E3 後中 2018

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(2)[1] $e^{\frac{x}{2}}\left(C_1\cos\frac{\sqrt{7}}{2}x + C_2\sin\frac{\sqrt{7}}{2}x\right)$

1. (1) $y' = \frac{1}{x+C}$, $y = \log|x+C|$ から $x+C = e^y$

∴ $y' = e^{-y}$

(2) 特性方程式 $\lambda^2 - \lambda + 2 = 0$　~~$(\lambda-2)(\lambda+1)=0$, $\lambda = -1, 2$~~　$\lambda = \frac{1\pm\sqrt{7}i}{2}$

~~$y = C_1e^{-x} + C_2e^{2x}$~~　[1] $e^{-x} = e^{\frac{x}{2}}e^{-\frac{3}{2}x}$, $e^{2x} = e^{\frac{x}{2}}e^{\frac{3}{2}x}$ より

~~$y = e^{\frac{x}{2}}\left(C_1\cosh\frac{3}{2}x + C_2\sinh\frac{3}{2}x\right)$~~　[2] $\frac{x}{2}$　[3] $\frac{3}{2}x$

$\eta = Ax^2 + Bx + C$, $\eta' = 2Ax + B$, $\eta'' = 2A$

∴ $2A - 2Ax - B + 2Ax^2 + 2Bx + 2C = 4x^2$

∴ $A = 2$, $B = 2$, $C = -1$　[4] 2　[5] 2　[6] −1

① の一般解 $y = $ ~~$C_1e^{-x} + C_2e^{2x}$~~ $+ 2x^2 + 2x - 1$

[7] $e^{\frac{x}{2}}\left(C_1\cos\frac{\sqrt{7}}{2}x + C_2\sin\frac{\sqrt{7}}{2}x\right) + 2x^2 + 2x - 1$

(3) [1] $\frac{1}{y}y' = \frac{2x}{1-x^2}$, $\log|y| = -\log(1-x^2) + C$

∴ $y = \frac{C}{1-x^2}$ $\left(\frac{C}{x^2-1}\text{ でもOK}\right)$

[2] $y(0) = C = 1$ より $y = \frac{1}{1-x^2}$

(4) [1] $y' = \frac{y}{x} - \frac{2x}{y}$, $u = \frac{y}{x}$ とおく. $y = xu$ より

$xu' + u = u - \frac{2}{u}$　∴ $uu' = -\frac{2}{x}$　$\frac{1}{2}u^2 = -2\log|x| + C$.

∴ $y^2 = x^2(C - 4\log|x|)$

[2] $y^2(1) = C = 1$ より $y^2 = x^2(1 - 4\log|x|)$

(5) [1] 特性方程式 $\lambda^2 + 2\lambda + 2 = 0$　∴ $\lambda = -1 \pm i$

∴ $y = e^{-x}(C_1\cos x + C_2\sin x)$

[2] 同 $\lambda^2 - 2\lambda + 1 = (\lambda-1)^2 = 0$　$\lambda = 1$ (重解)

∴ $y = (C_1 + C_2x)e^x$

[3] $\lambda^2 - 4\lambda + 1 = 0$　∴ $\lambda = 2 \pm\sqrt{4-1} = 2\pm\sqrt{3}$

$y = C_1e^{(2+\sqrt{3})x} + C_2e^{(2-\sqrt{3})x}$　又は

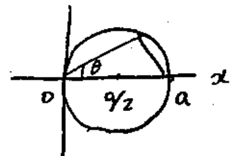
$y = e^{2x}(C_1\cosh\sqrt{3}x + C_2\sinh\sqrt{3}x)$

(6)

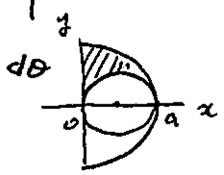
$W(\log x, x\log x) = \begin{vmatrix}\log x & x\log x\\ 1/x & 1+\log x\end{vmatrix} = (\log x)^2$

2. (1) 与式 $= \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} r^2\cos\theta\, dr\, d\theta$

$= \frac{2}{3}a^3\int_0^{\pi/2}\cos^4\theta\, d\theta = \frac{2}{3}a^3\cdot\frac{3}{4}\cdot\frac{\pi}{4} = \frac{\pi}{8}a^3$



(2) 与式 $= \int_0^{\pi/2}\int_{a\cos\theta}^{a} r^2\, dr\, d\theta = \frac{a^3}{3}\int_0^{\pi/2}(1-\cos^3\theta)\, d\theta$



$= \frac{a^3}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right) = \frac{a^3}{18}(3\pi - 4)$

(3) [1] 与式 $= \int_0^{2\pi}\int_0^a r\sqrt{a^2-r^2}\, dr\, d\theta = 2\pi\left[-\frac{1}{3}(a^2-r^2)^{3/2}\right]_0^a$

