MS2 後中 2018 (3章, 4章§1, §2 極座標)

2018年度試験答案用紙 番号 氏名

1.(1) 与式 $= \frac{1}{2}\left[x\sqrt{9-x^2}+9\sin^{-1}\frac{x}{3}\right]_0^{3/2} = \frac{9}{8}\sqrt{3}+\frac{9}{2}\sin^{-1}\frac{1}{2}$
$= \boxed{\frac{9}{8}\sqrt{3}+\frac{3}{4}\pi}$

(2) 与式 $= \int_{-1}^{0}\sqrt{2-(x+1)^2}\,dx = \int_0^1\sqrt{2-t^2}\,dt$
$= \frac{1}{2}\left[t\sqrt{2-t^2}+2\sin^{-1}\frac{t}{\sqrt{2}}\right]_0^1 = \boxed{\frac{1}{2}+\frac{\pi}{4}}$

(3) 与式 $= \frac{1}{2}\int_0^{\pi/2}\cos^8 t\,dt = \frac{1}{2}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{\pi}{4} = \boxed{\frac{35}{512}\pi}$
$t=2x$

(4) 与式 $= \left[-\frac{1}{10}(1-x^2)^5\right]_0^1 = \boxed{\frac{1}{10}}$

(5) 与式 $= \left[\log(\log 2x)\right]_e^{e^3} = \log(\log 2e^3)-\log(\log 2e)$
$= \log(\log 2+3)-\log(\log 2+1) = \boxed{\log\frac{\log 2+3}{\log 2+1}}$

(6) 与式 $= \int_0^{\pi/2}(\cos^4 x-\cos^6 x)\,dx = \frac{3}{4}\cdot\frac{\pi}{4}-\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{\pi}{4} = \boxed{\frac{\pi}{32}}$

2. $f(x) = \frac{a}{x}+\frac{b}{x^2}+\frac{c}{x-2}$
$ax(x-2)+b(x-2)+cx^2 = x^2+5x-2$
∴ $b=1,\ c=3,\ a=-2$ [1] -2 [2] 3
∴ $\int f(x)\,dx = \boxed{-2\log|x|-\frac{1}{x}+3\log|x-2|}$ (3)

3.(1) 与式 $= \int\frac{(e^x-x+1)'}{e^x-x+1}dx = \boxed{\log(e^x-x+1)}$ ∵ $e^x-x+1>0$

(2) 与式 $= \int\frac{\sin(2x-1)}{\cos(2x-1)}dx = -\frac{1}{2}\int\frac{\{\cos(2x-1)\}'}{\cos(2x-1)}dx$
$= \boxed{-\frac{1}{2}\log|\cos(2x-1)|}$

(3) 与式 $= -\int t^{1/3}dt = \boxed{-\frac{3}{4}\sqrt[3]{(\cos x+2)^4}}$
$t=\cos x+2$

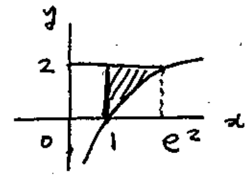
(4) 与式 $= (x+2)\log(x+2)-\int\frac{x+2}{x+2}dx = \boxed{(x+2)\log(x+2)-x}$

(5) 与式 $= x^2\sin x-2\int x\sin x\,dx$
$= x^2\sin x-2\left(-x\cos x+\int\cos x\,dx\right) = \boxed{x^2\sin x+2x\cos x-2\sin x}$

4. $y' = \boxed{\sinh\frac{x}{2}}$ (1), $1+(y')^2 = 1+\sinh^2\frac{x}{2} = \boxed{\cosh^2\frac{x}{2}}$ (2)
(2段) 4点
$\sqrt{1+(y')^2} = \sqrt{\cosh^2\frac{x}{2}} = \boxed{\cosh\frac{x}{2}}$ (3)
∴ $l = \int_0^4\cosh\frac{x}{2}dx = \left[2\sinh\frac{x}{2}\right]_0^4 = \boxed{2\sinh 2} = \boxed{e^2-e^{-2}}$ [4]

5.(1) $x' = e^t>0$ だから $S = \int_0^1|y|x'\,dt = \int_0^1(e^{2t}+1)e^t\,dt$
$= \int_0^1(e^{3t}+e^t)dt = \left[\frac{1}{3}e^{3t}+e^t\right]_0^1 = \boxed{\frac{1}{3}e^3+e-\frac{4}{3}}$

