SC4 後中 2019 (応数B, II)

試験答案用紙　　番号　　氏名

1. (1) [1] $\boxed{3.043}$ [2] $\boxed{3.007}$ [3] $\boxed{13.362}$

[4] $\boxed{1.833}$ (2点)

(2) X, Y は互いに独立. $E[X] = 1\times\frac{1}{2} + 2\times\frac{1}{6} + 0\times\frac{1}{3} = \frac{5}{6}$

$E[Y] = 2\times\frac{1}{2} + 3\times\frac{1}{6} + 0\times\frac{1}{3} = \frac{3}{2}$

[1] $E[Z] = E[XY] = E[X]E[Y] = \boxed{\frac{5}{4}}$

[2] X^2, Y^2 も互いに独立 $E[X^2] = 1^2\times\frac{1}{2} + 2^2\times\frac{1}{6} = \frac{7}{6}$

$E[Y^2] = 4\times\frac{1}{2} + 9\times\frac{1}{6} = \frac{7}{2}$

$\therefore E[Z^2] = E[X^2Y^2] = E[X^2]E[Y^2] = \boxed{\frac{49}{12}}$

[3] $V[Z] = E[Z^2] - (E[Z])^2 = \frac{49}{12} - \frac{25}{16} = \boxed{\frac{121}{48}}$

[4] X, Y は互いに独立なので $Cov[X,Y] = \boxed{0}$ (4点)

(3) [1] $E[X_n] = \frac{1}{13}(1+2+\cdots+13) = \frac{1}{13}\times\frac{1}{2}\times 13\times 14 = \boxed{7}$

[2] $E[X_n^2] = \frac{1}{13}(1^2+2^2+\cdots+13^2) = \frac{1}{13}\times\frac{1}{6}\times 13\times 14\times 27 = 63$ $\therefore V[X_n] = 63 - 7^2 = \boxed{14}$

[3] $E[\bar{X}] = E[X_n] = \boxed{7}$

[4] $V[\bar{X}] = \frac{1}{39}V[X_n] = \boxed{\frac{14}{39}}$

(4) $E[X] = 8\cdot\frac{1}{2} = \boxed{4}$ [1], $E[Y] = 12\cdot\frac{3}{4} = \boxed{9}$ [2]

$V[X] = 8\cdot\frac{1}{2}\cdot\frac{1}{2} = \boxed{2}$ [3], $V[Y] = 12\cdot\frac{3}{4}\cdot\frac{1}{4} = \boxed{\frac{9}{4}}$ [4]

$E[Z] = E[X] - E[Y] = 4 - 9 = \boxed{-5}$ [5]

$V[Z] = V[X] + V[Y] = 2 + \frac{9}{4} = \boxed{\frac{17}{4}}$ [6]

$E[W] = E[XY] = E[X]E[Y] = \boxed{36}$ [7]

(4点)

2. $P(W=m) = P(X+Y+Z=m) = \sum_{k=0}^{m} P(X+Y=k, Z=m-k)$

$= \sum_{k=0}^{m}\sum_{l=0}^{k} P(X=l)P(Y=k-l)P(Z=m-k)$

$= \sum_{k=0}^{m}\sum_{l=0}^{k} e^{-\lambda}\frac{\lambda^l}{l!} e^{-\mu}\frac{\mu^{k-l}}{(k-l)!} e^{-\nu}\frac{\nu^{m-k}}{(m-k)!}$

$= e^{-(\lambda+\mu+\nu)} \sum_{k=0}^{m} \frac{\nu^{m-k}}{(m-k)!}\left(\sum_{l=0}^{k}\frac{1}{k!}\, {}_kC_l\, \lambda^l \mu^{k-l}\right)$

$= e^{-(\lambda+\mu+\nu)} \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\nu^{m-k}(\lambda+\mu)^k$

$= e^{-(\lambda+\mu+\nu)} \sum_{k=0}^{m} \frac{1}{m!}\, {}_mC_k\, \nu^{m-k}(\lambda+\mu)^k = e^{-(\lambda+\mu+\nu)}\frac{(\lambda+\mu+\nu)^m}{m!}$

より W は $Po(\lambda+\mu+\nu)$ に従う. (7点)

3. (1) $k\geq 0$ で $\int_0^3\int_0^3 kxy\,dxdy = k\left(\int_0^3 x\,dx\right)^2 = \frac{81}{4}k = 1$

$\therefore k = \boxed{\frac{4}{81}}$

(2) $x<0,\ x>3$ のとき $f_1(x)=0$, $0\leqq x\leqq 3$ のとき

$f_1(x) = \frac{4}{81}\int_0^3 xy\,dy = \frac{4}{81}x\left[\frac{1}{2}y^2\right]_0^3 = \frac{2}{9}x$

$\therefore f_1(x) = \begin{cases} \frac{2}{9}x & (0\leqq x\leqq 3) \\ 0 & (x<0,\ x>3) \end{cases}$ (3点)

(3) $P(X\leqq 1, Y\geqq 1) = \int_0^1\int_1^3 \frac{4}{81}xy\,dy\,dx = \frac{4}{81}\left(\int_0^1 x\,dx\right)\left(\int_1^3 y\,dy\right)$

$= \frac{4}{81}\left[\frac{1}{2}x^2\right]_0^1\left[\frac{1}{2}y^2\right]_1^3 = \boxed{\frac{8}{81}}$

4. $E[T]=0$ だから $V[T] = E[T^2] = 2C\int_0^\infty t^2\left(1+\frac{t^2}{m}\right)^{-(m+1)/2}dt$

$\left(C = \frac{1}{\sqrt{m}B(m/2, 1/2)}\right)$ ここで $u = \frac{m}{t^2+m}$ と置換すれば

$t^2 = mu^{-1}(1-u)$, $\left(1+\frac{t^2}{m}\right)^{-(m+1)/2} = u^{(m+1)/2}$, $dt = -\frac{\sqrt{m}}{2}u^{-3/2}(1-u)^{-1/2}du$

だから $V[T] = 2C\frac{m\sqrt{m}}{2}\int_0^1 u^{(\frac{m}{2}-1)-1}(1-u)^{\frac{3}{2}-1}du$

$= \frac{m}{B(\frac{m}{2},\frac{1}{2})}B\left(\frac{m}{2}-1, \frac{3}{2}\right) = m\frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{1}{2})}\frac{\Gamma(\frac{m}{2}-1)\Gamma(\frac{3}{2})}{\Gamma(\frac{m+1}{2})}$

$= m\frac{\Gamma(\frac{m}{2}-1)\cdot\frac{1}{2}\Gamma(\frac{1}{2})}{(\frac{m}{2}-1)\Gamma(\frac{m}{2}-1)\Gamma(\frac{1}{2})} = \frac{m}{m-2}$ (7点)