ES4 前末. 2017

2016年度試験答案用紙　　番号　　氏名

1. $XT' = X''T$ より $\dfrac{T'}{T} = \boxed{\dfrac{X''}{X}}$ (1) $(=\lambda$ とおく) λ は定数 これから

$T' - \lambda T = 0$ ……④ , $\boxed{X'' - \lambda X}$ (2) $= 0$ ……⑤ $\lambda X - X''$ でも可

⑤ の一般解 (A, B : 任意定数)

$\lambda > 0 \Rightarrow X(x) = A\boxed{e^{\sqrt{\lambda}x}}$ (3) $+ B\boxed{e^{-\sqrt{\lambda}x}}$ (4) ∵ $e^{\sqrt{\lambda}x} \gtrless e^{-\sqrt{\lambda}x}$ ……⑥

$\lambda = 0 \Rightarrow X(x) = \boxed{Ax}$ (5) $+ B$ ……⑦

$\lambda < 0 \Rightarrow X(x) = A\cos\boxed{\sqrt{-\lambda}x} + B\sin\boxed{\sqrt{-\lambda}x}$ (6) ……⑧

条件 ② と恒等的に 0 でないから $X(0) = X(1) = 0$ となり これを満たすように A, B を求めると $\lambda > 0$, $\lambda = 0$ のときは $A = B = 0$ になってしまう.

$\lambda < 0$ のとき ⑧ より $X(0) = \boxed{A} = 0$, $X(1) = \boxed{A\cos\sqrt{-\lambda} + B\sin\sqrt{-\lambda}}$ (7) $= 0$

∴ $B \neq 0$, $\sin\boxed{\sqrt{-\lambda}}$ (8) $= 0$ ∴ $\sqrt{-\lambda} = m\pi$

$\lambda = \boxed{-m^2\pi^2}$ (9) ∴ $X(x) = B\sin\boxed{n\pi x}$ (10) $(n \in \mathbb{N})$

このとき ④ の一般解は $T(t) = Ce^{\lambda t} = C\boxed{e^{-m^2\pi^2 t}}$ (11) $(B \neq 0)$

(C は任意定数) 従って

$u_n(x,t) = \boxed{e^{-n^2\pi^2 t}}$ (11) $\sin\boxed{n\pi x}$ (10) は条件 ② を満たす ① の解

→ $u(x,t) = \sum_{n=1}^{\infty} C_n u_n(x,t)$ ……⑩ (C_n は任意定数)

これと条件 ③ より $C_n = \begin{cases} \boxed{0}\ (12) & (n = 2k) \\ \dfrac{1}{2k-1}\ (13) & (n = 2k-1) \end{cases}$

以上より ②, ③ を満たす ① の解は

$$u(x,t) = \boxed{\sum_{n=1}^{\infty} \frac{e^{-(2m-1)^2\pi^2 t}}{2m-1} \sin(2m-1)\pi x} \ (14)$$

$\sum_{n=1}^{\infty} \dfrac{1-(-1)^m}{2m} e^{-m^2\pi^2 t} \sin n\pi x$ でもOK.

(1) $\dfrac{X''}{X}$ (2) $X'' - \lambda X$ または $\lambda X - X''$

2. (1) $C_0 + \sum_{n=1}^{\infty}\left(a_m \cos\boxed{\dfrac{n\pi}{2}}x + b_m \sin\boxed{\dfrac{n\pi}{2}}x\right)$ [1]

$C_0 = \boxed{\dfrac{1}{4}}$ [2] $\displaystyle\int_{-2}^{2} f(x)dx = \frac{1}{4}\int_{-2}^{0}(-1)dx + \frac{1}{4}\int_{0}^{2} x\,dx = \frac{1}{4}\left([-x]_{-2}^{0} + \left[\frac{1}{2}x^2\right]_0^2\right)$

$= \dfrac{1}{4}(-2+2) = \boxed{0}$ [3]

$a_m = \boxed{\dfrac{1}{2}}$ [4] $\displaystyle\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\,dx = \frac{1}{2}\int_{-2}^{0}(-1)\cos\frac{n\pi}{2}x\,dx + \frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx$

$\displaystyle\int_{-2}^{0}\left(-\cos\frac{n\pi x}{2}\right)dx = \left[-\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]_{-2}^{0} = \boxed{0}$ [5]