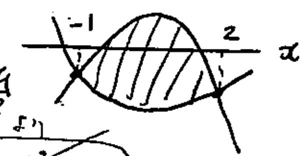
(2) [1] $S = \int_1^{e^2}(2-\log x)\,dx$
$= \left[2x-x\log x+x\right]_1^{e^2} = \boxed{e^2-3}$



[2] $V = 4\pi\cdot(e^2-1)-\pi\int_1^{e^2}(\log x)^2dx$
$= 4\pi(e^2-1)-\pi\left(\left[x(\log x)^2\right]_1^{e^2}-2\int_1^{e^2}\log x\,dx\right)$
$= 4\pi(e^2-1)-\pi(4e^2-2e^2-2) = \boxed{2\pi(e^2-1)}$

(3) $I_1 = \int\log x\,dx = \boxed{x\log x-x}$ [1]
$I_n = x(\log x)^n-n\int x(\log x)^{n-1}\cdot\frac{1}{x}dx = \boxed{x(\log x)^n}$ [2] $-$ n [3] I_{n-1}
$I_2 = x(\log x)^2-2I_1 = \boxed{x(\log x)^2-2x\log x+2x}$ [4]
$I_3 = \boxed{x(\log x)^3-3x(\log x)^2+6x\log x-6x}$ [5]

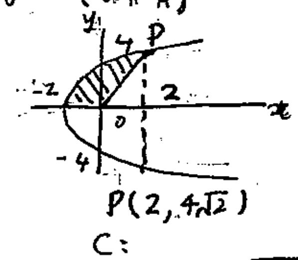
(4) 共有点 $x^2-\frac{4}{3}x-\frac{8}{3}+x^2-\frac{2}{3}x-\frac{4}{3} = 0 \to 2(x+1)(x-2)=0$
∴ $x = -1,\ 2$ [1] [2]
$-1\leqq x\leqq 2$ で $x^2-\frac{4}{3}x-\frac{8}{3}\leqq -x^2+\frac{2}{3}x+\frac{4}{3}$



$S = 2\int_{-1}^{2}(\cancel{-x^2+x+2})\,dx = 2\left[\cancel{-\frac{x^3}{3}+\frac{x^2}{2}}+2x\right]_{-1}^{2} = \boxed{9}$ [5]
$(-2x^2+2x+4)$ [3]
$-\frac{2}{3}x^3+x^2+4x$ [4]

(5) $x' = \frac{1}{2\sqrt{t}}>0\ (0<t<1)$ だから
$V = \pi\int_0^1(\sqrt{t}-t)^2\frac{1}{2\sqrt{t}}dt = \pi\int_0^1\left(\frac{1}{2}\sqrt{t}-t+\frac{1}{2}t^{3/2}\right)dt$
$= \pi\left[\frac{1}{3}t^{3/2}-\frac{t^2}{2}+\frac{1}{5}t^{5/2}\right]_0^1 = \boxed{\frac{\pi}{30}}$

6(1) 両辺に $1-\cos\theta$ をかけて
$r-r\cos\theta = 4$ $r\cos\theta = x$ より
$\sqrt{x^2+y^2} = x+4$. $x^2+y^2 = (x+4)^2$
∴ $y^2 = 8x+16$ $(x\geqq -2)$ (4点)



(2) $\alpha = \tan^{-1}2\sqrt{2}$ とおくと
$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{1}{3}$, このとき
$x = \frac{4}{1-\frac{1}{3}}\cdot\frac{1}{3} = 2$. また (1) から $y\geqq 0$ のとき $y = 2\sqrt{2}\sqrt{x+2}$
このとき $y = \sqrt{32} = 4\sqrt{2}$ だから
$S = \int_{-2}^{2}2\sqrt{2}\sqrt{x+2}\,dx-\frac{1}{2}\cdot 2\cdot 4\sqrt{2} = \frac{4\sqrt{2}}{3}\left[(x+2)^{3/2}\right]_{-2}^{2}-4\sqrt{2}$
$= \frac{20\sqrt{2}}{3}$ ($4\sqrt{2}$を引いていない: $\frac{32}{3}\sqrt{2}\to$ 4点) (6点)

