ is closed and convex in Y for each $t \in I$;

(2) $\bigcup_{t \in I} U(t)$ is bounded in Y;

(3) for any $t \in I$, $v \in \operatorname{Int} U(t)$, there exists an $\varepsilon_0 > 0$ such that

$$v \in \bigcap_{s \in (t, t+\varepsilon)} U(s) \quad \text{for any } 0 < \varepsilon \le \varepsilon_0.$$

It is clear from $H_4(1),(2)$ that U_{ad} is bounded and convex. Furthermore we have the following lemma.

<u>Lemma 5.1.</u> Let $H_4(1),(2)$ be satisfied. Then U_{ad} given by (5.2) is weakly closed and weakly compact in $L_p(I; Y)$.

(Proof) This lemma follows from Mazur's theorem and Eberlein-Smulian's theorem.

<u>Remark 5.1.</u> If $U(t)$ varies continuously with respect to the Hausdorff metric or $U(t)$ is monotone increasing, then the condition $H_4(3)$ is satisfied.

By Lemma 5.1 and H_3^w, Theorem 4.1 holds for the admissible set (5.2). Moreover if H_1 is satisfied, there is an optimal solution $(u,x) \in U_{ad} \times C(I; X)$ for J in (5.1). We now give the maximum principle for the cost J in (5.1) which is deduced from the optimality condition (4.2).

<u>THEOREM 5.1.</u> Let U_{ad} be given by (5.2) and H_4 be satisfied. Let $(u,x) \in U_{ad} \times C(I; X)$ be an optimal solution for J in (5.1). Then

$$\max_{v \in U(t)} \{ \langle B(t)v, p(t)\rangle - k_0(v,t)\} = \langle B(t)u(t), p(t)\rangle - k_0(u(t),t) \qquad \text{a.e. } t \in I, \quad (5.3)$$

where $p(t)$ is given by

$$p(t) = - G^*(T-t)d\phi_0(x(T)) - \int_t^T G^*(s-t)\partial_1 f_0(x(s),s)ds, \quad t \in I. \quad (5.4)$$

If X is reflexive, then $p(t)$ in (5.4) belongs to $C(I; X^*)$ and is the mild solution of (AS) in Theorem 4.1.

(Proof) Let $t \in (0, T)$ and $v \in \text{Int } U(t)$. Then by $H_4(3)$, the function

$$v_\varepsilon(s) = \begin{cases} u(s), & s \in I - (t, t+\varepsilon) \\ v, & s \in (t, t+\varepsilon) \end{cases}$$

belongs to U_{ad} for any $\varepsilon \in (0, \varepsilon_0]$. From (4.2) and Lebesgue's density theorem we have by letting $\varepsilon \to 0$ that for a.e. $t \in I$,

$$- \langle v, B^*(t)p(t)\rangle_{Y,Y^*} + k_0(v,t) \geq - \langle u(t), B^*(t)p(t)\rangle_{Y,Y^*} + k_0(u(t),t). \quad (5.5)$$

Let $t \in I$ be fixed for which $u(t) \in U(t)$ and (5.5) holds. Since the duality pairing $\langle v, B^*(t)p(t)\rangle_{Y,Y^*}$ is continuous in v, we have from (5.5) that (5.3) is true for such $t \in I$. The latter part of this theorem may be obvious.

We shall give some applications of Theorem 5.1. We consider the special cost functionals J_2-J_4 in Examples 5.1-5.3. Such costs are important in practical applications and are studied in [1,9,14,15,16] for systems without delay. We assume that U_{ad} is given by (5.2) and H_4 is satisfied in each examples below.

Example 5.1. (Special linearized Bolza problem) The cost J_2 is given by

$$J_2 = \langle x(T), \psi_0^* \rangle + \int_I \langle x(t), \psi_1^*(t)\rangle dt, \quad (5.6)$$

where $\psi_0^* \in X^*$ and $\psi_1^* \in L_1(I;\, X^*)$. Then we have

COROLLARY 5.1. Let $(u,x) \in U_{ad} \times C(I;\, X)$ be an optimal solution for J_2. Then

$$\max_{v \in U(t)} \langle B(t)v, p(t)\rangle = \langle B(t)u(t), p(t)\rangle \quad \text{a.e. } t \in I,$$

where $p(t)$ is given by

$$p(t) = - G^*(T-t)\psi_0^* - \int_t^T G^*(s-t)\psi_1^*(s)ds, \quad t \in I. \quad (5.7)$$

If X is reflexive, $p(t)$ in (5.7) belongs to $C(I;\, X^*)$ and satisfies

$$\begin{cases} \dfrac{dp(t)}{dt} + A_0^* p(t) + \displaystyle\int_{-h}^{0} d\eta^*(s) p(t-s) - \psi_1^*(t) = 0 \quad \text{a.e.} \quad t \in I, \\ p(T) = -\psi_0^*, \quad p(s) = 0 \quad s \in (T, T+h] \end{cases}$$

in the weak sense.

Example 5.2. (Terminal value control problem) Let X be a Hilbert space. As usual we identify X and X^*. The cost J_3 is given by

$$J_3 = \frac{1}{2} |x(T) - x_d|^2, \quad x_d \in X. \tag{5.8}$$

COROLLARY 5.2. Let $(u,x) \in U_{ad} \times C(I: X)$ be an optimal solution for J_3 in (5.8). Then

$$\max_{v \in U(t)} (B(t)v, p(t)) = (B(t)u(t), p(t)) \quad \text{a.e. } t \in I,$$

where $p(t)$ is given by

$$p(t) = G^*(T-t)(x_d - x(T)), \quad t \in I. \tag{5.9}$$

The adjoint state $p \in C(I; X^*)$ in (5.9) satisfies

$$\begin{cases} \dfrac{dp(t)}{dt} + A_0^* p(t) + \displaystyle\int_{-h}^{0} d\eta^*(s) p(t-s) = 0 \quad \text{a.e.} \quad t \in I \\ p(T) = x_d - x(T), \quad p(s) = 0 \quad s \in (T, T+h] \end{cases}$$

in the weak sense ($p(t)$ may be identically zero).

Example 5.3. (Minimum energy problem) Let X and Y be Hilbert spaces. The cost J_4 is given by

$$J_4 = \int_I (\lambda^2 |x(t)|^2 + |u(t)|_Y^2) dt, \tag{5.10}$$

where $\lambda > 0$. Then we have

COROLLARY 5.5. Let $(u,x) \in U_{ad} \times C(I; X)$ be an optimal solution for J_4. Then

$$\max_{v \in U(t)} \{ (B(t)v, p(t)) - |v|_Y^2 \} = (B(t)u(t), p(t)) - |u(t)|_Y^2 \quad \text{a.e. } t \in I,$$

where

$$p(t) = -\int_t^T G^*(s-t)(2\lambda^2 x(s))ds \qquad X^* = X, \qquad t \in I$$

satisfies

$$\begin{cases} \dfrac{dp(t)}{dt} + A_0^* p(t) + \displaystyle\int_{-h}^{0} d\eta^*(s)p(t-s) - 2\lambda^2 x(t) = 0 & \text{a.e. } t \in I \\ p(s) = 0 \quad s \in [T, T+h] \end{cases}$$

in the weak sense.

6. Bang-Bang Principle

Let the admissible set U_{ad} be given in Section 5. In this section we consider the terminal value cost J given by

$$J = \phi_0(x(T)), \tag{6.1}$$

where ϕ_0 satisfies $H_1(1)$ and $H_3(1)$. We investigate the possibility of the socalled bang-bang control for J in (6.1) under the time varying control domain $U(t)$. In general the bang-bang control does not hold for the retarded systems even in finite dimensional space [17,p.60]. However by restricting the cost J to the terminal value cost (6.1), we can prove that the bang-bang control is possible under some regularity condition for the adjoint system. Let X be reflexive in this section. Consider the adjoint system (AS) in (2.11). We denote by $p(t; q_0^*, q_1^*)$ the mild solution of (AS). Now we give the following condition

C_w: $q_0^* = 0$ in X^* follows from the existence of a set $E \subset I$ such that

$$\text{meas } E > 0 \quad \text{and} \quad p(t; q_0^*, 0) = 0 \quad \text{for all } t \in E. \tag{6.2}$$

We say that the adjoint system (AS) is weakly regular if the condition C_w is satisfied. Examples for which the system (AS) is weakly regular are given in [9,p.41], but such systems do not involve time delay.

Example 6.1. Consider the control system (CS) enjoying the following conditions i), ii) and iii):

i) A_0 generates an analytic semigroup;

ii) the Stieltjes measure η is given by $\eta(s) = -\chi_{(-\infty,-h]}(s)A_1$;

iii) the system (CS) is pointwise complete for all $t > 0$.

The condition iii) means that for any $f \in L_p^{loc}(R^+; X)$,

$$\mathrm{Cl}\,\{ x(t; f,g) : g \in X \times L_{p'}(I_h; X) \} = X \quad \text{for each} \quad t > 0,$$

where Cl M denotes the closure of M. If i), ii), iii) are satisfied, then the adjoint system of (CS) is weakly regular [10].