$= \frac{2}{3}\pi a^3$

[2] 与式 $= \int_0^{2\pi}\int_0^a r^3\cos^2\theta\, d\theta = \frac{a^4}{4}\int_0^{2\pi}\frac{1}{2}(1+\cos2\theta)\, d\theta$

$= \frac{a^4}{8}\left[\theta + \frac{1}{2}\sin2\theta\right]_0^{2\pi} = \frac{\pi}{4}a^4$

[3] 与式 $= \int_0^{2\pi}\int_0^a r^3\, dr\, d\theta = \frac{\pi}{2}a^4$

(5点)

3. $K \to \hat{K}$: $0\le u\le a$, $0\le v\le 1$, $0\le w\le 1$ — (1)

$J = u^2v$ (2)　∴ $I = \int_0^a\int_0^1\int_0^1 u^5v^2(1-v)^2(1-w)\, dw\, dv\, du$ (3)(4)

$= \left(\int_0^a u^5\, du\right)\left(\int_0^1 v^2(1-v)^2\, dv\right)\left(\int_0^1 (1-w)\, dw\right)$ (5)(6)(7)

$= \left[\frac{1}{6}u^6\right]_0^a\cdot\left[\frac{1}{3}v^3 - \frac{1}{2}v^4 + \frac{1}{5}v^5\right]_0^1\left[w - \frac{1}{2}w^2\right]_0^1 = \frac{a^6}{360}$ (8)

(5点)

4 (1) $\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$= \frac{15}{8}\sqrt{\pi}$

~~(2) $\Gamma(5) = 4! = 4\cdot3\cdot2 = 24$~~

(2) $B\left(\frac{3}{2},\frac{1}{2}\right) = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2}+\frac{1}{2})} = \frac{\frac{1}{2}(\Gamma(\frac{1}{2}))^2}{\Gamma(2)} = \frac{\pi}{2}$

2 (各2点)

5 (1) $x = \frac{1}{\sqrt{2}}(X - Y)$, $y = \frac{1}{\sqrt{2}}(X+Y)$ より　$x^2 - xy + y^2 \le 3$ →

$\frac{X^2}{2} + \frac{Y^2}{6} \le 1$, $y \ge x \to Y \ge 0$　∴ $\frac{X^2}{2} + \frac{Y^2}{6} \le 1$, $Y \ge 0$

(2) $J = 1$ より $I = -\sqrt{2}\iint_{\tilde{D}} Y\, dX\, dY = -\sqrt{2}\int_{-\sqrt{2}}^{\sqrt{2}}\frac{3}{2}(2 - X^2)\, dX$

$= -3\sqrt{2}\left[2X - \frac{1}{3}X^3\right]_0^{\sqrt{2}} = -3\sqrt{2}\cdot\frac{4}{3}\sqrt{2} = -8$

(2点)

6. (1) $J = \begin{vmatrix}a\cos\theta & -ar\sin\theta\\ b\sin\theta & br\cos\theta\end{vmatrix} = abr$

(2) $x = \frac{r}{\sqrt{2}}\cos\theta$, $y = r\sin\theta$ と変換すれば

$2x^2 + y^2 = r^2$ だから $D_R \to \widehat{D}_R$: $0 \le r \le R$, $0 \le\theta\le\frac{\pi}{2}$

(1) から $J = \frac{r}{\sqrt{2}}$ より

$I_R = \int_0^{\pi/2}\int_0^R \frac{\frac{1}{\sqrt{2}}r}{(1+r^2)^2}\, dr\, d\theta = \frac{1}{\sqrt{2}}\cdot\frac{\pi}{2}\left[-\frac{1}{2}\cdot\frac{1}{1+r^2}\right]_0^R$

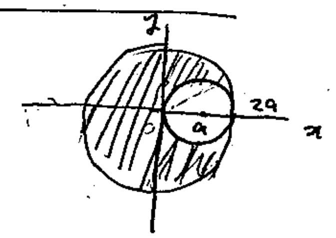
$= \frac{\pi}{4\sqrt{2}}\left(1 - \frac{1}{1+R^2}\right)$

(5点)

(3) $I = \lim_{R\to\infty} I_R = \frac{\pi}{4\sqrt{2}}$

7. $\iint x^2 + y^2 = 2ax \to (x-a)^2 + y^2 = a^2$ より

$\iint_D dx\, dy = \pi\cdot 4a^2 - \pi a^2 = 3\pi a^2$ (1)



対称性より $\bar{y} = 0$.　$x = r\cos\theta$, $y = r\sin\theta$

$\widehat{D}_1$: $2a\cos\theta \le r \le 2a$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$\widehat{D}_2$: $0 \le r \le 2a$, $\frac{\pi}{2} \le\theta\le\frac{3}{2}\pi$

