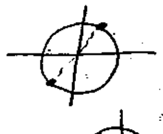
ES4 後中 2017

2016年度試験答案用紙

番号　　氏名

1.(1) [1] $1+\sqrt{3}i = 2e^{i\theta}$, $\theta = \frac{\pi}{3} + 2k\pi$



$\therefore \sqrt{1+\sqrt{3}i} = \pm\sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$ 　 $\pm\frac{\sqrt{2}}{2}(\sqrt{3}+i)$

[2] $-i = e^{i\theta}$, $\theta = \frac{3}{2}\pi + 2k\pi$

$\therefore \sqrt{-i} = \pm\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \pm\frac{1}{\sqrt{2}}(1-i)$

[3] $\log(1-i) = \log\sqrt{2} + i\left(2k + \frac{7}{4}\right)\pi$ 　$(k \in \mathbb{Z})$

[4] $\mathrm{Log}(\sqrt{3}-i) = \log 2 - i\frac{\pi}{6}$

(2) $z = -i + \frac{1-i}{2} = \frac{1}{2} - \frac{3}{2}i$ より

[1] $\frac{1}{2}$ 　[2] $-\frac{3}{2}$ 　[3] $\frac{\sqrt{10}}{2}$ 　[4] $\frac{1}{2} + \frac{3}{2}i$

(3) [1] $1+i = \sqrt{2}e^{i\theta}$, $\theta = \frac{\pi}{4}$ より

$\frac{1}{4}(1+i)^4 = \cancel{e^{i(\pi+2k\pi)}}$ 　$\cancel{(k \in \mathbb{Z})}$ 　$e^{i\pi} = \cos\pi + i\sin\pi$

[2] $\sqrt{3} - i = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 2e^{i\frac{11}{6}\pi}$

$= 2\left(\cos\frac{11}{6}\pi + i\sin\frac{11}{6}\pi\right)$ 　$e^{i\pi}$, -1 は不可

$2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$ は✕ 理由は

(4) [1] $w' = 6(z^2 - iz - 1)^5(2z - i)$

[2] $w' = -\frac{2}{(i-z)^3}\cdot(-1) = \frac{2}{(i-z)^3} = \frac{-2}{(z-i)^3}$ 　2項目

[3] $w' = \frac{z-i-z}{(z-i)^2} = \frac{-i}{(z-i)^2}$

(5) [1] 与式 $= (1+i+1-i)^2 = 4$

[2] 与式 $= \frac{(2+i)^2}{i} = \frac{3+4i}{i} = 4-3i$

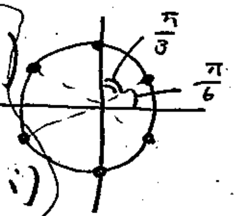
[3] 与式 $= \lim_{z\to i}\frac{(z+i)(z-i)}{(z+i)(2z-i)} = \lim_{z\to i}\frac{z-i}{2z-i} = \frac{-2i}{-3i}$

$= \frac{2}{3}$

(6) $z^6 = e^{i(\pi+2k\pi)}$ 　$\therefore z = e^{i\frac{\pi}{6}}e^{i\frac{k\pi}{3}}$ 　$(k = 0, 1, 2, \cdots, 5)$

$\therefore z = \pm i$, $\pm\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$, $\pm\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$\pm\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$, $\pm\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

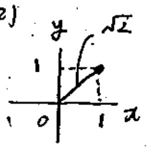


$\mathrm{Im}\, z < 0$ となるのは

$-i$, $\frac{-\sqrt{3}-i}{2}$, $\frac{\sqrt{3}-i}{2}$

2 (1) $\cosh\frac{i\pi}{3} = \frac{1}{2}\left(e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{3}}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$ (1)

(2) $\sinh\frac{i\pi}{3} = \frac{1}{2}\left(e^{i\frac{\pi}{3}} - e^{-i\frac{\pi}{3}}\right) = i\sin\frac{\pi}{3} = i\frac{\sqrt{3}}{2}$ (2)



$\log(1+i) = \log\sqrt{2}$ (3) $+ i\left(\frac{\pi}{4} + 2k\pi\right)$ (4) $(k \in \mathbb{Z})$

$(1+i)^i = \exp\{i\log(1+i)\} = \exp\left(-\frac{\pi}{4} - 2k\pi\right)$ (5) $+ i\log\sqrt{2}$ (6)

$= e^{-\frac{\pi}{4}-2k\pi}\left(\cos(\log\sqrt{2})\right.$ (7) $\left. + i\sin(\log\sqrt{2})\right)$ (8) 　4段

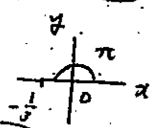
3 (1) $\tan w = z$ を解く. $X = e^{iw}$ と表せば $\tan w = \frac{\frac{1}{2i}\left(X - \frac{1}{X}\right)}{\frac{1}{2}\left(X + \frac{1}{X}\right)}$

$\tan w = \frac{1}{i}\frac{X^2-1}{X^2+1}$ なので $X^2 - 1 = iz(X^2+1)$ $\therefore X^2 = \frac{1+iz}{1-iz}$