The following assumption is needed in proving the bang-band principle.

H_5: $d\phi_0(x_u(T)) \neq 0$ in X^* for each $u \in U_{ad}$, where $x_u(t)$ is the trajectory corresponding to $u \in U_{ad}$.

THEOREM 6.1. Let the cost J be given by (6.1). Assume that the adjoint system (AS) is weakly regular and $B^*(t)$ is one to one for each $t \in I$. If H_5 is satisfied, then the optimal control $u(t)$ for J in (6.1) is a bang-bang control, i.e., $u(t)$ satisfies

$$u(t) \in \partial U(t) \quad \text{a.e.} \quad t \in I. \tag{6.3}$$

(Proof) This theorem is a consequence from the maximum principle (Theorem 5.1) and weak regularity.

Example 6.2. Let the assumptions in Theorem 6.1 be satisfied and let X be a Hilbert space. We consider two costs $J_3 = \frac{1}{2} | x(T) - x_d |^2$ and $J_5 = (x(T), \psi_0)$, $\psi_0 \in X$. If there exists no trajectory x_u, $u \in U_{ad}$ such that $x_u(T) = x_d$ ($\psi_0 \neq 0$ in X), then the optimal control $u(t)$ for J_3 (J_5) is a bang-bang control, i.e., $u(t)$ satisfies (6.3).

Let U be a convex set in Y. The convex set U is said to be strictly convex if $u, v, (u + v)/2 \in \partial U$ imply $u = v$. The following corollaries follow immediately from Theorem 6.1.

<u>COROLLARY 6.1.</u> Let the assumptions in Theorem 6.1 be satisfied and let $U(t)$ be strictly convex for all $t \in I$. Then the optimal control $u(t)$ for J in (6.1) is unique.

<u>COROLLARY 6.2.</u> Let the assumption in Theorem 6.1 be satisfied. Let Y be a Hilbert space and

$$U(t) = \{ u \in Y : |u - y(t)|_Y \leq r(t) \}, \quad t \in I, \tag{6.4}$$

where $y(\cdot) \in C(I; Y)$ and $r(\cdot) \in C(I; R^+ - \{0\})$. Then the optimal control $u(t)$ for J in (6.1) is unique and is given by

$$u(t) = r(t) \{ y(t) + \frac{\Lambda_Y^{-1} B^*(t) p(t)}{|\Lambda_Y^{-1} B^*(t) p(t)|_Y} \} \qquad \text{a.e. } t \in I,$$

where Λ_Y is the cannonical isomorphism of Y onto Y^* and

$$p(t) = G^*(T-t) d\phi_0(x(T)), \quad t \in I.$$

(Proof) Notice that the nonvoid closed ball in a Hilbert space is strictly convex and $U(t)$ in (6.4) is Hausdorff continuous in $t \in I$.

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On the Existence of Periodic Solutions to Nonlinear Abstract Parabolic Equations

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Introduction.

This paper conserns with the nonlinear parabolic evolution equation in a real Hilbert space H, which is of the form

(E) $(d/dt)u(t) \in -\partial\phi^{t}(u(t)) + f(t)$,

where $f \in L^{1}_{loc}(\mathbb{R}; H)$, ϕ^{t} $(t \in \mathbb{R})$ is a proper l.s.c. convex functional on H and $\partial\phi^{t}$ is the subdifferential of ϕ^{t}. Moreover we assume that both $\phi^{(\cdot)}$ and $f(\cdot)$ are T-perodic (i.e. $\phi^{t+T}=\phi^{t}$, $f(t+T)=f(t)$, $t \in \mathbb{R}$).

The existence of periodic solutions of (E) has been obtained by many authors under some assunptions on $\partial\phi^{t}$ (see (A)-(C) in Section 1).

In this paper we consider the existence of periodic solutions, assuming some conditions which differ from coercivity. We will show the existence of periodic solutions in case where $\partial\phi$ is odd (Theorem 1.1). Next we will give examples to see that some of the conditions, which are assumed in Theorem 1.1 or (C) in Section 1, are essential as far as to obtain a periodic solution of (E) (see Propositions 1.1-1.3).

1. Results

Let H be a real Hilbert space with inner product $(.,.)$ and norm $\| \quad \|$. We consider the existence of periodic solutions of

$$(E;\phi^t,f) \qquad (d/dt)u(t) \in -\partial\phi(u(t)) + f(t) \ .$$

Here ϕ^t, $t \in \mathbb{R}$, is a proper l.s.c.(lower semi-continuous) convex functional on H and $\partial\phi^t$ is the subdifferential of ϕ^t. Moreover we assume that

(1.1) both $\phi^{(\cdot)}$ and $f(.)$ are T-periodic and $f \in L^2(0,T;H)$.

In this paper we call u a solution of (E) with initial value u_0 if $u \in C^0([0,\infty);H) \cap W^{1,1}_{loc}((0,\infty);H)$, $u(0)=u_0$ and the relation (E) holds with u for a.e.$t \geq 0$. On the other hand, we call u is a T-periodic solution of (E) if $u \in W^{1,1}_{loc}([0,\infty):H)$, $u(t+T)=u(t), t \geq 0$, and the relation (E) holds for a.e.$t \geq 0$.

The following conditions are known to be sufficient for the existence of T-periodic solutions of (E):

(A) (Benilan and Brezis [2], Nagai[8], Yamada[11])

(i) For each $u_0 \in \overline{D(\phi^0)}$ there is a solution of (E) with initial value u_0.

(ii) $\liminf_{\|x\| \to \infty} \dfrac{\phi^t(x)-\phi^t(0)}{\|x\|} = \infty$ uniformly in $t \in [0, T)$.

(B) (Haraux[5] or [Theorem 1 in Lecture 21;6])

(i) $\phi^t \equiv \phi \quad (t \in \mathbb{R})$.

(ii) $\liminf_{\|x\| \to \infty} \dfrac{\phi(x)-\phi(0)}{\|x\|} = \delta > 0$.

(iii) $\int_0^T f(t)dt = 0$.

(C) ([Corollary 11 in Lecture 20; 6])

(i) $\phi^t \equiv \phi \quad (t \in \mathbb{R})$.

(ii) $\partial\phi$ is a linear operator.

(iii) $T^{-1}\int_0^T f(t)dt \in R(\partial\phi)$.

Our first result is the following:

Theorem 1.1. Suppose;

(1.2) $\phi^t \equiv \phi \quad (t \in \mathbb{R})$,

(1.3) ϕ is even (i.e. $\phi(-x)=\phi(x)$, $x \in H$),

(1.4) $f(t+\frac{T}{2}) = -f(t)$, $t \in \mathbb{R}$.

Then there is a T-periodic solution of (E).

Remarks 1.1. The assumption (1.3) of Theorem 1.1 differs from the topological condition (ii) of (B) (or (A)).

2. Condition (ii) of (C) is a special case of (1.3). In fact, if (ii) of (C) holds then one has $\phi(x) = 2^{-1}\|(\partial\phi)^{1/2}x\|^2$ for each $x \in D(\phi)$ $(=D((\partial\phi)^{1/2}))$, where $(\partial\phi)^{1/2}$ denotes the square root of self-adjoint operator $\partial\phi$ (cf.[Proposition 2.7;]).

3. In the case where both (1.3) and (1.4) hold, then we have $T^{-1}\int_0^T f(t)dt = 0 \in R(\partial\phi)$ (condition (iii) of (C)). To verify this we first note by (1.4) that one has $\int_0^T f(t)dt=0$. Next, by the convexity of ϕ and (1.3) we get the equality $\phi(0)= \min_H\phi$, which means that $0 \in \partial\phi(0) \subset R(\partial\phi)$.

On the other hand, if (1.3) holds then for each solution u of

(1.5) $(d/dt)u(t) \in -\partial\phi(u(t))$,

one has the convergence

(1.6) $\operatorname*{s-lim}_{t\to\infty} u(t) \in (\partial\phi)^{-1}(0)$.

(In general it is known that a solution of (1.5) may fail to converge strongly as $t\to\infty$ even if the set $(\partial\phi)^{-1}(0)$ is nonempty.) Moreover we can obtain the convergence (1.6) for solution u of (1.5) under the following condition ([4], [9]);

(1.3)' $\begin{cases} \text{there is a constant } c > 0 \text{ such that } \phi(-cx) \leq \phi(x) \\ \text{holds for each } x \in H. \end{cases}$

This condition is weaker than (1.3).