∴ $\iint_D x\, dx\, dy = \int_{-\pi/2}^{\pi/2}\int_{2a\cos\theta}^{2a} r^2\cos\theta\, dr\, d\theta + \int_{\pi/2}^{3\pi/2}\int_0^{2a} r^2\cos\theta\, dr\, d\theta$ (2)(3)(4)(5)

$= \int_{-\pi/2}^{\pi/2}\left[\frac{1}{3}r^3\right]_{2a\cos\theta}^{2a}\cos\theta\, d\theta + \left(\int_{\pi/2}^{3\pi/2}\cos\theta\, d\theta\right)\left(\int_0^{2a} r^2\, dr\right)$

第2項 $= [\sin\theta]_{\pi/2}^{3\pi/2}\cdot\left[\frac{1}{3}r^3\right]_0^{2a} = (-2)\cdot\frac{1}{3}\cdot 8a^3 = -\frac{16}{3}a^3$ (6)

第1項 $= 2\int_0^{\pi/2}\frac{8}{3}a^3(1-\cos^3\theta)\cos\theta\, d\theta = \frac{16}{3}a^3\left(\int_0^{\pi/2}(\cos\theta - \cos^4\theta)\, d\theta\right)$ (7)

$= \frac{16}{3}a^3\left(1 - \frac{3}{16}\pi\right) = \frac{16}{3}a^3 - \pi a^3$ (8)

∴ $\iint_D x\, dx\, dy = -\pi a^3$ (9)　∴ $\bar{x} = \frac{-\pi a^3}{3\pi a^2} = -\frac{a}{3}$ (10)

(1) 3π　(2) ~~$2a\cos\theta$~~ $2a$　(3) ~~$2a$~~ $2a\cos\theta$　(4) $r^2\cos\theta$　(5) $2a$

(6) $-\frac{16}{3}a^3$　(7) $\frac{8}{3}a^3(1-\cos^3\theta)\cos\theta$

(8) $\frac{16}{3}a^3\left(1 - \frac{3}{16}\pi\right) = \frac{16}{3}a^3 - \pi a^3$　(9) $-\pi a^3$

(10) $-\frac{a}{3}$

8. (1) $\int \frac{dT}{T-30} = -kt + C$, $\log|T-30| = -kt + C$

$\therefore$ $T = 30 + Ce^{-kt}$

(2) $t=0$ で $T=90$ だから $C=60$ $\therefore$ $T = 30 + 60e^{-kt}$

$T(\frac{1}{2}) = 30 + 60e^{-\frac{k}{2}} = 45$ より $e^{-\frac{k}{2}} = \frac{1}{4}$

$\therefore$ $T(1) = 30 + 60e^{-k} = 30 + 60 \cdot (\frac{1}{4})^2 = 33.75$ °C $= \frac{135}{4}$

(3行)

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9. (1) 特性方程式 $(\lambda+1)^2 = 0$ $\therefore$ $\lambda = -1$ (2重解)

$\therefore$ $y = (C_1 + C_2 x)e^{-x}$

(2) $\eta = (Ax+B)e^x$ と予想する (A, B: 定数)

$\eta' = (Ax + A + B)e^x$, $\eta'' = (Ax + 2A + B)e^x$

だから これを ① の y に代入して

$(4Ax + 4A + 4B)e^x = xe^x$ $\therefore$ $A = \frac{1}{4}$, $B = -\frac{1}{4}$

$\therefore$ $\eta = \frac{1}{4}(x-1)e^x$

(5行)

(3) $y = (C_1 + C_2 x)e^{-x} + \frac{1}{4}(x-1)e^x$

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10. $y = x + v$ ($v = v(x)$) として方程式に代入すると

$v' + \frac{x-1}{x} v = -\frac{v^2}{2x}$, $w = \frac{1}{v}$ とおくと $w' = -\frac{v'}{v^2}$ より

$w' + \frac{1-x}{x} w = \frac{1}{2x}$ …① $w' + \frac{1-x}{x} w = 0$ を解くと

$w = \frac{Ce^x}{x}$, C を $u = u(x)$ に置きかえて ① に代入すると

$\frac{e^x}{x} u' = \frac{1}{2x}$ $\therefore$ $u' = \frac{1}{2}e^{-x}$ → $u = -\frac{1}{2}e^{-x} + C$

$\therefore$ $w = \frac{Ce^x - 1}{2x}$ $\therefore$ $v = \frac{2x}{Ce^x - 1}$ 以上より

$y = x + \frac{2x}{Ce^x - 1}$

(~~7行~~)

(5行)

(1) $\frac{x-1}{x}$ または $(1 - \frac{1}{x})$ (2) $-\frac{1}{2x}$

(3) $\frac{1-x}{x}$ または $(\frac{1}{x} - 1)$ (4) $\frac{1}{2x}$

(5) $\frac{e^x}{x}$ (6) $-\frac{1}{2}e^{-x}$

(7) $Ce^x - 1$

(8) $x + \frac{2x}{Ce^x - 1}$