7. $dx = \boxed{3\sec^2\theta}$ (1) $d\theta$, $x^2+9 = 9(1+\tan^2\theta) = 9\sec^2\theta$ $\boxed{\sec\theta}$ (2)
∴ $I = \int\frac{3\sec^2\theta}{81\sec^4\theta}d\theta = \frac{1}{27}\int\boxed{\cos^2\theta}$ (3) $d\theta$
$= \frac{1}{54}\int(1+\boxed{\cos 2\theta}$ (4) $)d\theta = \boxed{\frac{1}{54}\left(\theta+\frac{1}{2}\sin 2\theta\right)}$ (5)
ここで $\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta = \tan\theta\cos^2\theta = \frac{\tan\theta}{1+\tan^2\theta} = \frac{3x}{x^2+9}$
より $I = \frac{1}{54}\left(\boxed{\tan^{-1}\frac{x}{3}+\frac{3x}{x^2+9}}\right)$ (6)
$J = \int\frac{x^2}{(x^2+9)^2}dx = \int x\cdot\frac{x}{(x^2+9)^2}dx = \boxed{-\frac{x}{2(x^2+9)}}$ (7) $+\frac{1}{2}\int\frac{dx}{x^2+9}$ $\boxed{\frac{1}{x^2+9}}$ (8)
$= \boxed{-\frac{x}{2(x^2+9)}+\frac{1}{6}\tan^{-1}\frac{x}{3}}$ (9)
よって
$I = \int\frac{dx}{(x^2+9)^2} = \frac{1}{9}\int\frac{x^2+9-x^2}{(x^2+9)^2} = \frac{1}{9}\left(\int\frac{dx}{x^2+9}-J\right)$
$= \frac{1}{9}\left(\boxed{\frac{1}{3}\tan^{-1}\frac{x}{3}}\text{ (10)}-J\right) = \frac{1}{9}\left(\frac{x}{2(x^2+9)}+\frac{1}{6}\tan^{-1}\frac{x}{3}\right)$
$= \boxed{\frac{1}{54}\left(\tan^{-1}\frac{x}{3}+\frac{3x}{x^2+9}\right)}$

8.(1) 与式 $= x\cos^{-1}x + \int \frac{x}{\sqrt{1-x^2}}dx$ $\left((\cos^{-1}x)' = -\frac{1}{\sqrt{1-x^2}}\right)$

$= \boxed{x\cos^{-1}x - \sqrt{1-x^2}}$

(2) 与式 $= x(\cos^{-1}x)^2 - \int x\cdot 2\cos^{-1}x\cdot\left(-\frac{1}{\sqrt{1-x^2}}\right)dx$

$= x(\cos^{-1}x)^2 + 2\int \frac{x}{\sqrt{1-x^2}}\cdot\cos^{-1}x\,dx$,

$\int \frac{x}{\sqrt{1-x^2}}\cos^{-1}x\,dx = -\sqrt{1-x^2}\cos^{-1}x + \int\sqrt{1-x^2}\cdot\left(-\frac{1}{\sqrt{1-x^2}}\right)dx$

$= -\sqrt{1-x^2}\cos^{-1}x - x$

$\therefore \int(\cos^{-1}x)^2dx = x(\cos^{-1}x)^2 - 2\sqrt{1-x^2}\cos^{-1}x - 2x$ (5点)

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9. $a_n = \frac{1}{n}\left(\frac{(3n)!}{(2n)!}\right)^{1/n}$ とおく. $b_1\cdots b_n = \prod_{k=1}^{n} b_k$ と表わせば 各 $b_k>0$

$\Rightarrow \log\prod_{k=1}^{n} b_k = \sum_{k=1}^{n}\log b_k$ より

$\log a_n = \log\left(\frac{(3n)!}{n^n(2n)!}\right)^{1/n} = \frac{1}{n}\log\frac{1}{n^n}(2n+1)(2n+2)\cdots(2n+n)$

$= \frac{1}{n}\log\prod_{k=1}^{n}\left(2+\frac{k}{n}\right) = \frac{1}{n}\sum_{k=1}^{n}\log\left(2+\frac{k}{n}\right)$

$\therefore \lim_{n\to\infty}\log a_n = \int_0^1\log(2+x)\,dx$

$= \left[(2+x)\log(2+x) - x\right]_0^1 = 3\log3 - 1 - 2\log2$

$= \log\frac{3^3}{2^2e}$ $\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\log a_n} = \frac{27}{4e}$ (7点)

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10. $x' = -3a\sin\theta\cos^2\theta$, $y' = 3a\sin^2\theta\cos\theta$ だから