We note that Theorem 1.1 does not hold if (1.3)' is assumed instead of (1.3). In fact we have the following proposition:

Proposition 1.1. There are l.s.c. convex functionals ϕ_1 and ψ on H and $f_1 \in L^2(0,T;H)$ with property (1.4) satisfying the following;

(1.7) $\partial\psi$ is a linear operator (condition (ii) of C)),

(1.8) $\min_H \phi_1 = \min_H \psi = 0$ and there is a constant $c_1 \geqq 1$ such that $\psi(x) \leqq \phi_1(x) \leqq c_1\psi(x)$ holds for $x \in H$, but

(1.9) there is no periodic solution of the equation (E) with $\phi^t = \phi_1$ and $f = f_1$.

Remark 1.4. ϕ_1 and f_1 satisfy (1.2), (1.4) in Theorem 1.: and the above-mentioned (1.3)'. In fact, if both (1.7) and (1.8) hold then it is seen that ϕ_1 satisfies (1.3)'. To verify this we note that ψ is even whenever (1.7) holds (see Remark 1.2). Thus we have only to see that, in general, if (1.8) holds with functional ϕ_1 and even convex functional ψ then (1.3)' holds with $\phi = \phi_1$ and $c = 1/c_1$. Since ψ is a even convex functional, one has $\psi(0)=\min_H \psi$. Hence, using (1.8) and eveness and convexit of ψ again, one has

$$\phi_1(-\frac{1}{c_1}x) \leqq c_1\psi(-\frac{1}{c_1}x) = c_1\psi(\frac{1}{c_1}x) = c_1\psi(\frac{c_1-1}{c_1}0 + \frac{1}{c_1}x)$$

$$\leqq c_1\{\frac{c_1-1}{c_1}\psi(0) + \frac{1}{c_1}\psi(x)\} = \psi(x) \leqq \phi_1(x)$$

for $x \in H$. This estimate means that ϕ_1 satisfies (1.3)' with $c = 1/c_1$.

As is seen in Remark 1.3, condition (1.4) yields that

$$(1.4)' \qquad \int_0^T f(t)dt = 0.$$

We note that Theorem 1.1 does not hold if one supposes (1.4)' instead of (1.4). In fact we have the following:

Proposition 1.2. There are l.s.c. convex functional ϕ_2 and $f_2 \in L^2(0,T;H)$ satisfying (1.3), (1.4)' and the following;

$$(1.10)\begin{cases}\text{there is no periodic solution of the equation (E) with } \phi^t \equiv \phi_2 \\ \text{and } f=f_2.\end{cases}$$

Finally we consider the following condition;

(1.11) $\phi^t = \psi + I_{K(t)}$, $t \in [0, T)$

with

(1.12) $\partial\psi$ is linear ((ii) of (C)), and

(1.13) $K(t)$, $t \in [0, T)$, is a closed linear subspace of H.

Here $I_{K(t)}$ denotes the indicator functional of $K(t)$, i.e.,

$$I_{K(t)}x = \begin{cases} 0 & \text{if } x \in K(t), \\ \infty & \text{otherwise.} \end{cases}$$

This condition is a generalization of assumptions (i) and (ii) in (C).

We have the following:

Proposition 1.3. There are l.s.c. convex functionals ϕ_3^t ($t \in [0, T)$) and $f_3 \in L^2(0,T;H)$ satisfying (1.11), (1.12), (1.13) and the following;

(1.14) $T^{-1}\int_0^T f_3(s)ds = 0 \in R(\partial\phi^t)$ for $t \in [0, T)$ ((iii) of (C)),

(1.15) for each $u_0 \in cl(D(\phi^0))$ $(=K(t))$ there is a solution $u \in W^{1,1}(0,\infty;H)$ of $(E;\phi_3^t,f_3)$ with $u(0)=u_0$.

(1.16) there is no periodic solution of $(E;\phi_3^t,f_3)$.

2. Proof of Theorem 1.1

For each $a \in \overline{D(\phi)}$ there is a unique solution u_a of (E) with initial value $u(0)=a$. We define a single-valued mapping S by $Sa = u_a(T)$ for $a \in \overline{D(\phi)}$.

To show that S has a fixed point in $\overline{D(\phi)}$ we use the following fixed point theorem;

Theorem A (Browder and Petryshyn [3]). Let S be a nonexpansive selfmapping of a nonempty closed convex set C of H. Then S has a fixed point in C if and only if for any $x_0 \in C$ the sequence of Picard iterates $\{x_n\}$ starting at x_0 (i.e. $x_{n+1}=Sx_n$) is bounded in H.

We have only to prove that for some $u_0 \in \overline{D(\phi)}$ the sequence of Picard iterates $\{u_n\}$ starting at u_0 is bounded. To show this we extend f on $[0,\infty)$ by $f(t)=f(t-[t/T]T)$ and let u be the solution of (E) with arbitraly initial-value u_0. Then the definition of $\{u_n\}$ means that $u_n=u(nT)$, $n \geq 0$. Hence it is sufficient to show that the set $\{u(t): t \geq 0\}$ is bounded in H.

In what follows we will show the boundedness of $\{u(t): t \geq 0\}$. We first note by (1.3) that the relation $\partial\phi(-x)=-\partial\phi(x)$ holds for each $x \in D(\partial\phi)$. Hence one has the relation

$$u'(t)-f(t) \in -\partial\phi(u(t)) = \partial\phi(-u(t))$$

for a.e.$t \geq 0$, where $u'(t)=(d/dt)u(t)$. Therefore, using (1.4) and the monotonicity of $\partial\phi$, we have

$$\begin{aligned}\frac{d}{dt}\|u(t+2^{-1}T) + u(t)\|^2 &= 2(u'(t+2^{-1}T)+u'(t), \ u(t+2^{-1}T)+u(t)) \\ &= 2(u'(t+2^{-1}T)-f(t+2^{-1}T)+u'(t)-f(t), \ u(t+2^{-1}T)-(-u(t))) \\ &\leq 0\end{aligned}$$

for a.e. $t \geq 0$. Hence

$$\text{(2.1)} \quad \|u(t+2^{-1}T)+u(t)\| \leq \|u(2^{-1}T)+u(0)\| \ (=c_1), \qquad t > 0.$$

On the other hand condition (1.3) also yields that $0 \in \partial\phi(0)$. Hence one has

$$\begin{aligned} \frac{d}{dt}\|u(t)\| &= \|u(t)\|^{-1}(u'(t), u(t)) \\ \text{(2.2)} \qquad &= \|u(t)\|^{-1}\{(u'(t)-f(t)-0, u(t)-0) + (f(t), u(t))\} \\ &\leq \|u(t)\|^{-1}\{0 + \|f(t)\| \|u(t)\|\} = \|f(t)\| \end{aligned}$$

for a.e. $t \geq 0$. Therefore we have

$$\text{(2.3)} \quad \|u(t+2^{-1}T)\| - \|u(t)\| \leq \int_t^{t+2^{-1}T} \|f(s)\| ds = \int_0^{2^{-1}T} \|f(s)\| ds \ (=c_2)$$

for $t > 0$.

Now we assume that the set $\{u(t): t \geq 0\}$ is unbounded. Then there is the sequence $\{t_n\}$ in $[0, \infty)$ defined by

$$t_n = \inf\{t \geq 0: \|u(t)\| \geq n\}.$$

Note that one has

$$\text{(2.4)} \quad \|u(s)\| < \|u(t_n)\| = n, \qquad 0 \leq s < t_n$$

for each $n \in \mathbb{N}$. Moreover it follows from (2.2) and (1.1) that $t_n \uparrow \infty$ as $n \to \infty$.

Fix an arbitraly $n \in \mathbb{N}$ with $t_n \geq 2^{-1}T$. Let $v(t)$ ($t \in [t_n-2^{-1}T, \infty)$) be the solution of the initial-value problem

$$\begin{cases} (d/dt)v(t) \in -\partial\phi(v(t)), & t \geq t_n-2^{-1}T, \\ v(t_n-2^{-1}T) = u(t_n-2^{-1}T). \end{cases}$$

Then one has estimates

$$(2.5)\qquad \|v(t_n)-u(t_n)\| \leq \int_{t_n-2^{-1}T}^{t_n} \|f(t)\|dt = \int_0^{2^{-1}T} \|f(t)\| dt \ (=c_2),$$

and

$$(2.6)\qquad \phi(v(t_n)) \leq \phi(v(t)), \qquad t \in [t_n-T,\ t_n).$$

We note that (1.3) and (2.6) together yield that

$$(2.7)\qquad (v(t_n),\ v'(s)) \leq -(v(s),\ v'(s)) \qquad \text{a.e.} s \in [t_n-2^{-1}T,\ t_n]$$

since by the definition of subdifferential one has

$$(-\overline{v(t_n)}-v(s),\ -v'(s)) \leq \phi(-v(t_n))-\bar{\phi}(v(s))$$
$$= \phi(v(t_n)) - \phi(v(s)) \leq 0.$$