もとに戻して $e^{i2w} = \frac{1+iz}{1-iz}$ 　$\therefore w = \frac{1}{2i}\log\frac{1+iz}{1-iz}$

定義式で $1 \pm iz \neq 0$ より $z \neq \pm i$ 　4行

(2) $\tan^{-1}(2i) = \frac{1}{2i}\log\frac{1-2}{1+2} = \frac{1}{2i}\log\left(-\frac{1}{3}\right)$



$= \frac{1}{2i}\{-\log 3 + i(2k\pi + \pi)\} = k\pi + \frac{\pi}{2} + \frac{i}{2}\log 3$ 　$(k \in \mathbb{Z})$

4 $f(z)$ の実部, 虚部を それぞれ u, v と表す.

(1) $u = -y$, $v = x$ なので $u_x = 0$, $v_y = 0$, $u_y = -1$, $v_x = 1$

$\therefore u_x = v_y$, $u_y = -v_x$ であり 明らかに u, v は全微分可能 C^1級なので

なので 正則. $f'(z) = u_x + iv_x = i$ 　3行

(2) $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$ なので $u_x = 3x^2 - 3y^2$,

$v_y = 3x^2 - 3y^2$, $u_y = -6xy$, $v_x = 6xy$ 　C^1級なので

$\therefore u_x = v_y$, $u_y = -v_x$ であり 明らかに u, v は全微分可能なので

正則. $g'(z) = u_x + iv_x = 3x^2 - 3y^2 + 6xyi$ 　4行

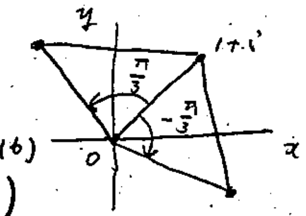
4 別解

(1) $f(z) = iz$ なので 正則で $f'(z) = i$

(2) $g(z) = z^3$ なので 正則で $g'(z) = 3z^2$

5 $1+i = \sqrt{2}$ (1) $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ (2) $= \sqrt{2}\exp\left(i\frac{\pi}{4}\right)$ (3)

$\therefore z = \sqrt{2}$ (4) $\exp\left(i\frac{\pi}{4} \pm i\frac{\pi}{3}\right)$ (5)



即ち $z = \sqrt{2}\exp\left(-i\frac{\pi}{12}\right)$

$= \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right)$ (6)

$= \sqrt{2}\left(\frac{\sqrt{3}+1}{2\sqrt{2}} - i\frac{\sqrt{3}-1}{2\sqrt{2}}\right) = \frac{\sqrt{3}+1}{2} - i\frac{\sqrt{3}-1}{2}$ (7)

$z = \sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$ (8) $= \frac{1-\sqrt{3}}{2} + i\frac{\sqrt{3}+1}{2}$ (9)

((7), (8) $\frac{\pi}{4} - \frac{\pi}{3}$, $\frac{\pi}{4} + \frac{\pi}{3}$ として加法定理を用いる)

(1) $\sqrt{2}$ 　(2) $\frac{\pi}{4}$ 　(3) $i\frac{\pi}{4}$ 　(4) $\sqrt{2}$ 　(5) $i\frac{\pi}{3}$ 　6段

(6) $-\frac{\pi}{12}$ 　(8) $\frac{7\pi}{12}$

6 (1) $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{z^2-1}{2zi}$

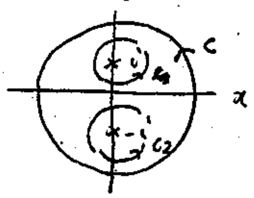
(2) $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$

(3) 与式 $= \frac{-i\left(z - \frac{1}{z}\right)}{1 - \left(z + \frac{1}{z}\right)} = \frac{-i(z^2-1)}{z - z^2 - 1}$ 　2段

7 (1) $\frac{a}{z-i} + \frac{b}{z+i} = \frac{(a+b)z + i(a-b)}{z^2+1}$ なので $\begin{cases} a+b = 3 \\ a-b = 1 \end{cases}$

$\therefore a = 2$ (1), $b = 1$ (2)

(2) $\int_C f(z)\,dz = 2\int_C \frac{1}{z-i}\,dz + \int_C \frac{1}{z+i}\,dz$



$= 2\cdot 2\pi i + 2\pi i = 6\pi i$ 　2段

8 (1) $\int_C z^3\,dz = \int_0^{2i} z^3\,dz = \left[\frac{1}{4}z^4\right]_0^{2i} = \frac{1}{4}(2i)^4 = 4$

証明 $z\left(-\frac{\pi}{2}\right) = i - i = 0$, $z\left(\frac{\pi}{2}\right) = i + i = 2i$)

(2) $z(0) = \pi$, $z(\pi/2) = i$ より

$\int_C \cos z\,dz = [\sin z]_\pi^i = \sin i = \frac{1}{2i}(e^{-1} - e) = i\sinh 1$ 　$\frac{i}{2}(e - e^{-1})$