$\displaystyle\int_0^2 x\cos\frac{n\pi x}{2}dx = \left[\frac{2}{n\pi}x\sin\frac{n\pi x}{2}\right]_0^2 - \frac{2}{n\pi}\int_0^2 \sin\frac{n\pi x}{2}dx$

$= \dfrac{4}{n^2\pi^2}\left[\cos\dfrac{n\pi x}{2}\right]_0^2 = \boxed{\dfrac{4}{n^2\pi^2}((-1)^m - 1)}$ [6]

∴ $a_m = \boxed{\dfrac{2}{n^2\pi^2}((-1)^m - 1)}$ [7]

$b_m = \dfrac{1}{2}\left(\displaystyle\int_{-2}^{0}(-1)\sin\frac{n\pi x}{2}dx + \int_0^2 x\sin\frac{n\pi x}{2}dx\right)$

$= \dfrac{1}{2}\left(\left[\dfrac{2}{n\pi}\cos\dfrac{n\pi x}{2}\right]_{-2}^{0} + \left[-\dfrac{2}{n\pi}x\cos\dfrac{n\pi x}{2}\right]_0^2 + \dfrac{2}{n\pi}\displaystyle\int_0^2\cos\frac{n\pi x}{2}dx\right)$

$= \dfrac{1}{2}\left(\boxed{\dfrac{2}{n\pi}(1-(-1)^m)}\ [8]\ \boxed{-\dfrac{4}{n\pi}(-1)^m}\ [9]\right) = \boxed{\dfrac{1}{n\pi}(1-3(-1)^m)}$ [10]

∴ $\boxed{\displaystyle\sum_{n=1}^{\infty}\left\{\frac{2((-1)^m-1)}{n^2\pi^2}\cos\frac{n\pi x}{2} + \frac{1-3(-1)^m}{m\pi}\sin\frac{n\pi x}{2}\right\}}$ [11]

(3) (1) で求めた Fourier 級数で $x=0$ を代入すると, 収束定理より

$\displaystyle\sum_{n=1}^{\infty}\frac{2((-1)^m-1)}{m^2\pi^2} = \frac{1}{2}\{f(+0)+f(-0)\} = -\frac{1}{2}$

$\displaystyle\sum_{n=1}^{\infty}\frac{-4}{(2k-1)^2\pi^2} = -\frac{1}{2}$ ∴ $\displaystyle\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2} = \sum_{n=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$ ③

(2) $C_m = \dfrac{1}{2}(a_m - ib_m)$ $(m \neq 0)$ だから $C_0 = 0$ (1) より

$C_m = \dfrac{1}{n^2\pi^2}((-1)^m - 1) - \dfrac{i}{2n\pi}(1-3(-1)^m)$ $(m \neq 0)$

∴ $\displaystyle\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left\{\frac{1-3(-1)^m}{2n\pi i} - \frac{1-(-1)^m}{n^2\pi^2}\right\}e^{i\frac{n\pi x}{2}}$ ③

3 (1) $F(u) = \displaystyle\int_{-\infty}^{0} e^{x}e^{-iux}dx = \int_{-\infty}^{0} e^{(1-iu)x}dx$

$= \left[\dfrac{1}{1-iu}e^{(1-iu)x}\right]_{-\infty}^{0} = \boxed{\dfrac{1}{1-iu}}$ ∵ $|e^{(1-iu)x}| = e^{x} \to 0$ $(x \to \infty)$

(2) $F(u) = \displaystyle\int_a^b e^{-iux}dx = \left[\frac{i}{u}e^{-iux}\right]_a^b$

$= \boxed{\dfrac{i}{u}(e^{-ibu} - e^{-iau})}$

(3) 公式より $F(u) = \boxed{\sqrt{\pi a}\,\exp\left(-\dfrac{au^2}{4}\right)}$

(4) $g(x) = e^{-2x^2}$, $G(u) = \mathfrak{F}[g(x)]$ とおけば $g'(x) = -4xe^{-2x^2}$

∴ $f(x) = -\dfrac{1}{4}g'(x)$ ∴ $F(u) = -\dfrac{1}{4}\mathfrak{F}[g'(x)] = -\dfrac{iu}{4}G(u)$

$= \boxed{-\dfrac{iu}{4}\sqrt{\dfrac{\pi}{2}}e^{-\frac{u^2}{8}}}$

4. (1) $\mathfrak{F}[f_{xx}] = (iu)^2F(u,t) = -u^2F(u,t)$ より

$\boxed{\dfrac{\partial F}{\partial t} = -u^2F(u,t)}$

(2) $\dfrac{1}{F}\dfrac{\partial F}{\partial t} = -u^2$ だから $\log|F| = -\displaystyle\int u^2dt + c(u)$