By (2.7) and (2.4) we have

$$(v(t_n),\ v(t_n)-v(t_n-2^{-1}T)) = \int_{t_n-2^{-1}T}^{t_n} (v(t_n),\ v'(s))ds$$

$$(2.8)\qquad \leq \int_{t_n-2^{-1}T}^{t_n} (-v(s),\ v'(s))ds$$

$$= 2^{-1}\{\|v(t_n-2^{-1}T)\|^2 - \|v(t_n)\|^2\}$$

$$\leq 2^{-1}\|v(t_n-2^{-1}T)\|^2 = 2^{-1}\|u(t_n-2^{-1}T)\|^2 \leq 2^{-1}n^2.$$

Put $y=v(t_n)-u(t_n)$ and $z=v(t_n-2^{-1}T)+u(t_n)$ $(=u(t_n-2^{-1}T)+u(t_n))$ Then estimates (2.1)and (2.5) yield that $\|y\| \leq c_2$ and $\|z\| \leq c$ respectively. Hence

$$(v(t_n),\ v(t_n)-v(t_n-2^{-1}T))$$

$$\begin{aligned}(2.9)\qquad &= (u(t_n)+y,\ \ u(t_n)+y+u(t_n)-z)\\ &\geq 2\|u(t_n)\|^2 - (c_1+c_2)\|u(t_n)\| - c_2(c_2+c_1)\\ &= 2n^2 - (c_1+c_2)n - c_2(c_2+c_1).\end{aligned}$$

Now by (2.8) and (2.9) one has

$$2n^2 - (c_1+c_2)n - c_2(c_2+c_1) \leq 2^{-1}n^2 .$$

Since c_1 and c_2 are independent of n, this estimate is a contradiction. Therefore the set $\{u(t) : t \geq 0\}$ is bounded.

Consequently applying Theorem A we conclude that there is a periodic solution of (E).

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A CHARACTERIZATION OF THE M-ACCRETIVITY

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X and Y denote Banach spaces with the norm $\|\cdot\|_X$ and $\|\cdot\|_Y$. We omit X or Y in the norm if confusion does not occur. X" denotes the second dual space of X.

First, for multi-valued operator $A:X \to Y$, we shall define its (*)-derivative $\delta A(x,x'): X'' \to Y$ at $(x,x') \in$ the graph of A, which is a generalization of Gâteaux derivative in a certain sense. Using this derivative, we characterize the m-accretivity.

1. Definition of (*)-derivative

Throughout this report, we denote by $\{h_\alpha\}$, a directed family of positive numbers which tends to 0. We shall define (*)-derivative $\delta A(x,x'); X'' \to Y$ at $(x,x') \in$ the graph of A as follows. We define $\delta A(x,x')z \ni y$ for $z \in X''$, $y \in Y$, if there are a Lipschitz continuous curve $v(t) \in D(A)$ and a C^1-curve $w(t) \in Av(t)$ on $[0,T]$ for some $T > 0$, such that

$$(1.1) \qquad w^*\text{-}\lim_{\alpha} \frac{v(h_\alpha)-x}{h_\alpha} = z \text{ in } X'' \quad \text{for some } \{h_\alpha\},$$

$$\lim_{h \downarrow 0} \frac{w(h)-x'}{h} \; (= w'(0)) = y \text{ in } Y.$$

REMARK 1. As direct consequences we have:

$v(0) = x$ and $w(0) = x'$ for $v(t)$ and $w(t)$ in the above.

$D(\delta A(x,x'))$ is the set of all the elements z in (1.1).

If $\delta A(x,x')z \ni y$, then $\delta A(x,x')(az) \ni ay$ for every $a \geq 0$.

REMARK 2. In general, $\delta A(x,x')$ may be a multivalued operator.

REMARK 3. If A is single-valued and $A(x + ty)$ is defined and C^1-class in t on $[0,t_0)$ for sufficiently small $t_0 > 0$, then $\delta A(x,Ax)z \ni \frac{d^+}{dt} A(x + tz)|_{t=0}$.

EXAMPLE. Let Ω be a domain in R^n with $C^{2,\alpha}$-class compact boundary. We take $X = C^{2,\alpha}(\Omega)$ and $Y = C^{\alpha}(\omega)$. We consider the following following elliptic operator with the oblique boundary condition;

$$Au = \sum_{i,j=1}^{n} a_{ij}(x,u,Du)D_{ij}u + b(x,u,Du) \cdot (D_i u = \frac{\partial u}{\partial x_i}, Du = (D_1 u, \cdots, D_n u)),$$

$$D(A) = \{u \in C^{2,\alpha}(\Omega);\ c(x,u,Du) = 0 \text{ on } \partial\Omega\}$$

where a_{ij}, $b \in C^2(\Omega\times R\times R^n)$, $c \in C^3(\Omega\times R\times R^n)$ and satisfy the following assumptions:

$$a_{ij}(x,y,z)\xi_i\xi_j \geq \lambda(\|y\|,\|z\|)|\xi|^2 > 0,$$

$$|c_y(x,y,z)| > \kappa(\|y\|, \|z\|) > 0 \text{ on } \partial\Omega,$$

$$|c_z(x,y,z)\cdot\nu| > \kappa(\|y\|,\|z\|) > 0 \text{ on } \partial\Omega,$$

where ν is an outer normal vector from Ω. In this case, $D(A)$ is not necessarily convex nor open, and therefore A is not Gâteux differ differentiable in general. But A has a linear (*)-derivative $\delta A(u,Au)$ for every $u \in D(A)$:

$$\delta A(u,Au)v = \sum_{i,j}(D_y a_{ij})(x,u,Du)D_{ij}u$$

$$+ \sum_{i,j,k}(D_{z_k} a_{ij})(x,u,Du)D_{ij}u\, D_k v + D_y b(x,u,Du)v$$

$$+ \sum_k D_{z_k} b(x,u,Du)D_k v.$$

$$D(\delta A(u,Au)) = \{v \in C^{2,\alpha}(\bar{\Omega});\ D_y c(x,u,Du)v + \quad D_{z_k} c(ux,u,DU)$$

$$+ \sum_k D_{z_k} c(x,u,Du)D_k v = 0 \text{ on } \partial\Omega\}.$$

2. Theorems and proofs

THEOREM 1. Let $A: X \to Y$ be a closed (multi-valued) operator with A^{-1} single-valued. Let L be a non-negative constant. Then the following three conditions are equivalent.

1°) $R(A) = Y$ and $\| A^{-1}x - A^{-1}y\|_X \le L\|x - y\|_Y$ for every x, $y \in Y$.

2°) $R(\delta A(x, x')) = Y$ and $\|\delta A(x,x')^{-1}y\|_X \le L\|y\|_Y$ for every $x \in D(A)$, $x' \in Ax$ and $y \in Y$.

3°) $\overline{R(\delta A(x, x'))} = Y$ and $\| \delta A(x,x')^{-1}y\|_X \le L\|y\|_Y$ for every $x \in D(A)$, $x' \in Ax$ and $y \in R(\delta A(x,x'))$.

REMARK 4. In order that Theorem 1 holds, $w(t)$ need only be differentiable on $[0,T]$ with $\|w'(t)\|$ upper semi-continuous for some positive T in the definition of (*)-derivative.

Proof of Theorem 1. "1°) implies 2°)". Let $x \in D(A)$, $x' \in Ax$ and $y \in Y$. We put $v(t) = A^{-1}(x' + ty)$ and $w(t) = x' + ty \in Av(t)$. Then it holds $\| \frac{1}{h}(v(h) - x)\| \le L$. Therefore, we have $w^*\text{-}\lim_{h_\alpha \downarrow 0} \frac{1}{h_\alpha}(v(h_\alpha) - x) = z$ for some $\{h_\alpha\}$ and $z \in X'$. This fact implies that $\delta A(x,x')z \ni w'(0) = y$. Hence, we have that $R(\delta A(x,x')) = Y$.

On the other hand, suppose that $\delta A(x,x')z = y$. Then, there are a Lipchitz continuous curve $v(t) \in D(A)$ and a C^1-curve $w(t) \in Av(t)$ with $w^*\text{-}\lim_{h_\alpha \downarrow 0} \frac{1}{h_\alpha}(v(h_\alpha) - x) = z$ for some $\{h_\alpha\}$ and $\lim_{h \downarrow 0} \frac{1}{h}(w(h) - x') = y$. Since A^{-1} is single-valued, we have that

$$\left\|\frac{1}{h_\alpha}(v(h_\alpha) - x)\right\| = \left\|\frac{1}{h_\alpha}(A^{-1}w(h_\alpha) - A^{-1}x')\right\| \le L\left\|\frac{1}{h_\alpha}(w(h_\alpha) - x')\right\|$$

Letting $h_\alpha \downarrow 0$ in the last inequality, we obtain that $\|z\| \le L\|y\|$.

"2°) implies 3°)". This is trivial.

To prove "3°) implies 1°)", it suffices to show the following lemma.

LEMMA. Let $A: X \to Y$ be a closed (multi-valued) operator. Let $x_0 \in D(A)$, $y_0 \in Ax_0$, $L \ge 0$, $r > 0$ and $y^* \in B_r(y_0)$, where $B_r(y_0)$ is the open ball of radius r centered y_0. If A^{-1} is single-valued on $R(A) \cap B_r(y_0)$, then the following 3°)' implies 1°)'.