$S = \frac12\int_0^{2\pi}(xy' - x'y)\,d\theta$

$= \frac12\int_0^{2\pi}(3a^2\sin^2\theta\cos^4\theta + 3a^2\sin^4\theta\cos^2\theta)\,d\theta$

$= \frac32 a^2\int_0^{2\pi}\sin^2\theta\cos^2\theta\,d\theta = \frac38 a^2\int_0^{2\pi}\sin^2 2\theta\,d\theta$

$= \frac{3}{16}a^2\int_0^{2\pi}(1-\cos4\theta)\,d\theta = \frac{3}{16}a^2\left[\theta - \frac14\sin4\theta\right]_0^{2\pi} = \frac38\pi a^2$

(5点)

<別解>

6.(2) $r = \frac{4}{1-\cos\theta} = \frac{2}{\sin^2\frac{\theta}{2}}$ $\left(\frac12(1-\cos\theta) = \sin^2\frac{\theta}{2}\right)$

$\therefore S = \frac12\int_\alpha^\pi r^2d\theta = 2\int_\alpha^\pi \mathrm{cosec}^4\frac{\theta}{2}\,d\theta = 4\int_\beta^{\pi/2}\mathrm{cosec}^4u\,du$

$u = \frac{\theta}{2},\ \beta = \frac{\alpha}{2}$

$= 4\int_\beta^{\pi/2}\mathrm{cosec}^2u\cdot\mathrm{cosec}^2u\,du = 4\int_\beta^{\pi/2}(1+\cot^2u)\,\mathrm{cosec}^2u\,du$

$= -4\int_c^0(1+v^2)\,dv = 4\left[v + \frac13v^3\right]_0^c = 4\left(c + \frac{c^3}{3}\right)\cdots①$

$v = \cot u,\ c = \cot\beta = \cot\frac{\alpha}{2}$

ここで $\tan\alpha = 2\sqrt2$ から $\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac13$ $\left(0<\alpha<\frac{\pi}{2}\right)$

$\therefore \tan\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1}{\sqrt2}$ $\therefore c = \cot\frac{\alpha}{2} = \sqrt2$

これを①に代入して $S = \frac{20}{3}\sqrt2$

8.(2) $t = \cos^{-1}x$ と置換する $(0\le t\le\pi \ \therefore \sin t\ge0)$

$x = \cos t$ から $dx = -\sin t\,dt$

$\therefore \int(\cos^{-1}x)^2dx = -\int t^2\sin t\,dt = t^2\cos t - 2\int t\cos t\,dt$

$= t^2\cos t - 2\left(t\sin t - \int\sin t\,dt\right) = t^2\cos t - 2t\sin t - 2\cos t$

ここで $\sin t = \sqrt{1-\cos^2t} = \sqrt{1-x^2}$ となる

与式 $= x(\cos^{-1}x)^2 - 2\sqrt{1-x^2}\cos^{-1}x - 2x$

10. 対称性から $0\le\theta\le\frac{\pi}{2}$ の部分と x軸および y軸で囲まれた部分の面積を4倍する. $0<\theta<\frac{\pi}{2}$ で $x' = -3a\sin\theta\cos^2\theta < 0$, $y>0$ より

$S = 4\int_0^{\pi/2}a\sin^3\theta\cdot3a\sin\theta\cos^2\theta\,d\theta = 12a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta$

$= 12a^2\int_0^{\pi/2}(\sin^4\theta - \sin^6\theta)\,d\theta = 12a^2\left(\frac34\cdot\frac{\pi}{4} - \frac56\cdot\frac34\cdot\frac{\pi}{4}\right)$

$\cos^2\theta = 1 - \sin^2\theta$

$= \frac38\pi a^2$

<その他>

3 (1) $e^x - x + 1 > 0$ は気づきにくいと思うので今回は絶対値がついていても減点しなかった.

3 (4) $t = x+2$ と置換すると

与式 $= \int\log t\,dt = t\log t - t$

$= \boxed{(x+2)\log(x+2) - x - 2}$

これで正解

4. (2) は $1+\sinh^2\frac{x}{2}$ ではだめ.

$1 + ((1))^2 = (2)$ となっているから (2) が $1+\sinh^2\frac{x}{2}$ ではおかしい、わざわざ (2) を出してきた理由を考える.

7 (6) $\tan^{-1}\frac{x}{3} + \frac12\sin\left(2\tan^{-1}\frac{x}{2}\right)$ は 1点とした.