∴ $F = \Phi(u)e^{-tu^2}$

ここで $\Phi(u) = F(u,0) = \mathfrak{F}[f(x,0)] = \mathfrak{F}[e^{-x^2}]$

$= \sqrt{\pi}e^{-\frac{u^2}{4}}$ だから $\boxed{F = \sqrt{\pi}e^{-\frac{u^2}{4}}e^{-tu^2}}$ ④

(3) $F = \sqrt{\pi}e^{-\left(\frac{1}{4}+t\right)u^2} = \sqrt{\pi}e^{-\frac{4t+1}{4}u^2}$

一方 $\mathfrak{F}\left[e^{-\frac{x^2}{4t+1}}\right] = \sqrt{\pi(4t+1)}\,e^{-\frac{4t+1}{4}u^2}$

∴ $\mathfrak{F}^{-1}[F(u,t)] = \dfrac{1}{\sqrt{4t+1}}e^{-\frac{x^2}{4t+1}}$ ③

奇関数なので 正弦変換 $S(u)$ は

5 (1) $S(u) = 2\displaystyle\int_0^2\left(-\frac{x}{2}\right)\sin ux\,dx = \left[\frac{x}{u}\cos ux\right]_0^2 - \frac{1}{u}\int_0^2\cos ux\,dx$

$= \dfrac{2}{u}\cos 2u - \dfrac{1}{u^2}[\sin ux]_0^2 = \boxed{\dfrac{2u\cos 2u - \sin 2u}{u^2}}$

(2) $F(u) = -iS(u)$ より $F(u) = \boxed{\dfrac{i(\sin 2u - 2u\cos 2u)}{u^2}}$

$= \boxed{\dfrac{1-2ui}{2u^2}e^{i2u} - \dfrac{1+2ui}{2u^2}e^{-i2u}}$

6.(1) [1] $C_f(u) = 2\int_0^\infty (e^{-x}\cos x)\cos ux\,dx$

$= \int_0^\infty e^{-x}\{\cos(x+ux)+\cos(x-ux)\}\,dx$

積を和・差に直す公式

$= \int_0^\infty e^{-x}\cos(u+1)x\,dx + \int_0^\infty e^{-x}\cos(u-1)x\,dx$ 2

$= \left[\frac{e^{-x}}{1+(u+1)^2}\{-\cos(u+1)x+(u+1)\sin(u+1)x\}\right]_0^\infty$

$+ \left[\frac{e^{-x}}{1+(u-1)^2}\{-\cos(u-1)x+(u-1)\sin(u-1)x\}\right]_0^\infty$ 5

$= \frac{1}{1+(u+1)^2}+\frac{1}{1+(u-1)^2} = \frac{2u^2+4}{\{1+(u+1)^2\}\{1+(u-1)^2\}}$ 8

$= \frac{2u^2+4}{u^4+4}$ ← 加点 +1

8

[2] $F(u) = C_f(u) = \boxed{\frac{2u^2+4}{u^4+4}}$

(2) [1] $S_g(u) = 2\int_0^\infty e^{-x}\sin x\sin ux\,dx$

$= \int_0^\infty e^{-x}\{\cos(x-ux)-\cos(x+ux)\}\,dx$

$= \int_0^\infty e^{-x}\cos(u-1)x\,dx - \int_0^\infty e^{-x}\cos(u+1)x\,dx$ 2

$= \left[\frac{e^{-x}}{1+(u-1)^2}\{-\cos(u-1)x+(u-1)\sin(u-1)x\}\right]_0^\infty$

$- \left[\frac{e^{-x}}{1+(u+1)^2}\{-\cos(u+1)x+(u+1)\sin(u+1)x\}\right]_0^\infty$ 5

$= \frac{1}{1+(u-1)^2}-\frac{1}{1+(u+1)^2} = \frac{4u}{\{1+(u-1)^2\}\{1+(u+1)^2\}}$ 8

$= \frac{4u}{u^4+4}$ ← 加点 +1

8

[2] $G(u) = -iS_g(u) = \boxed{\frac{-i4u}{u^4+4}} = \frac{4u}{i(u^4+4)}$

$-i\left\{\frac{1}{1+(u-1)^2}-\frac{1}{1+(u+1)^2}\right\}$

ちらも 正解