1°)' $y^* \in A(\overline{B_{L\|y-y_0\|}(x_0)})$.

3°)' $y^* - y \in \overline{R(\delta A(x, y))}$ and $\|\delta A(x,y)^{-1}z\| \le L\|z\|$ for every $x \in D(A) \cap B_{Lr}(x_0)$, $y \in Ax \cap B_r(y_0)$ and $z \in R(\delta A(x,y))$.

Proof. Let ε be an arbitrary number such that $0 < 2\varepsilon < r - \|y^* - y_0\|$.

1) First, we show that for arbitrary $a > 0$, $x \in D(A) \cap B_{Lr}(x_0)$ and $y \in Ax \cap B_r(y_0)$, there exist $h \in (0, \frac{1}{2a})$, $x' \in D(A)$ and $y' \in Ax'$ which satisfy the following;

(2.1) $\|x' - x\| \le hL(a\|y^* - y\| + \frac{\varepsilon}{a})$,

(2.2) $\|y' - y - ha(y^* - y)\| < \frac{h\varepsilon}{a}$.

From the assumption 3°)', there exist $z \in X''$ and $y'' \in Y$ such that

(2.3) $\quad \|y'' - y\| < \dfrac{\varepsilon}{2a^2}$ and

$$a(y'' - y) \in R(\delta A(x,y)).$$

Therefore, there are a Lipshitz continuous curve $v(t) \in D(A)$ and a C^1-curve $w(t) \in Av(t)$ with $w(0) = y$ and $w'(0) = a(y'' - y)$, on $[0,h']$ for some $h' > 0$. Thus, there exists $h \le \min\{h', \frac{1}{2a}\}$ such that

$$\|w(t) - y - ta(y'' - y)\| < \frac{t\varepsilon}{2a},$$

$$\|w'(t) - a(y'' - y)\| < \frac{\varepsilon}{2a}$$

for $0 \le t \le h$. Hence, using (2.3), we have that

(2.4) $\quad \|w(t) - y - ta(y^* - y)\| < \dfrac{t\varepsilon}{a}$,

(2.5) $\quad \|w'(t) - a(y^* - y)\| < \dfrac{\varepsilon}{a}$

for $0 \le t \le h$. From the definition of Gâteaux derivative, for each t with $0 \le t \le h$, we have $\{h_\alpha\}$ such that

$$\delta A(v(t), w(t))w^*\text{-}\lim_{\alpha} \frac{1}{h_\alpha}(v(t + h_\alpha) - v(t)) \ni w'(t).$$

Thus, by the assumption 3°)', it follows that

(2.6) $\quad \|w^*\text{-}\lim_{\alpha} \dfrac{1}{h_\alpha}(v(t + h_\alpha) - v(t))\| \le L\|w'(t)\|$ for $0 \le t \le h$.

Let $f \in F(v(h) - x)$. The mapping: $t \to (v(t) - x, f)$ is Lipschitz continuous of $[0,h]$ to R. Therefore, it is differentiable a.e. t and satisfies the following;

$$(2.7)\qquad \|v(h) - x\|^2 = \int_0^h \frac{d}{dt}(v(t) - x, f)dt$$

$$\le hL(a\|y^* - y\| + \frac{\varepsilon}{a})\|v(h) - x\|.$$

In the last inequality, we used (2.5) and (2.6). Hence, if we set $x' = v(h)$ and $y' = w(h)$, then (2.7) and (2.4) imply (2.1) and (2.2) respectively.

2) Let Ω be the first uncountable ordinal number, and W be the well-ordered set $\{\alpha ; \alpha < \Omega\}$. We shall define $h_\alpha \ge 0$, $x_\alpha \in D(A) \cap B_{Lr}(x_0)$ and $y_\alpha \in Ax_\alpha \cap B_r(y_0)$ by transfinite induction, and we put $t_\alpha = \sum_{\beta \le \alpha} h_\beta$.

(1) We set $h_0 = 0$.

(2) Let $\alpha \in W$. Assume that h_β, $x_\beta \in D(A)$ and $y_\beta \in Ax_\beta$ has already been defined for $\beta < \alpha$ such that

$$(2.8)\qquad 0 \le h_\beta \le \frac{1}{2}(1 - \sum_{\gamma<\beta} h_\gamma) \text{ if } \beta > 0,$$

$$(2.9)\qquad \|x_\beta - x_\gamma\| \le L(t_\beta - t_\gamma)(\|y^* - y_0\| + 2\varepsilon) \text{ for } \gamma \le \beta,$$

$$(2.10)\qquad \|(1 - t_\gamma)(y_\beta - y^*) - (1 - t_\beta)(y_\gamma - y^*)\|$$

$$\le 2\varepsilon(1 - t_\beta)(1 - t_\gamma)(t_\beta - t_\gamma) \text{ for } \gamma \le \beta.$$

We note that if $t_\beta < 1$, then (2.10) is equivalent to (2.10)';

$$(2.10)'\qquad \|(y_\beta - y^*)/(1 - t_\beta) - (y_\gamma - y^*)/(1 - t_\gamma)\| \le 2\varepsilon(t_\beta - t_\gamma)$$

for $\gamma \le \beta$. We also note that (2.8) and (2.9), (2.10) with $\gamma = 0$ yield the following inequalities;

$$0 \le t_\beta \le 1,$$

(2.11) $\|x_\beta - x_0\| \le Lt_\beta(\|y^* - y_0\| + 2\varepsilon) < Lr,$

(2.12) $\|y_\beta - y_0\| \le t_\beta(\|y^* - y_0\| + 2\varepsilon) < r,$

(2.13) $\|y_\beta - y^*\| \le (1 - t_\beta)(\|y^* - y_0\| + 2\varepsilon t_\beta) \le (1 - t_\beta)r.$

Therefore, it follows that $x_\beta \in D(A) \cap B_{Lr}(x_0)$ and $y_\beta \in Ax_\beta$ $B_r(y_0)$. We define h_α, x_α and y_α in two cases.

<1> In case that $\alpha = \alpha' + 1$ with $\alpha' \in W$, we further consider two cases.

(a) If $t_{\alpha'} = 1$, then we set $h_\alpha = 0$, $x = x_{\alpha'}$ and $y = y_{\alpha'}$.

(b) If $t_{\alpha'} < 1$, then we set $h_\alpha = h$, $x_\alpha = x'$ and $y_\alpha = y'$ in 1) with $a = 1/(1 - t_{\alpha'})$, $x = x_{\alpha'}$ and $y = y_{\alpha'}$. Then it follows that

$$0 < h_\alpha < \frac{1}{2}(1 - t_{\alpha'}), \quad 0 < t_\alpha < 1,$$

(2.14) $\|x_\alpha - x_{\alpha'}\| \le h_\alpha L\{\|y^* - y_{\alpha'}\|/(1 - t_{\alpha'}) + \varepsilon(1 - t_{\alpha'})\},$

(2.15) $\|y_\alpha - y_{\alpha'} - \frac{h_\alpha}{1-t_{\alpha'}}(y^* - y_{\alpha'})\| < h_\alpha\varepsilon(1 - t_{\alpha'}).$

We shall prove that (2.8), (2.9) and (2.10)' hold for $\beta = \alpha$. (2.8) is trivial. From (2.13) with $\beta = \alpha'$ and (2.14), we have

(2.16) $\|x_\alpha - x_{\alpha'}\| \le h_\alpha L(\|y^* - y_0\| + 2\varepsilon).$

From (2.9) with $\beta = \alpha'$ and (2.16), it follows that (2.9) holds for $\beta = \alpha$. Since $h_\alpha < \frac{1}{2}(1 - t_{\alpha'})$, we have that $\frac{1-t_{\alpha'}}{1-t_\alpha} \le 2$. Therefore, we have from (2.15) that

$$(2.17) \quad \left\| \frac{y_\alpha - y^*}{1 - t_\alpha} - \frac{y_{\alpha'} - y^*}{1 - t_{\alpha'}} \right\| < \frac{h_\alpha \varepsilon (1 - t_{\alpha'})}{1 - t_\alpha} \le 2h_\alpha \varepsilon.$$

From (2.10)' with $\beta = \alpha'$ and (2.17), it follows that (2.10)' holds for $\beta = \alpha$. Hence, we can continue induction.