1点

2段

9. (1) $w = \dfrac{1}{\overline{9-2i}} = \dfrac{1}{9+2i} = \boxed{\dfrac{9}{85} - \dfrac{2}{85}i}$

(2) $w = \dfrac{1}{\bar{z}}$ から z を消くと $z = \dfrac{1}{\bar{w}} = \dfrac{w}{|w|^2}$

$z = \dfrac{u}{u^2+v^2} + i\dfrac{v}{u^2+v^2}$ ∴ [1] $x = \boxed{\dfrac{u}{u^2+v^2}}$ [2] $y = \boxed{\dfrac{v}{u^2+v^2}}$

(3) $z-3i = x+(y-3)i$ だから $|z-3i|^2 = 4$ の $\boxed{x^2+(y-3)^2}$ [1] $= 4$ ①

$x^2+y^2-6y+9=4$ で $x^2+y^2 = \boxed{\dfrac{1}{u^2+v^2}}$ [2] だから

① は $\dfrac{1}{u^2+v^2} - \dfrac{6v}{u^2+v^2} + 5 = 0$ ∴ $\boxed{5}$ [3] $(u^2+v^2) - \boxed{6}$ [4] $v + 1 = 0$

$u^2+v^2-\dfrac{6}{5}v = -\dfrac{1}{5}$, $u^2+\left(v-\boxed{\dfrac{3}{5}}\right)^2 = \boxed{\dfrac{4}{25}}$ [6]

従って中心は点 $\dfrac{3}{5}i$ [7] 半径 $\boxed{\dfrac{2}{5}}$ [8] [5] の円になる

(8行) → 6段

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10 $x = \boxed{\dfrac{1}{2}}$ [1] $(z+\bar{z})$, $y = \boxed{-\dfrac{i}{2}}$ [2] $(z-\bar{z})$ だから $ax+by+c=0$

に代入すると $\dfrac{a}{2}(z+\bar{z}) - \dfrac{ib}{2}(z-\bar{z}) + c = 0$

$\left(\dfrac{1}{2}(a-bi)\right)$ [3] $z + \left(\dfrac{1}{2}(a+bi)\right)$ [4] $\bar{z} + c = 0$

$\alpha = \dfrac{1}{2}(a-bi)$ とすれば $\dfrac{1}{2}(a+bi) = \boxed{\bar{\alpha}}$ (5)

$x+2y+6$ は $a=1, b=2$ $(c=6)$ なので $\alpha = \dfrac{1}{2}(1-2i)$ より

$(\boxed{1-2i}$ [6]$)z + (\boxed{1+2i}$ [7]$)\bar{z} + 12 = 0$ となる

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11. $C_1 : z = t$ $(0 \leqq t \leqq 1)$, $C_2 : z = 1+ti$ $(0 \leqq t \leqq 1)$

$C_3 : z = (1-t)+(1-t)i$ $(0 \leqq t \leqq 1)$ とすれば $C = C_1+C_2+C_3$

で C_1 上で $z'=1$, $Im(z)=0$, C_2 上で $z'=i$, $Im(z)=t$

C_3 上で $z'=-(1+i)$, $Im(z)=1-t$ だから

$\displaystyle\int_C Im(z)dz = i\int_0^1 t\,dt - (1+i)\int_0^1 (1-t)\,dt$

$= i\left[\dfrac{t^2}{2}\right]_0^1 - (1+i)\left[t-\dfrac{t^2}{2}\right]_0^1 = -\dfrac{1}{2}$

(6行)

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9. $w = \dfrac{1}{\bar{z}} = \dfrac{z}{z\bar{z}} = \dfrac{z}{|z|^2}$ である

(3) [1] $x^2+(y-3)^2$ (展開してもよい)

[2] $\dfrac{1}{u^2+v^2}$ [3] 5 [4] 6

[5] $\dfrac{3}{5}$ [6] $\dfrac{4}{25}$ または $\left(\dfrac{2}{5}\right)^2$

[7] $\dfrac{3}{5}i$ [8] $\dfrac{2}{5}$, $\sqrt{\dfrac{4}{25}}$ は ×

→ 9(3) [7] は <u>数値と書いてあるから</u> $(0, \frac{3}{5})$ は ×

→ [1] $\dfrac{1}{2}$ [2] $-\dfrac{i}{2}$ または $\dfrac{1}{2i}$

(3) $\dfrac{1}{2}(a-bi)$ (4) $\dfrac{1}{2}(a+bi)$ → $\dfrac{a}{2}-\dfrac{b}{2}i$, $\dfrac{a}{2}+\dfrac{b}{2}i$ は ×

(5) $\bar{\alpha}$ [6] $1-2i$ [7] $1+2i$

☆ 1 (3) [2]

$2\left(\cos\dfrac{\pi}{6} - i\sin\dfrac{\pi}{6}\right)$

は

$2\left\{\cos\left(-\dfrac{\pi}{6}\right) + i\sin\left(-\dfrac{\pi}{6}\right)\right\}$

なので $\arg z$ の指定にあっていない。

極形式とは

$z = r(\cos\theta + i\sin\theta)$, $r = |z|$, $\theta = \arg z$