〈2〉 In case that α is a limit ordinal number, we set $h_\alpha = 0$. Then, it is easily seen that $t_\alpha = \sup_{\beta<\alpha} t_\beta$. Let $\beta_n \uparrow \alpha$ as $n \to \infty$. It follows from (2.9) that $\{x_{\beta_n}\}$ is a Cauchy sequence in X. From (2.10) with $\gamma = \beta_m \le \beta = \beta_n$ for $t_\alpha < 1$, or with $\gamma = 0 \le \beta = \beta_n$ for $t_\alpha = 1$, it follws that $\{y_{\beta_n}\}$ is a Cauchy sequence in Y. Hence, there are $x_\alpha \in X$ and $y_\alpha \in Y$ such that

$$x_{\beta_n} \to x_\alpha \text{ in X as } n \to \infty, \quad y_{\beta_n} \to y_\alpha \text{ in Y as } n \to \infty.$$

Since $A: X \to Y$ is a closed operator, the above convergences imply that $x_\alpha \in D(A)$ and $y_\alpha \in Ax_\alpha$. Also from (2.9) and (2.10), we see that x_α and y_α are independent of the choice of a subsequence $\{\beta_n\}$. Letting $\beta \uparrow \alpha$ in (2.9) and (2.10), it follows that (2.9) and (2.10) hold for $\beta = \alpha$. Since (2.8) is trivial, we can continue induction.

3) We set $t^* = \sup_{\alpha \in W} t_\alpha (\le 1)$. Since t_α is nondecreasing as to α, we can take a nondecreasing sequence $\{\alpha_n\}$ in W such that $t_{\alpha_n} \uparrow t^*$ as $n \to \infty$. Since α_n is nondecreasing, α_n converges to a ordinal number α^* as $n \to \infty$. Noting that $\alpha_n < \Omega$, and that Ω is an

uncountable ordinal number, we have that $\alpha^* < \Omega$, i.e., $\alpha^* \in W$. Therefore, it follows that $t^* = t_{\alpha^*}$. If $t^* < 1$, we have from 2) (2) <1> (b) that $t_{\alpha^*+1} = t_{\alpha^*} + h_{\alpha^*+1} > t_{\alpha^*}$, which contradicts the maximality of t_{α^*}, because $\alpha^* + 1 \in W$. Hence, we obtain that $t^* = 1$. From the definition of t_α, we easily see that there is a sequence $\{\beta_n\}$ in W such that $t_{\beta_n} < 1$ and $t_{\beta_n} \to 1$ as $n \to \infty$. Letting $n \to \infty$ in (2.13) with $\beta = \beta_n$, we have that

(2.18) $\quad y_{\beta_n} \to y^*$ in Y as $n \to \infty$.

From (2.9) with $\beta = \beta_n$, $\gamma = \beta_m$ for $m \le n$, we have that $\{x_{\beta_n}\}$ is a Cauchy sequence in X. Hence, there is $x_\varepsilon \in X$ such that

(2.19) $\quad x_{\beta_n} \to x_\varepsilon$ in X as $n \to \infty$.

Since $A: X \to Y$ is a closed operator, (2.18) and (2.19) imply that $x_\varepsilon \in D(A)$ and $y^* \in Ax_\varepsilon$. Since A^{-1} is single-valued on $R(A) \cap B_r(y_0)$, $x^* = x_\varepsilon = A^{-1}y^*$ is uniquely determined. From (2.11), we have that

$$\|x^* - x_0\| \le L(\|y^* - y_0\| + 2\varepsilon).$$

Letting $\varepsilon \to 0$, we have that $x^* \in \overline{B_{L\|y^*-y_0\|}(x_0)}$, which completes the proof.

The next theorem, which is the purpose of this report follows immediately from Theorem 1.

THEOREM 2. Let A be a closed operator in X with $(I + \lambda A)^{-1}$ single-valued for sufficiently small $\lambda > 0$. Then the following three conditions are equivalent.

1°) A is m-accretive.

2°) $R(\delta(I + \lambda A)(x,x')) = X$ and $\|\delta(I + \lambda A)(x,x')z\| \geq \|z\|$ for sufficiently small $\lambda > 0$, $\forall x \in D(A), \forall x' \in Ax$ and $\forall z \in D(\delta(I + \lambda A)(x,x'))$.

3°) $\overline{R(\delta(I + \lambda A)(x,x'))} = X$ and $\|\delta(I + \lambda A)(x,x')z\| \geq \|z\|$ for sufficiently small $\lambda > 0$, $\forall x \in D(A)$, $\forall x' \in Ax$ and $\forall z \in D(\delta(I + \lambda A)(x,x'))$.

Existence of Solutions to Second Order Semilinear

Differential Equations

Kenichi Fukuda

I. Introduction We shall consider the problem

$$(1.1)\qquad u'' - (A_1 + A_2)u' + A_2 A_1 u = f(u),\ u(0)=\phi\ ,\ u'(0)=\psi.$$

in a Banach space X , where A_i (i = 1,2) is the infinitesimal generator of a C_0-semigroup $T_i(t)$ and f is continuously Gâteaux differentiable on X .

One stnadard approach to the second order equation of a form

$$(1.2)\qquad u'' + A u' + B u = f(u),\quad u(0)=\phi,\ u'(0)=\psi\ ,$$

is to reduce it to a first order system in some space $X_E \times X$, where X_E ($\subset$ X) has an energy norm. One disadvantage to this approach is that the space X_E depends on the particular equation and it is not easy to find a suitable norm.

Our approach is to factor the problem(1.2) into the form of (1.1) and then reduce it to the first order system

$$(1.3)\qquad \frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} u_2 \\ f(u_1) \end{bmatrix}$$

in $X \times X$. While this method may at first seem unnatural, it usefulness will be demonstrated by applying it to a large class of e equations(semilinear version of wave equation, strong damped and damped, telegraph equations and etc.) using only the quadratic equation. Moreover, the factoring procedure eliminates the need to find an energy norm suitable to the problem.

Our aim is to discuss the existence of strong solutions ~~to~~

to (1.1).

Continuous function u(t) is said to be a mild solution to (1.1) if itsatisfies

$$(1.4)\quad \begin{aligned} u(t) &= T_1(t)\phi + \int_0^t T_1(t-\tau)T_2(\tau)(\psi - A_1\phi)\, d\tau \\ &+ \int_0^t \int_0^\tau T_1(t-\tau)T_2(\tau-s)\ f(u(s))\, d\tau\, ds, \quad \phi\in D(A_1) . \end{aligned}$$

We say that a function u(t) is a strong solution to (1.1) if it satisfies

$$(1.5)\quad u(t) \in C^1(I;X)\cap C(I:D(A_1)),$$
$$u\ (t) - A_1u(t)\in C^1(I:X)\cap C(I:D(A_2)) ,$$

and

$$(1.6)\quad (\frac{d}{dt} - A_2)(\frac{d}{dt} - A_1)\ u(t) = f(u(t)),\quad t\in I.$$

The existence of strong solutions is discussed in[3]in the case that a mild solution u(t) and f(u(t)) is twice continuously differrentiable, through tedious calculations.

II. **Main results** Throughout this paper, F is continuously Gâteaux differentiable on X and satisfies (H1) or (H2);

(H1) For any $r>0$ there exists $L(r)$ such that $|f(x) - f(y)| \leqq L(r)\ |x - y|$ for $|x|, |y| \leqq r$.

(H2) For any x in X there exists a neighborhood of x on which f is Lipschitz continuous.

Mild solutions to (1.3) is defined through the equation

$$(2.1)\quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} T_1(t)u_1(0) \\ T_2(t)u_2(0) \end{bmatrix} + \int_0^t \begin{bmatrix} T_1(t-s)\ u_2(s) \\ T_2(t-s)\ u_1(s) \end{bmatrix} ds .$$

Mild solution u(t) of (1.4) is obtained by el minating the second component of (2.1). Conversely, $u_1(t)= u(t)$ and $u_2(t) = T_2(t)(\psi - A_1\phi) + \int_0^t T_2(t-s)\ f(u(s))\ ds$ satisfy (2.1).

To consider the existence of mild solutions, we shall mention

Theorem 2.1 Suppose that A is the infinitesimal generator of a C_0-semigroup and F satisfies (H1) or (H2). Let $\mathcal{T}(U_0)$ be the maximal time of existence of mild solution to $\dot{U}(t) = [A + F]U(t)$, $U(0)=U_0$. Then we have

(1) If $\mathcal{T}(U_0) < \infty$, then $\lim_{t \nearrow \mathcal{T}(U_0)} \|U(t)\| = \infty$,

(respectively, if $\mathcal{T}(U_0) < \infty$, then $\lim_{t \nearrow \mathcal{T}(U_0)} \bigcap_{t<s<\mathcal{T}(U_0)} \overline{\{U(\xi) \mid s < \xi < \mathcal{T}(U_0)\}} = \phi$)

(2) $\mathcal{T}(U_0) < \lim_{V \to U_0} \mathcal{T}(V)$, $D_t = \{ V \mid t < \mathcal{T}(V)\}$.

The proof of this theorem is demonstrated in [1] and [6].

Applying this results to (1.3) and (2.1) and eliminating the second component, we have

Theorem 2.2 Let $\mathcal{T}(\phi, \psi)$ be the maximal time of existence of solution u(t) to (1.4). Then we have

(1) If $\mathcal{T}(\phi,\psi) < \infty$, then $\lim_{t \nearrow \mathcal{T}(\phi,\psi)} \|u(t)\| = \infty$,

(resp. if $\mathcal{T}(\phi,\psi) < \infty$, $\lim_{t \nearrow \mathcal{T}(\phi,\psi)} \bigcap_{<s<\mathcal{T}(\phi,\psi)} \overline{\{u(\xi) \mid s < \xi < \mathcal{T}(\phi,\psi)\}} = \phi$.)

From Theorem 2.1, we can define a local semiflow (or semigroup) S(t) by letting $S(t)U_0 = U(t)$, $D(S(t)) = D_t$, where U(t) is a mild solution to $\frac{d}{dt}U(t) = [A + F]U(t)$, $U(0)=U_0$.

The next result is obtained in [2].

Theorem 2.3 Let A be the infinitesimal generator of a C_0-semigroup and F be continuously Gâteaux differentiable on X. Then we have

(1) S(t) is continuously Gâteaux differentiable on D(S(t)).

(2) $S(t)U_0$ $(U_0 \in D(A))$ is a C^1-solution to

$$\frac{d}{dt} U(t) = [A + F] U(t), \quad U(0) = U_0$$

and satisfies

$$\frac{d}{dt} S(t)U_0 = dS(t)(U_0)\,[A + F]\,U_0$$
$$= [A + F]\,S(t)U_0 \quad .$$

The differentiability of $U \mapsto S(t)U$ is also discussed in [5] and [6] in the condition that F is continuously Fréchet differentiable on X.

It is trivial that $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ and $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} u_2 \\ f(u_1) \end{bmatrix}$ satisfy the the assumptions of Theorem 2.3. So we can apply this to our problem.

We now discuss strong solutions to (1.1), The following theorem is anounced in [3].

<u>Theorem</u> Suppse that $u(t) \in C^2([0,T]:X)$ and satisfies (1.4). Also assume that $f(v(t)) \in C^2$ whenever $v(t) \in C^2$. Then $u(t)$ is a strog solutions (1.1) for $\phi \in D(A_2A_1)$ and $\psi \in D(A_1)$.

However, the assumptions of this Theorem are seen to be rather strong and the proof is completed after tedious calculations. So we apply Theorem 2.3 and then have

Theorem 2.4. Assume that A_1 and A_2 are infinitesimal generator and f is continuously Gâteaux differentiable on X. If $\phi \in D(A_1)$, $\psi - A_1\phi \in D(A_2)$ then a solution to (1.4) is a strong solution (1.1).

<u>Theorem 2.4′</u> Under the same assumption as in theorem 2.4.,

the equation (1.1) has a unique strong solutions on the maximal interval of existence of solution to (1.4).

We can prove these theorems through applyinp theorem 2.3, and eliminating the second component from (1.3) and (1.4).

III. Examples We now consider the several applications. All of the examples will be done in complex Hilbert space $L^2(\Omega)$, where Ω is either a smooth bounded region in R^n or all of R^n . For details, we reffer to [3].

Example 3.1. The semilinear telegraph equation

(3.1) $u_{tt} + \alpha u_t - \Delta u = f(u) \quad (\alpha > 0)$

in $L^2(R^n)$ (or $L^2(\Omega)$, where Ω is smooth bounded in R^n).

Example 3. 2. The strongly damped equation

(3.2) $u_{tt} + \alpha \Delta u_t - \Delta u = f(u) \quad (\alpha > 0) \quad u(0) = \phi.\ u\ (0) = \psi$

in $L^2(R^n)$

Example 3.3. The strong damped equation in one demention

(3.3) $u_{tt} + \alpha u_{tx} - u_{xx} = f(u),\ \alpha \in R\ ,\ x \in R,$
$u(0.x) = \phi(x), u_t(0.x) = \psi(x),\ \phi, \psi \in L^2(R)$.

Other examples are mentioned in [3]

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非線型Schrödinger方程式のL^2-解

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§1. 序

次のような非線型Schrödinger方程式のCauchy問題を考える。

(1.1) $i\frac{\partial u}{\partial t} = -\Delta u + \lambda|u|^{r-1}u$, $t>0$, $x\in\mathbb{R}^n$,

(1.2) $u(0,x) = u_0(x)$.

但し、λは実数定数である。問題(1.1)-(1.2)の解の存在については様々な研究がなされている。$u_0\in H^1(\mathbb{R}^n)$の時はGinibre & Velo [2] が $u(t)\in C(\mathbb{R}^+;H^1(\mathbb{R}^n))$ の弱解を作っており、$u_0\in H^2(\mathbb{R}^n)$ の時はBaillon, Cazenave & Figueira [1] が $u(t)\in C(\mathbb{R}^+;H^2(\mathbb{R}^n))\cap C^1(\mathbb{R}^+;L^2(\mathbb{R}^n))$ の強解

を作っている。　さらに、滑らかな初期値 u_0 に対しては、早稲田大学の Tsutsumi & Hayashi [6] の両氏が古典解を構成している。　ところが、Strauss は文献 [4] の中で、$p=1+\frac{4}{n}$ の時に $u_0 \in L^2(\mathbb{R}^n)$ に対して問題 (1.1)-(1.2) の局所解に相当するものを構成している。　この論文では、$u_0 \in L^2(\mathbb{R}^n)$ に対して問題 (1.1)-(1.2) の弱解を作るという観点から、Strauss の結果をさらに発展させてみたい。　この論文の主定理は次のようなものである。

<u>定理 1</u>　$1 < p < 1+\frac{4}{n}$ とする。

その時、任意の $u_0 \in L^2(\mathbb{R}^n)$ に対して次の

ようなクラスの問題 (1.1)-(1.2) の弱解 $u(t)$ が一意的に存在する。

(1.3) $u(t) \in C(\mathbb{R}^+; L^2(\mathbb{R}^n)) \cap L^r_{loc}(\mathbb{R}^+; L^{p+1}(\mathbb{R}^n))$,

(1.4) $\frac{\partial u}{\partial t}(t) \in L^{r/p}_{loc}(\mathbb{R}^+; H^{-2}(\mathbb{R}^n))$,

(1.5) $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$, $t \geq 0$.

但し、$r = \frac{4(p+1)}{n(p-1)}$ である。

注意 1　定理 1 の解は $t \in \mathbb{R}^-$ に対しても、もちろん構成できる。

§2. 補題.

ここでは、定理 1 を証明するのに必要ないくつかの補題を述べる。以後、$r = \frac{4(p+1)}{n(p-1)}$ とし $U(t) = e^{i\Delta t}$ とする。$L^2(\mathbb{R}^n)$、$H^1(\mathbb{R}^n)$ などの $\mathbb{R}^n$ 上で定義されている関数空

間は、単に L^2、H^1 などと書くことにする。

補題 2　$1 < p < 1 + \frac{4}{n}$ とする。

その時、次の評価が成立する。

(2.1) $\|U(\cdot)v\|_{L^r(\mathbb{R};L^{p+1})} \le \delta \|v\|_{L^2}$, $v \in L^2$.

但し、δ は n と p だけに依存する正定数である。

この補題 2 については [5] を参照せよ。

補題 3　次の評価が成立する。

(2.2) $\|U(t)v\|_{L^q} \le C|t|^{-\frac{n}{2}+\frac{n}{q}} \|v\|_{L^{\bar{q}}}$, $v \in L^{\bar{q}}$.

但し、$1/q + 1/\bar{q} = 1$, $q \ge 2$ で、定数 C は n と q だけに依存する。

補題3については[2]を参照せよ。

次のような積分方程式を考える。

(2.3) $u(t) = U(t)u_0 - i\int_0^t U(t-\tau) f(u(\tau))d\tau,$
$t \geq 0.$

但し、$f(v) = \lambda |v|^{p-1} v$ である。この積分方程式は Cauchy 問題 (1.1)-(1.2) の積分方程式版である。この積分方程式 (2.3) の解の存在について次のような Ginibre & Velo [2] による結果がある。

<u>補題4</u>　$1 < p < 1 + \frac{4}{n}$ とする。

その時、任意の $u_0 \in H^1$ に対して (2.3) の解 $u(t) \in C(\mathbb{R}^+; H^1)$ が一意的に存在して

(2.4)　$\|u(t)\|_{L^2} = \|u_0\|_{L^2}$　, $t \geq 0$

を満たす。

注意 2　　$\lambda>0$ の時は、$1<p<\frac{n+2}{n-2}$ で補題 4 が成立する。詳細は Ginibre & Velo [2] を参照せよ。

補題 5　　ある $T>0$ に対して、$u(t)\in C([0,T];H^{-2})$ かつ ある $K>0$ に対して、

(2.5)　$\|u(t)\|_{L^2}\leq K$　, a.a. $t\in[0,T]$

と仮定する。その時、実は、$u(t)\in \bar{C}_w([0,T];L^2)$ で、すべての $t\in[0,T]$ に対して不等式 (2.5) が成立する。但し、$C_w([0,T];L^2)$ は L^2 の弱位相で連続な関数の集合を表わす。

補題 5 は基本的な関数解析の結果をつかって証明できる。

§3. 証明.

$h(x)$ を $\|h\|_{L^1}=1$ であるような C_0^∞-関数とする。良く知られているように、$h_j(x)=j^n h(jx)$, $j=1,2,\cdots$, と置いて軟化子を作る。次のような積分方程式を考える。

$$(3.1)\quad u_j(t)=U(t)h_j*u_0-i\int_0^t U(t-\tau)f(u(\tau))d\tau,$$
$$t\geq 0,\ j=1,2,\cdots.$$

ここで、$*$ はたたみ込み (convolution) を表わす。補題4よりすべての j に対して、(3.1) の解 $u_j(t)\in C(\mathbb{R}^+;H^1)$ が存在し（ここでは初期値は h_j*u_0 として考える）、次の等式を満たす。

$$(3.2)\quad \|u_j(t)\|_{L^2}=\|h_j*u_0\|_{L^2}\leq\|u_0\|_{L^2},$$
$$t\geq 0,\ j=1,2,\cdots.$$

$\rho=\|u_0\|_{L^2}$ とおき、M を次のような関数

の集合とする。

(3.3) $M = \{ v(t) \in L^\infty(0,T;L^2) \cap L^r(0,T;L^{p+1})\ ;$

$$\|v\|_{L^\infty(0,T;L^2)} \le \rho,$$

$$\|v\|_{L^r(0,T;L^{p+1})} \le 2\delta\rho \ \}.$$

但し、ここでTは後で決定される小さな正定数である。 Mは$L^r(0,T;L^{p+1})$の強位相で閉じていることを注意しておく。

まず、集合Mの中で(2.3)の局所解を構成する。 $s>0$に対して、

(3.4) $$u_j^s(t) = \begin{cases} 0 & , \ t \notin [0,s], \\ u_j(t) & , \ t \in [0,s] \end{cases}$$

とおく。 (3.1)の両辺に対して$L^r(0,s;L^{p+1})$ノルムをとると、

(3.5) $$\|u_j\|_{L^r(0,s;L^{p+1})} \le \delta\rho + C\left\|\int_0^t |t-\tau|^{-\frac{n}{2}+\frac{n}{p+1}} \|f(u_j(\tau))\|_{L^{1+1/p}}\, d\tau\right\|_{L^r(0,s)}$$

$$\leq \delta\rho$$

$$+c\left\| \int_{-\infty}^{\infty} |t-\tau|^{-\frac{n}{2}+\frac{n}{p+1}} \| u_j^s(\tau) \|_{L^{p+1}}^p \, d\tau \right\|_{L^r(\mathbb{R})}$$

$$\leq \delta\rho + c \| u_j^s \|_{L^{q_1}(\mathbb{R};L^{p+1})}^p$$

$$\leq \delta\rho + c \| u_j \|_{L^{q_1}(0,s;L^{p+1})}^p , \quad s \in [0,T].$$

但し、 $q_1 = \dfrac{4p(p+1)}{n+4-(n-4)p}$ である。 ここで、(3.5) の最初の不等式では (2.1)、(2.2) をつかい、 3番目の不等式では一般化された Young の不等式 ([3] を参照せよ) をつかった。$1 < p < 1+\frac{4}{n}$ の時、 $0 < q_1 < r$ であるから Hölder の不等式より

(3.6) $\| u_j \|_{L^{q_1}(0,s;L^{p+1})}$

$$\leq \left(\int_0^s d\tau \right)^{1/q_2} \| u_j \|_{L^r(0,s;L^{p+1})}$$

$$\leq T^{1/q_2} \| u_j \|_{L^r(0,s;L^{p+1})} , \quad s \in [0,T].$$

但し、 $q_2 = \dfrac{2(n+2)-2(n-2)p}{n+4-(n-4)p}$ である。

(3.5)、(3.6)より

(3.7) $\|u_j\|_{L^r(0,s;L^pH)}$

$$\leq \delta\rho + C_0 T^{p/q_2}\|u_j\|^p_{L^r(0,s;L^pH)},\ s\in[0,T],$$
$$j=1,2,\cdots.$$

但し、$C_0=C_0(n,p)$である。ここで、

Tを

(3.8) $T<(2C_0(2\delta\rho)^{p-1})^{-q_2/p}$

となるように十分小さくとると、代数方程式

$C_0T^{p/q_2}y^p+\delta\rho-y=0$を満たす正数$y$が

2つ存在し、小さい方は$2\delta\rho$よりも小さい。

そこで、小さい方のyをy_0とする。

$\|u_j\|_{L^r(0,s;L^pH)}=X_j(s)$と置くと(3.7)より

(3.9) $X_j(s)\leq\delta\rho+C_0T^{p/q_2}X_j(s)^p,\ s\in[0,T],$

$$X_j(0)=0,$$
$$j=1,2,\cdots.$$

(3.9)はもしTが(3.8)を満たすような十分小さくとられていれば

(3.10) $X_j(s) \le y_0 \le 2\delta\rho$, $s \in [0,T]$, $j=1,2,\cdots$

であることを示している。従ってFatouの補題により

(3.11) $\|u_j\|_{L^r(0,T;L^{p+1})} \le 2\delta\rho$, $j=1,2,\cdots$.

一方、(3.2)より

(3.12) $\|u_j\|_{L^\infty(0,T;L^2)} \le \rho$, $j=1,2,\cdots$

だから、(3.11)と合わせると、Tを(3.8)が成立するほど十分小さく選ぶなら、

(3.13) $u_j \in M$, $j=1,2,\cdots$

ということになる。以後、Tは(3.8)が成立するほど十分小さいとする。任意のu_j, $u_k \in M$に対して(3.5)、(3.6)のように計算すると

(3.14) $\|u_j - u_k\|_{L^r(0,T;L^{p+1})}$

$$\le \delta \|h_j * u_0 - h_k * u_0\|_{L^2}$$

$$+ \overline{C}_0 T^{p/q_2} \cdot 2(2\delta\rho)^{p-1} \|u_j - u_k\|_{L^r(0,T;L^{p+1})}.$$

ここで、$q_2 = \dfrac{2(n+2)-2(n-2)p}{n+4-(n-4)p}$, $\overline{C}_0 = \overline{C}_0(n,p)$

である。Tを

(3.15) $\overline{C}_0 T^{p/q_2} \cdot 2(2\delta\rho)^{p-1} \le \frac{1}{2}$

となるように十分小さく取れば、(3.14)より

(3.16) $\|u_j - u_k\|_{L^r(0,T;L^{p+1})}$

$$\le 2\delta \|h_j * u_0 - h_k * u_0\|_{L^2}$$

$$\longrightarrow 0 \quad (j, k \longrightarrow \infty).$$

従って、(3.13)、(3.16)より、次のようなクラスの(2.3)の解 $u(t)$ を得る。

(3.17) $u(t) \in L^\infty(0,T;L^2) \cap L^r(0,T;L^{p+1})$,

(3.18) $\|u(t)\|_{L^2} \le \|u_0\|_{L^2}$, a.a. $t \in [0,T]$.

$(\cdot\,,\cdot)$ を L^2-内積とし、形式的に(2.3)

の両辺を t で微分すると、任意の
$\varphi\in\mathcal{S}$ に対して

(3.19) $\frac{d}{dt}(u(t),\varphi)=(U(t)u_0,-i\Delta\varphi)$

$-i(f(u(t)),\varphi)-i\int_0^t(U(t-\tau)f(u(\tau)),-i\Delta\varphi)d\tau$

$=-i(f(u(t)),\varphi)-(u(t),-i\Delta\varphi)\quad,t\in[0,T].$

$1<p<1+\frac{4}{n}$ の時、$r>p$ だから、(3.17)
と(3.19)より、

(3.20) $\frac{\partial u}{\partial t}(t)\in L^{r/p}(0,T;H^{-2}).$

(3.17)と(3.20)より、$u(t)\in C([0,T];H^{-2})$
だから、これと(3.18)を合わせると補題5
が適用できて

(3.21) $u(t)\in C_w([0,T];L^2)$

かつ不等式(3.18)はすべての $t\in[0,T]$ に対
して成立する。 このクラスの(2.3)の解の一意
性は通常の方法で証明できる。 また $U(t)$

がユニタリーグループであることより、tに関して負の方向へ(2.3)を一意的に解くことができることから、(3.18)でtと0の役割を入れかえることができる。即ち、初期時刻をt、初期値を$u(t)$としてtに関して負の方向に(2.3)を解けば(3.18)と逆向きの不等式が得られる。これより(1.5)が成立することが示せた。(1.5)と(3.21)より$u(t) \in C([0,T];L^2)$ということになる。存在時間Tは、$\|u_0\|_{L^2}$の大きさとp, nにしか依存しない((3.8), (3.15)を参照)ので、a priori bound (1.5)より、解は時間に関して大域的に拡張できる。

(証明終)

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